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Computing the Quantum Cohomology of Some Fano Threefolds and Its Semisimplicity.

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Sunto. – *Nella presente nota si calcola una presentazione esplicita dell'anello di coomologia quantica «small» per alcune threefold di Fano, ottenute scoppiando una o due curve lisce in $\mathbb{P}^3$ o nella quadrica liscia. Usando sistematicamente l'associatività del prodotto quantico, si rende necessario calcolare esplicitamente soltanto un sottoinsieme molto piccolo ed enumerativo della famiglia degli invarianti di Gromov-Witten. Successivamente, si mostra che tali varietà soddisfano la congettura di Dubrovin sulla semisemplicità della coomologia quantica, sia mediante una semplice verifica sulle presentazioni in precedenza calcolate, sia mostrando che una threefold di Fano liscia X con $b_3(X) = 0$ ammette un sistema eccezionale completo di generatori per la categoria derivata dei fasci coerenti. I dettagli si trovano nel preprint* [4] *e saranno pubblicati altrove.*

Summary. – *We compute explicit presentations for the small Quantum Cohomology ring of some Fano threefolds which are obtained as one- or two-curve blow-ups from $\mathbb{P}^3$ or the smooth quadric. Systematic usage of the associativity property of quantum product implies that only a very small and enumerative subset of Gromov-Witten invariants is needed. Then, for these threefolds the Dubrovin conjecture on the semisimplicity of Quantum Cohomology is proven by checking the computed Quantum Cohomology rings and by showing that a smooth Fano threefold X with $b_3(X) = 0$ admits a complete exceptional set of the appropriate length. Details are contained in the preprint* [4] *and will be published elsewhere.*

Smooth Fano threefolds have been classified by Iskovskih, Mori and Mukai [8, 9, 10, 11] in 106 deformation classes. The second Betti number b_2 ranges from 1 to 10; according to the classification, we denote with M_n^b the n-th element of the list of Fano threefolds M having $b_2(M) = b$.

We compute an explicit presentation for the following 13 Fano threefolds obtained as the blow-up of the the projective space or the quadric along one or two smooth curves:

THEOREM 1. – *The small Quantum Cohomology ring of the Fano threefold M_n^2 with $n = 21, 22, 26, 27, 29, 30, 33$ is isomorphic to the polynomial quotient ring*

$$\mathbb{C}[E, H, q_0, q_1]/(f_1^C - f_1^Q, f_2^C - f_2^Q),$$

where the relations are described in table 2.

SKETCH OF THE PROOF. – With the help of a specific software, we compute all the relations between Gromov-Witten invariants which arise from Associativity identities between three divisors. Precisely, the (small) quantum product of two cohomology classes α_1, α_2 is defined as

$$\alpha_1 * \alpha_2 := \alpha_1 \cup \alpha_2 + \sum_{t, \beta} I_\beta(\alpha_1, \alpha_2, T_t) \widehat{T}_t q^\beta$$

where $\{T_i\}$ is a basis of the vector space $\mathsf{H}^*(M, \mathbb{C})$, $\{\widehat{T}_i\}$ its Poincaré dual and β ranges over all the effective classes in $\mathsf{H}_2(M, \mathbb{Z})$. The associativity relations $P(i, j, k) = T_i * (T_j * T_k) - (T_i * T_j) * T_k = 0$ can be read as polynomials in $\mathbb{C}[x_1, \ldots, x_N]$, where $x_1, \ldots, x_N$ are the *essential* Gromov-Witten invariants, i.e. those which cannot be simplified by Divisor Axiom and do not vanish trivially because of the virtual dimension. We consider the ideal $J_A \subset \mathbb{C}[x_1, \ldots, x_N]$ generated by all these *associativity* relations.

Such relations are not enough, i.e., the affine variety $A \subset \mathbb{C}^N$ defined by J_A has positive dimension; so we have to find another ideal J_G of *geometric* relations such that the ideal $J_A + J_G$ determines a single point in $\mathbb{C}^N$, that is, such that the problem of determining the values of all the Gromov-Witten invariants involved in the quantum products is completely solved.

We summed up in table 1 some information about A; this variety can be thought as the object which describes associativity constraints. By comparing the dimension of A with the number N of essential invariants, we can observe that in all the cases that we studied, associativity relations play a rather big role in determining the values of Gromov-Witten invariants: indeed, we need only a very small set of additional geometric relations.

One of these geometric relations is given by the blow-up construction, as in the following:

LEMMA 1. – *In a threefold $\widetilde{X}$ which is the blow-up along a curve, the class F of an exceptional fiber is enumerative and the value of the Gromov-Witten invariant $I_F(\varphi)$ is -1, where φ is the cohomology class of an exceptional fiber.*

Apart from this standard invariant, we needed a few other geometric rela-

TABLE 1. – *Amount of geometric information needed in order to determine Quantum Cohomology. L denotes a line, C a conic, T a twisted cubic and U a rational quartic.*

Threefold	N	$\dim A$	$\deg A$
M^2_{22} = blow-up of $\mathbb{P}^3$ along U	24	2	3
M^2_{27} = blow-up of $\mathbb{P}^3$ along T	14	2	2
M^2_{30} = blow-up of $\mathbb{P}^3$ along C	14	3	1
M^2_{33} = blow-up of $\mathbb{P}^3$ along L	10	2	1
M^2_{21} = blow-up of Q^3 along U	24	3	5
M^2_{26} = blow-up of Q^3 along T	24	3	3
M^2_{29} = blow-up of Q^3 along C	14	3	2
M^3_{12} = blow-up of $\mathbb{P}^3$ along $L \sqcup T$	81	3	9
M^3_{18} = blow-up of $\mathbb{P}^3$ along $L \sqcup C$	81	3	6
M^3_{25} = blow-up of $\mathbb{P}^3$ along $L \sqcup L$	52	3	3
M^3_{10} = blow-up of Q^3 along $C \sqcup C$	81	3	23
M^3_{15} = blow-up of Q^3 along $L \sqcup C$	81	3	7
M^3_{20} = blow-up of Q^3 along $L \sqcup L$	81	3	5

tions, that are the following:

(1) $\qquad I_{L_0-F}(\varrho, \bullet) = \deg \mathcal{C} \qquad$ for $n = 22, 27, 30, 33$,

(2) $\qquad I_{L_0-2F}(\varrho, \varrho) = 1 \qquad$ for $n = 30$,

(3) $\qquad I_{L_0}(\varrho, \bullet) = 1$ and $I_{L_0-F}(\varrho, \varphi) = 1 \qquad$ for $n = 21, 26, 29$,

where we denoted by $\bullet$, φ and ϱ the cohomology class respectively of a point, of an exceptional fiber and of the pullback of a generic line class in $\mathbb{P}^3$ or Q^3, and we put $L_0 = (\varrho)_*$ and $F = (\varphi)_*$, where $(-)_*\colon \mathsf{H}^p(M, \mathbb{C}) \to \mathsf{H}_p(M, \mathbb{C})$ denotes Poincaré duality.

As an example, we report some of these specific computations in the following:

LEMMA 2. – $I_{L_0-F}(\varrho, \bullet) = \deg \mathcal{C}$ *for* $X = M^2_k$, $k = 22, 27, 30, 33$.

PROOF OF THE LEMMA. – Let $\ell \in \varrho$ and $x \in \bullet$ be generic representatives. Irreducible curves $\mathcal{C}'$ having class $L_0 - F$ are strict transforms of lines intersecting $\mathcal{C}$ in exactly one point. The map sending $\mathcal{C}' \in \overline{M}_{0,0}(X, L_0 - F)$ to $\mathcal{C}' \cap \mathcal{C}$ is a fibration over $\mathcal{C}$ with bidimensional fibers.

If $\deg \mathcal{C} > 1$, the class $L_0 - F$ contains also reducible curves; we want to show that these curves do not contribute to the invariant, i.e., that no line intersecting $\mathcal{C}$ with multiplicity greater than 1 can meet both x and ℓ.

Indeed, choose a generic plane $\mathbb{P}^2$ containing ℓ. The projection $\pi_x\colon \mathbb{P}^3 \to \mathbb{P}^2$

maps $\mathcal{C}$ into a plane rational curve $\mathcal{C}_0$ with $m = \frac{1}{2}(d-1)(d-2)$ nodes $x_1, \ldots, x_m$. The lines $\overline{x_i x}$ are exactly all the lines passing through x and ℓ and intersecting $\mathcal{C}$ with multiplicity greater than 1. Genericity of x implies that none of these m lines meets ℓ, so that no reducible curve in class $L_0 - F$ is involved in the computation of this Gromov-Witten invariant.

This implies both the enumerativeness of the invariant, since we have seen above that $\dim \overline{M}_{0,0}(X, L_0 - F) = 3$, and the possibility to compute it by considering only irreducible curves, that is, by counting the number of lines ℓ whose strict transforms meets $\mathcal{C}$ in exactly one point.

The cone projecting $\mathcal{C}$ from x intersects ℓ in $d = \deg \mathcal{C}$ points, who belong to the lines ℓ_i $(i = 1, \ldots, d)$ in the cone. The strict transforms of the ℓ_i are thus all the rational curves belonging to class $L_0 - F$ and meeting the strict transforms of x and ℓ; this proves the lemma. ∎

Finally, we «quantize» the presentation in a standard way (see e. g. [7]), obtaining a presentation for the Quantum Cohomology ring; the results are summed up in table 2. ∎

THEOREM 2. – *The small Quantum Cohomology ring of the Fano threefold* M_n^3 *for* $n = 10, 12, 15, 18, 20, 25$ *is isomorphic to the polynomial quotient ring*

$$\mathbb{C}[E_1, E_2, H, q_0, q_1, q_2]/(f_1^C - f_1^Q, f_2^C - f_2^Q, f_3^C - f_3^Q),$$

where the relations are described in table 3.

PROOF. – Analogous to the $b_2 = 2$ case, only a bit easier since classical cohomology has a presentation where all relations are quadratic polynomials in the divisors. The presentations are summed up in table 3. ∎

After these computations, with the help of another *ad-hoc* software we compute Quantum Cohomology for the 7 Fano threefolds $S_k \times \mathbb{P}^1$, $k = 2, \ldots, 8$, starting from existing computations [1].

Then we focus on the following conjecture, which has been proved by Bayer and Manin [3] for Del Pezzo surfaces, and holds for $\mathbb{P}^n$ because of Beĭlinson's Theorem and well-known Quantum Cohomology computations.

CONJECTURE (Dubrovin, [6] 4.2.2 (1); [3]; [2]). – *Let X be a (smooth complex compact) variety. The even Quantum Cohomology ring of X is generically semisimple if and only if*

(i) *X is Fano and*

(ii) *the bounded derived category of coherent sheaves on X admits a complete exceptional set of length* $\sum_q \dim \mathsf{H}^{q,q}(X)$.

TABLE 2. – *Relations in the small Quantum Cohomology rings of threefolds described in Theorem* 1.

X	f_i^C	f_i^Q
M_{21}^2	EH^2	$-8EHq_0+10H^2q_0-28Eq_0^2-6Hq_0^2$ $+8Hq_0q_1+16q_0^2q_1$
	$E^2-\frac{5}{2}EH+2H^2$	$-2Eq_0+3Hq_0+Eq_1+2q_0q_1$
M_{22}^2	EH^2	$-6EHq_0+10H^2q_0-15Eq_0^2-6Hq_0^2$ $-4Eq_0q_1+18Hq_0q_1+4q_0q_1^2$
	$E^2-\frac{7}{2}EH+4H^2$	$-\frac{5}{2}Eq_0+5Hq_0+Eq_1+3q_0q_1$
M_{26}^2	EH^2	$-4EHq_0+\frac{7}{2}H^2q_0-6Eq_0^2-2Hq_0^2+6Hq_0q_1$
	$E^2-\frac{7}{3}EH+\frac{3}{2}H^2$	$-\frac{5}{6}Eq_0+\frac{5}{6}Hq_0+Eq_1+\frac{1}{2}q_0q_1$
M_{27}^2	EH^2	$-3Eq_0+8Hq_0+3q_0q_1$
	$E^2-\frac{10}{3}EH+3H^2$	$Eq_1+\frac{1}{3}q_0$
M_{29}^2	EH^2	$4Hq_0$
	$E^2-2EH+H^2$	Eq_1
M_{30}^2	EH^2	$-2Eq_0+2Hq_0+2q_0q_1$
	$E^2-3EH+2H^2$	Eq_1
M_{33}^2	EH^2	q_0
	$E^2-2EH+H^2$	Eq_1

(conforming to the conjecture, we will call such length *appropriate*).

By direct computation, we check semisimplicity of the Quantum Cohomology for the above threefolds plus the quadric Q^3, the threefolds V_5, V_{22}, and other 12 threefolds for which a presentation was already known [1, 5, 13, 14].

Finally, we examine the other part of the conjecture. From [12] and the classification of Fano threefolds we deduce the following

PROPOSITION 1. – *Let X be a smooth Fano threefold with $b_3(X)=0$; then the bounded derived category of coherent sheaves on X admits a complete exceptional set of the appropriate length.*

Putting all the above pieces together, we prove the conjecture for 36 of the 59 Fano threefolds having only even cohomology:

TABLE 3. – *Relations in the small Quantum Cohomology rings of threefolds described in Theorem* 2.

X	f_i^C	f_i^Q
M^3_{10}	E_1E_2	$-2E_1q_0-2E_2q_0+4Hq_0+4q_0^2$
	$E_1^2-2E_1H+H^2$	$E_1q_1+2q_0q_1$
	$E_2^2-2E_2H+H^2$	$E_2q_2+2q_0q_2$
M^3_{12}	E_1E_2	$-2E_1q_0-4E_2q_0+8Hq_0+3q_0q_2$
	$E_1^2-2E_1H+H^2$	$E_1q_1+q_0q_1$
	$E_2^2-\frac{10}{3}E_2H+3H^2$	$-\frac{1}{3}E_1q_0-\frac{2}{3}E_2q_0+\frac{4}{3}Hq_0+\frac{1}{3}q_0q_1+E_2q_2+2q_0q_2$
M^3_{15}	E_1E_2	$-2E_1q_0-E_2q_0+2Hq_0$
	$E_1^2-E_1H+\frac{1}{2}H^2$	$-E_1q_0-\frac{1}{2}E_2q_0+Hq_0+E_1q_1+q_0q_1+\frac{1}{2}q_0q_2$
	$E_2^2-2E_2H+H^2$	$E_2q_2+q_0q_2$
M^3_{18}	E_1E_2	$-2E_2q_0+2Hq_0+2q_0q_2$
	$E_1^2-2E_1H+H^2$	E_1q_1
	$E_2^2-3E_2H+2H^2$	E_2q_2
M^3_{20}	E_1E_2	$-E_1q_0-E_2q_0+Hq_0$
	$E_1^2-E_1H+\frac{1}{2}H^2$	$-\frac{1}{2}E_1q_0-\frac{1}{2}E_2q_0+\frac{1}{2}Hq_0+E_1q_1+\frac{1}{2}q_0q_1+\frac{1}{2}q_0q_2$
	$E_2^2-E_2H+\frac{1}{2}H^2$	$-\frac{1}{2}E_1q_0-\frac{1}{2}E_2q_0+\frac{1}{2}Hq_0+\frac{1}{2}q_0q_1+E_2q_2+\frac{1}{2}q_0q_2$
M^3_{25}	E_1E_2	q_0
	$E_1^2-2E_1H+H^2$	E_1q_1
	$E_2^2-2E_2H+H^2$	E_2q_2

THEOREM 3. – *The following Fano threefolds verify Dubrovin conjecture:*
$\mathbb{P}^3$, Q^3, V_5, V_{22};
M^2_k *with* $k=21, 22, 24, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36$;
M^3_k *with* $k=10, 12, 15, 17, 18, 20, 24, 25, 27, 28, 30, 31$;
$\mathbb{P}^1\times S_k$ *with* $k=2, \dots, 8$.

On the other hand, Fano threefolds having $b_3>0$ present additional difficulties.

Precisely, many of them are known not to have a complete exceptional set of the appropriate length; thus, the Dubrovin conjecture is equivalent to non-semisimplicity of the big Quantum Cohomology ring. The computation of this latter ring is much more complicated, since it involves infinitely many Gromov-Witten invariants; also, the relation between the small ring and non-semisimplicity of the big ring is rather implicit, contrarily to the case of Fano threefolds having $b_3 = 0$.

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