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Almost Symplectic Structures and Harmonic Morphisms.

JEAN-MARIE BUREL

Sunto. – *In questo articolo, introduciamo la nozione di applicazione armonica simplettica fra varietà addomesticate e otteniamo qualche proprietà. Nel caso in cui le varietà siano quasi hermitiane, otteniamo un nuovo metodo per costruire applicazioni armoniche con fibre minimali. In fine, presentiamo un esempio di tali applicazioni fra spazi proiettivi.*

Summary. – *In this paper, we introduce the notion of symplectic harmonic maps between tamed manifolds and establish some properties. In the case where the manifolds are almost Hermitian manifolds, we obtain a new method to contruct harmonic maps with minimal fibres. We finally present examples of such applications between projectives spaces.*

1. – Introduction.

The study of minimal submanifolds is an important question in differential geometry. Useful tools for constructing such submanifolds are harmonic morphisms which are solutions of an overdetermined system of partial differential equations. More precisely, they are smooth maps $\phi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds which preserve Laplace's equation in the sense that if $f : U \subset N \to \mathbb{R}$ is a harmonic function with $\phi^{-1}(U)$ non-empty then $f \circ \phi : \phi^{-1}(U) \subset M \to \mathbb{R}$ is a harmonic function. Equivalently, [6], [8], they have been characterized as harmonic maps which are semi-conformal, where ϕ semi-conformal means that for any $x \in M$ ($d\phi(x) \neq 0$), the restriction of $d\phi(x)$ to the orthogonal complement of $\ker d\phi(x)$ in $T_x M$ is conformal and surjective. The conformal factor is called the *dilation* and denoted by $\lambda(x)(>0)$. We extend the function λ over critical points by giving it the value 0. The semi-conformal property is equivalent to:

$$h(d\phi(X), d\phi(Y)) = \lambda^2 g(X, Y)$$

for any horizontal vector fields X and Y. In [1], Baird and Eells gave a crucial geometric characterisation of harmonic morphisms. Introducing the stresstensor energy for harmonic maps,

$$S_\phi = e_\phi g - \phi^* h$$

where e_ϕ is the energy density, they establish that, if $\dim N = 2$, a semi-conformal map is harmonic and so a harmonic morphism if and only if the regular fibres are minimal. If $\dim N \geqslant 3$, a harmonic morphism of dilation λ has minimal fibres if and only if it is horizontally homothetic i.e. $\mathrm{grad}_{\mathcal{H}} \lambda^2(x) = 0$ with $x \in \overline{M} = M \setminus C_\phi$ where $\mathrm{grad}_{\mathcal{H}} \lambda^2(x)$ denotes the orthogonal projection of $\mathrm{grad}(\lambda^2(x))$ onto the horizontal space $\mathcal{H} = (\ker d\phi)^\perp$.

So we see that, in different situations, looking for a harmonic morphism is equivalent to finding a semi-conformal map with minimal fibres. It has been observed that the property of a map being a harmonic morphism is invariant under certain biconformal changes in the metric (cf. [9], [10]). A certain flexibility of the structure appears and so we can imagine that there exists a less rigid structure with respect to which it is possible to define a harmonic morphism. We therefore introduce the notion of *symplectic harmonic map*: this notion only needs an almost symplectic form ω, i.e. a non-degenerate 2-form not necessarily closed on a $2m$-dimensional manifold, and an almost complex structure J. The interesting fact of this definition is that it is the missing element for a holomorphic map between almost Hermitian manifolds to have minimal fibres. We recall that a map $\phi : (M^{2m}, J^M) \to (N^{2n}, J^N)$ between almost complex manifolds is called holomorphic when $J^N \circ d\phi = d\phi \circ J^M$. This approach in terms of almost symplectic structure and almost complex structure allows us to construct new families of harmonic maps with minimal fibres between almost Hermitian manifolds.

2. – Preliminaries.

Throughout this paper, we assume that all our objects such as manifolds, maps, etc., are smooth. We use the notation $\Gamma(E)$ to denote the space of smooth sections of a bundle $E \to M$. The natural framework for these ideas is the one of *tamed manifolds*, terminology due to Gromov [7], i.e. a smooth $2m$-dimensional manifold M endowed with an almost symplectic 2-form ω and an almost complex structure J such that

$$\omega(X, JX) > 0 \qquad \forall X \not\equiv 0 \in \Gamma(TM).$$

The property that $\omega(X, JX) > 0$ can be interpreted geometrically by saying that the orientation of the manifold given by ω coincides with that given by J. If (M^{2m}, ω, J) is a tamed manifold, then (g, J) is an almost Hermitian structure where

$$g(X, Y) = \frac{1}{2}(\omega(X, JY) + \omega(Y, JX)), \ X, Y \in \Gamma(TM) \tag{1}$$

and the *associated* 2*-form* Ω is defined by

$$\Omega(X, Y) = g(JX, Y), \qquad X, Y \in \Gamma(TM). \tag{2}$$

The structure J is *tamed* by ω if and only if the metric defined by (1) is positive definite. We first note.

LEMMA 1. – *If* (M^{2m}, ω, J) *is a tamed manifold and* Ω *denotes the associated 2-form defined by* (2), *then the following are equivalent*:

i) $\omega = \Omega$;

ii) ω *is compatible with* J;

iii) $\omega(X, Y) = g(JX, Y)$, $X, Y \in \Gamma(TM)$;

iv) $g(X, Y) = \omega(X, JY)$, $X, Y \in \Gamma(TM)$.

In this case, the manifold M is said to be *almost Hermitian.*

3. – Symplectic harmonicity.

In this section we introduce the new notion of *symplectic harmonicity*. We reformulate the minimality of the fibres of a holomorphic map in terms of differential form. Throughout this section, we shall assume that $m \geqslant n$.

DEFINITION 1. – *Let* (M^{2m}, ω^M, J) *be a tamed manifold and let* $\phi : (M^{2m}, \omega^M, J) \to (N^{2n}, J^N)$ *be a holomorphic map, then* ϕ *is* $(m-n)$*-symplectic harmonic along a regular fibre* $F = \phi^{-1}(y)$, $y \in N$ if, at any point x of F,

$$d(\omega^M)^{m-n}(e_1, \dots, e_{(m-n)}, J^M e_1, \dots, J^M e_{(m-n)}, .) = 0 \tag{3}$$

where $(\omega^M)^{m-n}$ *denotes the* $(m-n)$*-fold exterior product of* Ω *and* $\{e_i, J^M e_i\}_{i=1,\dots,m-n}$ *are tangent to the fibres.*

When ϕ is $(m-n)$-symplectic harmonic along any regular fibre, we will call ϕ a *symplectic harmonic* map.

Note that the definition is independent of the choice of the frame. Consider $\{u_1, \dots, u_m, J^M u_1, \dots, J^M u_m\}$, for $i = 1, \dots, m$

$$\begin{cases} u_i \quad\ = B_{ij} e_j + C_{ij} J^M e_j \\ J^M u_i = -C_{ij} e_j + B_{ij} J^M e_j \end{cases} \tag{4}$$

where the coefficients B_{ij}, C_{ij} satisfy:

$$\begin{cases} B_{ik}B_{il}+C_{ik}C_{il}=\delta_{kl} \\ B_{ik}C_{il}-B_{il}C_{ik}=0\,. \end{cases} \tag{5}$$

Substituing in (3) and applying (5), we deduce the independence by linearity of the forms.

As an example we have that any holomorphic map from an almost Kähler manifold to an almost complex manifold, is symplectic harmonic.

PROPOSITION 1. – *Any holomorphic map from a* $(1,2)$*-symplectic tamed manifold* (M^{2m},ω,J) *to an almost complex manifold* (N^{2m-2},J^N)*, is symplectic harmonic.*

PROOF. – Let (M^{2m},ω,J) be a $(1,2)$-symplectic tamed manifold and let ϕ be a holomorphic map. A manifold is $(1,2)$-symplectic if $(\nabla_X J)Y+(\nabla_{JX}J)JY=0$ for all $X,Y\in\Gamma(TM)$ or equivalently $d\omega(A,B,C)=0$ for any $A\in T^{1,0}M^{2m}$, $B,C\in T^{0,1}M^{2m}$ where $T^{\mathbb{C}}M^{2m}=T^{1,0}M^{2m}\oplus T^{0,1}M^{2m}$ is the decomposition of the complexified tangent bundle into $\pm i$-eigenspaces. Let $A=X-iJX$, $B=Y+iJY$, $C=Z+iJZ$. Then the real and imaginary parts of $d\omega(A,B,C)=0$ vanish if and only if

$$d\omega(X,Y,Z)-d\omega(X,JY,JZ)+d\omega(JX,Y,JZ)+d\omega(JX,JY,Z)=0\,. \tag{6}$$

Interchanging X and Y in (6) and combining the two, we have that $d\omega(A,B,C)=0$ is equivalent to $d\omega(X,JY,Z)=d\omega(JX,Y,Z)$.

Setting $Y=X$ in $d\omega(X,JY,Z)=d\omega(JX,Y,Z)$ yields $d\omega(X,JX,Z)=0$. On the other hand if $d\omega(X,JX,Z)=0$ is satisfied, replacing X by $X+Y$ then X by $X-Y$, and combining the two equations so obtained, we deduce that $d\omega(X,JY,Z)=d\omega(JX,Y,Z)$. Thus we obtain the equivalence between M being $(1,2)$-symplectic and $d\omega(X,JX,.)=0$. Let $\{e_1,J^Me_1\}$ be a local frame for the vertical space $\mathcal{V}=\ker d\phi$, then we have $d\omega(e_1,J^Me_1)=0$ i.e. ϕ is a symplectic harmonic map. ■

EXAMPLE 1. – Let P be a holomorphic polynomial

$$\begin{aligned} P:\mathbb{C}^2 &\to \mathbb{C} \\ (z,w) &\mapsto P(z,w) \end{aligned}$$

The vector field $Z=-P_w\partial_z+P_z\partial_w$ is in the kernel of dP, and so is $\overline{Z}$. Let $F:U\subset\mathbb{R}^2\to\mathbb{R}$ and let $u=|z|^2$, $v=|w|^2$. Define the non-degenerate 2-form ω by

$$\omega=F(|z|^2,\ |w|^2)(dz\wedge d\bar{z}+dw\wedge d\overline{w})=F(u,v)(dz\wedge d\bar{z}+dw\wedge d\overline{w})\,.$$

The polynomial P is *symplectic harmonic* with respect to ω if and only if

$d\omega(Z, \overline{Z}, .) = 0$ i.e.

$$\begin{cases} F_u \bar{z} \overline{P_z} + F_v \overline{w} \overline{P_w} = 0 \\ F_u z P_z + F_v w P_w = 0 \end{cases}$$

system which has a solution given by $F(u, v) = \alpha(u - v)$ with α a function of u and v only. ■

When M and N are both almost Hermitian, the notion of symplectic harmonicity can be expressed as follows.

THEOREM 1. – *Let $\phi : (M^{2m}, \Omega^M, J^M) \to (N^{2n}, \Omega^N, J^N)$ be a holomorphic map between almost Hermitian manifolds, then ϕ is symplectic harmonic if and only if the regular fibres of ϕ are minimal.*

PROOF. – Denote by $\Omega^M = \omega^M$ (cf. Lemma 1) the associated 2-form on M. We can locally choose $\{e_i, J^M e_i\}_{i=1, \dots, m-n}$ as a local orthonormal frame which spans the vertical space. Denote by $\{\theta_i, J^M \theta_i\}$ be the dual frame.

We have $d(\Omega^M)^{m-n} = (m-n)(\Omega^M)^{m-n-1} \wedge d\Omega^M$; locally Ω^M is characterized as the form

$$\Omega^M = \sum_{i=1}^{m-n} \theta_i \wedge J^M \theta_i + A \tag{7}$$

where A is a 2-form which vanishes on $\mathcal{V} = \ker d\phi$. We denote by ∇^M the Levi-Civita connection on M. Since $\theta_i(.) = g(e_i, .)$, we deduce that $d\theta_i(e_i, X) = -g(e_i, [e_i, X])$ i.e. $d\theta_i(e_i, X) = g(X, \nabla^M_{e_i} e_i)$. For any $X \in \Gamma(\mathcal{H} = \mathcal{V}^\perp)$, since $\mathcal{V}$ is integrable, we have

$$\begin{aligned} d(\Omega^M)^{m-n}(e_1, \dots, J^M e_{m-n}, X) &= \pm(m-n)! \sum_{i=1}^{m-n} d\Omega^M(e_i, J^M e_i, X) \\ &= \pm(m-n)! \sum_{i=1}^{m-n} d\theta_i(e_i, X) + dJ^M \theta_i(J^M e_i, X) \\ &= \pm(m-n)! \sum_{i=1}^{m-n} g^M(X, \nabla^M_{e_i} e_i + \nabla^M_{J^M e_i} J^M e_i). \end{aligned}$$

But the right-hand-side represents $2(m-n)$-times the mean curvature of the fibres of ϕ in the direction of X, thus the result follows. ■

The above proposition shows that the notion of *symplectic harmonic* maps is closely related to the one of *p-harmonic* maps. A p-harmonic map is a natural generalisation of the one of harmonic (or 2-harmonic) map. We remind that a p-harmonic map is a critical point of the p-energy functionnal i.e.

$$E_p(\phi) = \frac{1}{p}\int_M |d\phi|^p v_g.$$

We refer to [2] and the references therein for details. We deduce

THEOREM 2. – *Let $\phi : (M^{2m}, \Omega^M, J^M) \to (N^{2n}, \Omega^N, J^N)$ be a semi-conformal holomorphic submersion between almost Hermitian manifolds. Then the following relations are equivalent*

(i) *ϕ is $(m-n)$-symplectic harmonic,*

(ii) *ϕ is $2n$-harmonic*

(iii) *ϕ has minimal fibres.*

Applying Lemma 1.2 in [2], we have that a symplectic harmonic submersion between almost Hermitian manifolds, there exists a metric in the conformal class of the associated metric with respect to which the map can be rendered harmonic.

PROOF. – The equivalence between (ii) and (iii) is a direct consequence of Theorem 2.5 in [2]. Moreover from Theorem 1, ϕ being symplectic harmonic, is equivalent to the minimality of the fibres, and the result follows. ∎

Just as the rendering problem for harmonic maps is interesting – that is, given a smooth map, when can we find metrics with respect to which it is either harmonic or at least homotopic to a harmonic map? [5] – similarly we can pose an analogeous question concerning symplectic harmonic maps.

Given a holomorphic map from an almost complex manifold to a tamed manifold, when can we find an almost symplectic form ω such that J is tamed by ω, with respect to which the map is symplectic harmonic?

The following result shows that this can always be done locally.

PROPOSITION 2. – *Let $\phi : (M^{2m}, J^M) \to (N^{2n}, \omega^N, J^N)$ be a holomorphic submersion from an almost complex manifold to a tamed manifold. At any point $x \in M$, there exists a neighbourhood $U \subset M$ of x and a non-degenerate 2-form ω defined on U such that (U, ω, J^M) is a tamed manifold and such that the map $\phi|_U$ is symplectic harmonic.*

PROOF. – Let $x \in M$ and let $U \subset M$ be an open neighbourhood of x. We define locally, a system of coordinates $(x_1, \dots, x_{2m})$ such that $\{\partial x_1, \dots, \partial x_{2(m-n)}\}$ generates the vertical space V_y with $y \in U$. This is possible since ϕ is a submersion and is therefore locally equivalent to a projection of the form $(W \subset N)$ $W \times F \to W$. Now define a 2-form ω on U by

$$\omega = dx_1 \wedge dx_2 + \ldots + dx_{2(m-n)-1} \wedge dx_{2(m-n)} + \phi^* \omega^N.$$

This 2-form ω is clearly non-degenerate. Furthermore, we can suppose the coordinates suitably chosen, so that, on a possibly smaller neighbourhood, ω tames J^M. Given a frame $\{v_1, \ldots, v_{2(m-n)}\}$ of V_x, we may assume that $\partial x_j(x) = v_j$ with $j = 1, \ldots, 2(m-n)$. At the point x, we can now choose a basis of the form:

$$\{v_1, J^M v_1, \ldots, v_{(m-n)}, J^M v_{(m-n)}\}.$$

Thus at x, J^M is tamed by ω, by continuity it must also be tamed in a neighbourhood of x. With this choice of ω, the map $\phi|_U$ is $(m-n)$-symplectic harmonic. ■

4. – Symplectic minimal submanifold.

In this section, we extend the notion to immersions and introduce the one of *symplectic minimal submanifolds*. The following lemma is elementary.

LEMMA 2. – *Any holomorphically immersed submanifold* $(M^{2m}, J^M) \overset{\phi}{\hookrightarrow} (N^{2n}, \omega^N, J^N)$ *of a tamed manifold is also a tamed manifold with respect to the induced structure*

$$\omega^M = \phi^* \omega^N \ \textit{and}\ J^M = J^N|_{TM}.$$

From now on, we assume that *any holomorphically immersed submanifold* of a tamed manifold is endowed with this induced structure. By analogy with the case for Riemannian manifolds, it is natural to introduce the following.

DEFINITION 2. – *A holomorphically immersed submanifold* $(M^{2m}, J^M) \overset{\phi}{\hookrightarrow} (N^{2n}, \omega^N, J^N)$ *of a tamed manifold is called symplectic minimal if*

$$d(\omega^N)^m (d\phi(e_1), J^N d\phi(e_1), \ldots, d\phi(e_m), J^N d\phi(e_m), .) = 0 \tag{8}$$

with $\{e_i, J^M e_i\}_{i=1,\ldots,m}$ *a local frame of* TM.

A first example is that any regular pseudo-holomorphic map ([7]) in a symplectic tamed manifold (i.e. $d\omega^N = 0$) is a symplectic minimal surface.

The notions of symplectic minimal submanifolds and symplectic harmonic maps are related as follows.

THEOREM 3. – *Let* $\phi : (M^{2m}, \omega^M, J^M) \to (N^{2n}, \omega^N, J^N)$ $(m \geqslant n)$ *be a holomorphic map what is submersive almost everywhere between tamed manifolds,* ϕ *is symplectic harmonic if and only if* ϕ *has regular symplectic minimal fibres.*

PROOF. – Let $F = \phi^{-1}(y)$, $y \in N$ be a regular fibre, we have $\dim F = 2(m-n) = 2k$. Since ϕ is holomorphic, we can define an almost complex structure $J^F = J^M|_{TF}$. Let $\{e_i, J^M e_i\}_{i=1,\dots,k}$ be a local frame tangent to F. The canonical inclusion $i : F \hookrightarrow M$ is a holomorphic immersion. From now on, we identify F with its image in M. If ϕ is symplectic harmonic along F, we have

$$d(\omega^M)^k(e_1, J^F e_1, \dots, e_k, J^F e_k, .) = 0 .$$

This is equivalent to

$$d(\omega^M)^k(di(e_1), J^M di(e_1), \dots, di(e_k), J^M di(e_k), .) = 0 .$$

This relation means that F is symplectic harmonic for i. The converse is obvious. ■

When M is an almost Hermitian manifold, we have

THEOREM 4. – *An isometrically and holomorphically immersed submanifold of an almost Hermitian manifold is symplectic minimal if and only if it is minimal.*

PROOF. – Let $\phi : (M^{2m}, \Omega^M, J^M) \hookrightarrow (N^{2n}, \Omega^N, J^N)$ be a holomorphic isometric immersion between almost Hermitian manifolds with associated 2-forms Ω^M and Ω^N. The structure (g^M, J^M) is induced by the structure (g^N, J^N) where $g^N(X, Y) = \Omega^N(X, J^N Y)$. We identify M with its image by ϕ. Denote by ∇^N (resp. ∇^M) the Levi-Civita connection on N (resp. on M). Let $\{e_i, J^M e_i\}_{i=1,\dots,m}$ be a local orthonormal frame of TM. We complete by $2(n-m)$ linearly independent vector fields $\{e_j, J^N e_j : j = m+1, \dots, n\}$. For any $X \in \Gamma(TM)$, since X is a linear combinaison of (e_k, Je_k), $k = 1, \dots, m$, we have

$$d(\Omega^N)^m(e_1, J^M e_1, \dots, e_m, J^M e_m, X) = 0 .$$

Denote by $TM^\perp$ the normal bundle of M with respect to g^N and ∇^N the Levi-Civita connection on N. We have $d(\Omega^N)^m = m(\Omega^N)^{m-1} \wedge d\Omega^N$. Locally we can write Ω^N in the form:

$$\Omega^N = \sum_{i=1}^{m} \theta_i \wedge J^M \theta_i + A$$

where A is a 2-form which vanishes on TM and $\theta_i(.) = g^N(e_i, .)$. For any $X \in \Gamma(TM^{\perp})$, a similar computation as in the proof of Theorem 1, gives

$$d(\Omega^N)^m(e_1, \dots, J^M e_m, X) = \pm m!\, g^N(X, \tau(\phi))$$

where $\tau(\phi)$ represents the tension field of ϕ. It follows that the submanifold M is symplectic minimal if and only if it is minimal. ■

5. – Symplectic harmonic maps to a surface.

According to the geometric characterisation of Baird and Eells [1], A case of particular interest is when the dimension of the codomain equals to 2. The notion of symplectic harmonicity greatly simplifies the problem of finding harmonic morphisms reducing the second order equation of harmonicity to two first order conditions related respectively to the almost symplectic structure and the almost complex structure. From now on, we assume that $dimN = 2$. In the case where the domain is an almost Hermitian manifold, we have a new method to construct harmonic morphisms to surfaces.

THEOREM 5. – *Let ϕ be a holomorphic map from an almost Hermitian manifold (M^{2m}, Ω, J) to a surface, ϕ is symplectic harmonic if and only if ϕ is harmonic. Moreover, in this case, the map ϕ is a harmonic morphism.*

PROOF. – From Theorem 3, the map ϕ is symplectic harmonic if and only if the regular fibres are minimal. The geometric interpretation of harmonic morphisms [1] implies that ϕ is harmonic if and only if the regular fibres are minimal. It follows that ϕ is symplectic harmonic if and only if it is harmonic. The rank of $d\phi$ is, at a point $x \in M$, 0 or 2. Let x be a point of M with rank $d\phi(x) = 2$. If $X \in \ker d\phi(x)$ then $d\phi \circ (JX) = J^N \circ d\phi(X) = 0$, so JX is in $\ker d\phi(x)$ and $\ker d\phi$ can be identified with $\mathbb{C}^{m-1}$. So we obtain that $(\ker d\phi(x))^{\perp} \cong \mathbb{C}$ and $d\phi|_{(\ker d\phi(x))^{\perp}}$ commutes with J and is homothetic. In particular ϕ is horizontally weakly conformal. Since ϕ is harmonic and semi-conformal, ϕ is a harmonic morphism. ■

Applying our method, we can contruct new examples of harmonic maps with minimal fibres. In [4], we prove that the standard conformal class of $\mathbb{S}^4$ contains a family of metrics $g_{k,l}$ parametrized by a pair of positive integers (k, l) such that for each pair, there exists a non-constant harmonic morphism from $(\mathbb{S}^4, g_{k,l})$ to the Riemann sphere $\mathbb{S}^2$. When $M = \mathbb{C}P^2$, there is no globally defined harmonic morphism from $\mathbb{C}P^2$ to $\mathbb{C}P^1$ when $\mathbb{C}P^2$ is endowed with the Fubini-Study metric [11]. We prove.

THEOREM 6. – *There exist non-constant harmonic morphisms with respect to a family of metrics* $\{g_{k,l}\}$ *conformally equivalent to the Fubini-Study metric parametrized by pairs of positive integers* (k, l) *from* $\mathbb{C}P^2 \setminus \mathbb{C}P^1$ *to* $\mathbb{C}P^1$.

PROOF. – The complex projective space $\mathbb{C}P^2$ can be parametrized in the form $\mathbb{S}^3 \times [0, \pi/2]$ so that each point has for homogeneous coordinate $[\cos s, \sin sy]$ in $\mathbb{C}^3$ where $y \in \mathbb{S}^3 \subset \mathbb{C}^2$ and $s \in [0, \pi/2]$. The Fubini-Study metric on $\mathbb{C}P^2$ is given by

$$ds^2 + g^s_{\mathbb{S}^3}$$

where $g^s_{\mathbb{S}^3}$ is the metric on $\mathbb{S}^3$ obtained by rescaling the Euclidean metric on the fibre direction of the Hopf fibration with factor $\sin^2 s \cos^2 s$ and on its orthogonal complement by factor $\sin^2 s$. If $y = (\cos t e^{ia}, \sin t e^{ib})$ with $t \in [0, \pi/2]$, $a, b \in [0, 2\pi[$, the Fubini-Study metric on $\mathbb{C}P^2$ is given by

$$ds^2 + \sin^2 s \{\cos^2 s (\cos^2 t da - \sin^2 t db)^2 + \cos^2 t \sin^2 t (da + db)^2 + dt^2\}$$

i.e. $g_{\mathbb{C}P^2} = ds^2 + \sin^2 s \cos^2 s g^{\mathcal{V}} + \sin^2 s g^{\mathcal{H}}$. We construct a semi-conformal map $\Phi_{k,l}$ from $\mathbb{C}P^2$ to $\mathbb{C}P^1$ from the composition of two semi-conformal maps, ϕ from $\mathbb{C}P^2$ to $\mathbb{S}^3 = \mathbb{S}^0 * \mathbb{S}^2$ and $\phi_{k,l}$ from $\mathbb{S}^3 = \mathbb{S}^1 * \mathbb{S}^1$ to $\mathbb{S}^2$. To make possible such a composition we will define a new system of coordinates on an open dense subset of $\mathbb{C}P^2$, to identify the joint of $\mathbb{S}^0 * \mathbb{S}^2$ with the joint $\mathbb{S}^1 * \mathbb{S}^1$. Consider $\phi : \mathbb{C}P^2 \to \mathbb{S}^3$ defined by

$$\phi([\cos s, \sin sy]) = (\cos \alpha(s), \sin \alpha(s)\, H(y)) \tag{9}$$

where H is the Hopf map: $\mathbb{S}^3 \to \mathbb{S}^2$ and α a smooth function of s chosen such that the map ϕ is semi-conformal i.e. $\alpha(s) = 2 \arctan (A \tan^2 s/2)$. Without loss of generality, we may take $A = 1$. Critical set occurs when $s = 0$. We now make a change of coordinates $u(s, t)$, $\psi(s, t)$ setting

$$\cos \alpha(s) + i \sin \alpha(s) \cos 2t = \cos u(s, t)\, e^{i\psi(s, t)}$$

$$\sin \alpha(s) \sin 2t = \sin u(s, t).$$

Let $\beta : [0, \pi/2] \to [0, \pi/2]$ be a smooth function of u such that $\beta(0) = 0$ and $\beta(\pi/2) = \pi/2$. In the new system of coordinates, we define the map $\phi_{k,l}$: $\mathbb{S}^3 \to \mathbb{S}^2$ by

$$\phi_{k,l}(\cos u e^{i\psi}, \sin u e^{i(a+b)}) = (\cos \beta(u), \sin \beta(u)\, e^{i(k\psi + l(a+b))})$$

where $\beta(u)$ is chosen such that $\phi_{k,l}$ is semi-conformal i.e. (cf. [3] Example 4.10)

$$\beta(u) = 2\arctan\left(\frac{l-p}{l+p}\right)^{l/2}\left(\frac{k+p}{k-p}\right)^{k/2}$$

with $p(u) = \sqrt{k^2\sin^2 u + l^2\cos^2 u}$.

When $k = l = 1$, the map $\phi_{k,l}$ is the Hopf map. The map $\Phi_{k,l}$ is a globally defined continuous map from $\mathbb{C}P^2$ to $\mathbb{C}P^1$ with smooth and semi-conformal with respect to the Fubini-Study metric. We now render the map $\Phi_{k,l}$ harmonic, so a harmonic morphism with respect to a conformal metric to $g_{\mathbb{C}P^2}$. On $M^4 = \mathbb{C}P^2 \backslash (\{s = 0, \pi/2\} \cup \{u = 0\})$, we define the following frame

$$Y_1 = \frac{2}{\sin 2s}(\partial_a - \partial_b)$$

$$Y_2 = \frac{2\sin\alpha}{\sin s\sqrt{k^2\sin^2 u + l^2\cos^2 u}}((k(\sin^2 t\partial_a + \cos^2 t\partial_b) - l\partial_\psi)$$

$$X_1 = \frac{2\sin\alpha}{\sin s}\partial_u$$

$$X_2 = \frac{4\sin\alpha}{\sin s\sin 2u\sqrt{k^2\sin^2 u + l^2\cos^2 u}}(k\sin^2 u + l\cos^2 u(\sin^2 t\partial_a + \cos^2 t\partial_b))\,.$$

The map $\Phi_{k,l}$ determines an almost complex structure J defined as follows: J preserves $\mathcal{V}$, is orthogonal with respect to the standard conformal class of $\mathbb{C}P^2$, and induces the natural orientation on $\mathbb{C}P^2$. The almost complex structure J is given by $JY_1 = Y_2$ and $JX_1 = X_2$. We then introduce a non-degenerate 2-form ω defined by

$$\omega = f^2(u,\psi)\{2\sin\alpha\cos s[l\cos^2 ud\psi\wedge(\cos^2 tda - \sin^2 tdb) +$$
$$k\sin^2 uda\wedge db] + \sin u\cos u(kdu\wedge\psi + ldu\wedge(da + db))\}$$

where f is an unknown function to determine. Note that $\sin\alpha$, $\cos t$, $\sin t$ depend on u, ψ.

Since the form ω and the almost complex structure J are compatible, M^4 is an almost Hermitian manifold. The map Φ_{kl} is symplectic harmonic (so a harmonic morphism) if and only if

$$d\omega(Y_1, JY_1, .) = 0\,.$$

The last condition gives the following system

$$2\frac{f'_\psi}{f} = -\frac{(\sin\alpha\cos s)'_\psi}{\sin\alpha\cos s}$$

$$2\frac{f'_u}{f} = -\frac{l^2(\sin\alpha\cos s\cos^2 u)'_u + k^2(\sin\alpha\cos s\sin^2 u)'_u}{\sin\alpha\cos s(l^2\cos^2 u + k^2\sin^2 u)}$$

which has a solution given by

$$f^2(u,\psi) = \frac{1}{\cos s\sin\alpha(k^2\sin^2 u + l^2\cos^2 u)}\,.$$

The associated metric is thus expressed by

$$g = \frac{2}{\cos s\sqrt{(k^2\sin^2 2t + l^2\cos^2 2t)(\sin^4 s)/4 + l^2\cos^2 s}}\,g_{\mathbb{C}P^2}.$$

This metric is singular when $s = \pi/2$. The value $s = \pi/2$ corresponds to the focal manifold $\{[0, y]\,|\,y\in\mathbb{S}^3\}$: copy of $\mathbb{C}P^1$. Thus we obtain a family of harmonic morphisms with respect to g from $\mathbb{C}P^2\backslash\mathbb{C}P^1$ to $\mathbb{C}P^1$. ■

In higher dimensions, the technique still applies. We give a simplest proof of Theorem 4.11 in [3].

THEOREM 7. – *There exists a family of harmonic morphisms from* $\mathbb{S}^3\times\mathbb{S}^3\backslash(T^2\cup T^2)$ *to* $\mathbb{S}^2$ *parametrized by quadruples of non-zeros integers* (k, l, m, n).

REMARK 1. – *The* 2*-form* ω *cannot be extended over all* $\mathbb{S}^3\times\mathbb{S}^3$, *the notion of symplectic harmonicity is defined almost everywhere. However, the associated metric* g *defined by* (1) *is smooth.*

PROOF. – Consider the manifold $\mathbb{S}^3\times\mathbb{S}^3$ parametrized in the form

$$((\cos s e^{ia}, \sin s e^{ib}), (\cos t e^{ic}, \sin t e^{id}))\,,$$

where $s, t\in[0,\pi/2]$ and $a, b, c, d\in[0, 2\pi[$. Consider the map $\phi:\mathbb{S}^3\times\mathbb{S}^3\to\mathbb{S}^2$ defined by:

$$\phi((\cos s e^{ia}, \sin s e^{ib}), (\cos t e^{ic}, \sin t e^{id})) = (\cos\alpha(s,t), \sin\alpha(s,t)\, e^{i(ka+lb+mc+nd)})\,,$$

where $\alpha(s,t)$ is chosen such that ϕ is semi-conformal i.e. (cf. [3] example 4.10)

$$\frac{1}{\sin^2\alpha}(\dot\alpha_s^2 + \dot\alpha_t^2) = \frac{k^2}{\cos^2 s} + \frac{l^2}{\sin^2 s} + \frac{m^2}{\cos^2 t} + \frac{n^2}{\sin^2 t}\,.$$

Complementary critical and singular sets occur on the union of tori $(s = t = 0) \cup (s = t = \pi/2)$ and $(s = 0, t = \pi/2) \cup (s = \pi/2, t = 0)$ respectively.

Define a frame on $M^6 = \mathbb{S}^3 \times \mathbb{S}^3 \backslash (\{s = 0, \pi/2\} \cup \{t = 0, \pi/2\})$

$$Y_1 = \frac{1}{\sqrt{\dot{\alpha}_s^2 + \dot{\alpha}_t^2}} (\dot{\alpha}_t \partial_s - \dot{\alpha}_s \partial_t)$$

$$Y_2 = \frac{1}{\sqrt{l^2 \cos^2 s + k^2 \sin^2 s}} (l\partial_a - k\partial_b)$$

$$Y_3 = \frac{1}{\sqrt{n^2 \cos^2 t + m^2 \sin^2 t}} (n\partial_c - m\partial_d)$$

$$Y_4 = \frac{1}{B} [(n^2 \cos^2 t + m^2 \sin^2 t)(k \sin^2 s\partial_a + l \cos^2 s\partial_b) - (l^2 \cos^2 s + k^2 \sin^2 s)(m \sin^2 t\partial_c + n\cos^2 t\partial_d)]$$

$$X_1 = \frac{1}{\sqrt{\dot{\alpha}_s^2 + \dot{\alpha}_t^2}} (\dot{\alpha}_s \partial_s + \dot{\alpha}_t \partial_t)$$

$$X_2 = \frac{1}{A}\left[\frac{\sin t \cos t}{\sin s \cos s}(k \sin^2 s\partial_a + l \cos^2 s\partial_b) + \frac{\sin s \cos s}{\sin t \cos t}(m \sin^2 t\partial_c + n \cos^2 t\partial_d)\right]$$

$A = \sqrt{\cos^2 s \sin^2 s(n^2 \cos^2 t + m^2 \sin^2 t) + \cos^2 t \sin^2 t(l^2 \cos^2 s + k^2 \sin^2 s)}$ and $B = A\sqrt{l^2 \cos^2 s + k^2 \sin^2 s}\ \sqrt{n^2 \cos^2 t + m^2 \sin^2 t}$.

The map ϕ determines an almost complex structure J defined by $JY_1 = Y_3$, $JY_2 = Y_4$ and $JX_1 = X_2$. We then introduce a non-degenerate 2-form ω on M^6.

$$\omega = f^2(s, t)\left\{\frac{(\dot{\alpha}_t ds - \dot{\alpha}_s dt) \wedge (n \cos^2 t dc - m \sin^2 t dd)}{\sqrt{\dot{\alpha}_s^2 + \dot{\alpha}_t^2}\sqrt{n^2 \cos^2 t + m^2 \sin^2 t}} + \frac{A}{C}(l^2 \cos^2 s + k^2 \sin^2 s)\, da \wedge db - \frac{A}{C}(l \cos^2 s da - k \sin^2 s db) \wedge (m dc + n dd) + \frac{\sin 2t \sin 2s}{4A\sqrt{\dot{\alpha}_s^2 + \dot{\alpha}_t^2}}(\dot{\alpha}_s ds + \dot{\alpha}_t dt) \wedge (k da + l db + m dc + n dd)\right\}$$

where $C = 2(l^2 \cos^2 s + k^2 \sin^2 s)\sqrt{n^2 \cos^2 t + m^2 \sin^2 t}$ and f is an unknown function to determine.

Since ω and J are compatible, M^6 is an almost Hermitian manifold. The map ϕ is symplectic harmonic so a harmonic morphism with respect to the associated metric (1) provided

$$d(\omega \wedge \omega)(Y_1, JY_1, Y_3, JY_3, .) = 0 . \tag{10}$$

Let $h^4(s,t) = f^4(s,t)\dfrac{A}{2\sqrt{\dot\alpha_s^2 + \dot\alpha_t^2}(l^2\cos^2 s + k^2\sin^2 s)(n^2\cos^2 t + m^2\sin^2 t)}$.

Condition (10) is satisfied if and only if

$$[h^4(n^2\cos^2 t + m^2\sin^2 t)(l^2\cos^2 s + k^2\sin^2 s)\ \dot\alpha_t]'_t + [h^4(n^2\cos^2 t + m^2\sin^2 t)(l^2\cos^2 s + k^2\sin^2 s)\ \dot\alpha_s]'_s = 0 .$$

A solution is thus given by

$$h^4(s,t) = \frac{\sin s \sin t \cos s \cos t}{\sin\alpha(s)[(n^2\cos^2 t + m^2\sin^2 t)(l^2\cos^2 s + k^2\sin^2 s)]^{3/2}} .$$

Applying the semi-conformal equation, we deduce that

$$f^2(s,t) = \frac{\sqrt{2}}{(n^2\cos^2 t + m^2\sin^2 t)^{1/4}(l^2\cos^2 s + k^2\sin^2 s)^{1/4}} .$$

The associated metric takes the form

$$g = \frac{\sqrt{2}}{(n^2\cos^2 t + m^2\sin^2 t)^{1/4}(l^2\cos^2 s + k^2\sin^2 s)^{1/4}} g_{\mathbb{S}^3\times\mathbb{S}^3} \qquad \blacksquare$$

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