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Ultraweakly Compact Operators and Dual Spaces.

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Sunto. – *In questo articolo si introduce e si caratterizza la classe di tutti gli operatori ultradebolmente compatti, definiti negli spazi di Banach per mezzo dei loro operatori coniugati. Si analizza la relazione esistente fra un operatore ultradebolmente compatti e il suo coniugato. Si presentano esempi di operatori appartenenti a questa classe. Inoltre, si studia la connessione fra la compattezza ultradebole di* $T \in L(X, Y)$ *e i sottospazi minimali di* Y' *e si presenta un risultato relativo alla fattorizzazione degli operatori ultradebolmente compatti.*

Summary. – *In this paper, the class of all bounded ultraweakly compact operators in Banach spaces is introduced and characterised in terms of their first and second conjugates. We analize the relationship between an ultraweakly compact operator and its conjugate. Examples of operators belonging to this class are exhibited. We also investigate the connection between ultraweak compactness of* $T \in L(X, Y)$ *and minimal subspaces of* Y' *and we present a result of factorisation for ultraweakly compact operators.*

1. – Introduction and preliminaries.

We use standard notations: X, Y are Banach spaces, B_X the closed unit ball of X, X' the dual space of X, $L(X, Y)$ the set of all bounded linear operators from X into Y, T', $N(T)$ and $R(T)$ the conjugate operator, the null space and the range of $T \in L(X, Y)$, respectively, J_X (or simply, J) the canonical embedding of X into X'', I_X the identity operator on X. If $A \subset X$ and $L \subset X'$, then $A^{\perp} := \{x' \in X' : x'(x) = 0 \text{ for all } x \in A\}$, ${}_{\perp}L := \{x \in X : x'(x) = 0 \text{ for all } x' \in L\}$; A is called relatively $\sigma(X, L)$-compact if its $\sigma(X, L)$-closure in X is $\sigma(X, L)$ -compact, A is $\sigma(X, L)$-sequentially compact if every sequence in A contains a subsequence converging in the topology $\sigma(X, L)$ to some element of X. If A, B are closed subspaces of X, then we denote by $T|_A$ the restriction of $T \in L(X, Y)$ to A and $\delta(A, B) := \inf\{\|a - b\| : a \in A, \|a\| = 1, b \in B\}$ is called the inclination of A to B.

A subspace L of X' has positive characteristic if for some $\alpha > 0$ and all

$x \in X$ we have $\sup\{|x'(x)| : x' \in B_L\} \geqslant \alpha\|x\|$. The largest α with this property is called the characteristic $r(L)$ of L and by [6, Theorem 7] it is equal to $\sup\{\beta \geqslant 0 : \beta B_{X'} \subseteq \overline{B_L}^{\sigma(X', X)}\}$. A closed, $\sigma(Y', Y)$-dense subspace N of Y' induces a Hausdorff topology $\sigma(Y, N)$ on Y. In [8], $\sigma(Y, N)$ is called an ultraweak topology on Y, where an extension of the Eberlein-Smulian theorem is obtained under certain conditions on N. In this paper, we study some other aspects of the ultraweak topologies on Y in relation to operators in $L(X, Y)$.

The following lemmas are elementary and help to read Definition 3 and the next results.

LEMMA 1. – Let E and N be closed subspaces of Y and Y' respectively such that $JE \cap N^{\perp} = \{0\}$. Then $JE + N^{\perp}$ is closed if and only if $\delta(JE, N^{\perp}) > 0$.

PROOF. – It is easy to see that $\delta(JE, N^{\perp}) = 1/\|P\|$, where P is the projection from $JE \oplus N^{\perp}$ onto JE with null space $N^{\perp}$ and by closed graph theorem $JE \oplus N^{\perp}$ is closed if and only if P is continuous. ■

LEMMA 2. – *Let N be a closed, $\sigma(Y', Y)$-dense subspace of Y'. Then $JY \oplus N^{\perp}$ is closed if and only if N has positive characteristic.*

PROOF. – Since the $\sigma(Y', Y)$-density of N is equivalent to $JY \cap N^{\perp} = \{0\}$ and by [6, Theorem 9] the characteristic of N coincides with the inclination of JY to $N^{\perp}$, the desired result follows from Lemma 1. ■

Following [1], we recall some facts about bounded operators in Banach spaces. Let V be a finite dimensional subspace of Y'' such that $JY \cap V = \{0\}$ and let $T \in L(X, Y)$. Then TB_X is relatively $\sigma(Y, {}_{\perp}V)$-compact if and only if $R(T'') \subseteq JY \oplus V$. We observe that such a V coincides with $({}_{\perp}V)^{\perp}$ (as $\dim V < \infty$), ${}_{\perp}V$ is $\sigma(Y', Y)$-dense (as $JY \cap V = JY \cap ({}_{\perp}V)^{\perp} = \{0\}$) and by Lemma 2 we have that $r({}_{\perp}V) > 0$.

These observations suggest the following notion:

DEFINITION 3. – *Let M be a closed, $\sigma(Y', Y)$-dense subspace of Y' with positive characteristic. We say that $T \in L(X, Y)$ is $M^{\perp}$-weakly compact if TB_X is relatively $\sigma(Y, M)$-compact and T is called ultraweakly compact if T is $M^{\perp}$-weakly compact for some M as above.*

The corresponding classes of operators will be denoted by $M^{\perp} - WC(X, Y)$ and $UWC(X, Y)$ respectively.

When $M = Y'$ our definition of a $M^{\perp}$-weakly compact operator coincides with the standard notion of a weakly compact operator and we have the following characterisation:

THEOREM 4. – *Let $T \in L(X, Y)$. Then the following properties are equivalent:*

a) *T is weakly compact.*

b) *$R(T'') \subseteq JY$.*

c) *T' is $\sigma(Y', Y) - \sigma(X', X'')$ continuous.*

PROOF. – See e.g. [7]. ■

In Section 2 we shall obtain characterisations analogous to those of Theorem 4. The properties corresponding to a), b) and c) are:

a') T is $M^\perp$-weakly compact.

b') $R(T'') \subseteq JY \oplus M^\perp$.

c') T' is $\sigma(Y', JY \oplus M^\perp) - \sigma(X', X'')$ continuous.

Also, we analize the relationship between an ultraweakly compact and its conjugate operator.

Section 3 describes examples and investigate special cases; in particular, the upper-semi-Fredholm operators and the strictly singular operators.

Let $N \subset Y'$ be a subspace. According to J. Dixmier [6], we call N minimal if N is closed, $\sigma(Y', Y)$-dense and there exists no proper subspace $L \subset N$ with both these properties.

A description of minimal subspaces $N \subset Y'$ in terms of compactness is given in

THEOREM 5. – *Let N be a closed, $\sigma(Y', Y)$-dense subspace of Y'. Then the following properties are equivalent:*

a) *N is minimal.*

b) *$Y'' = JY \oplus N^\perp$*

c) *B_Y is relatively $\sigma(Y, N)$-compact. In this case $\overline{B_Y}^{\sigma(Y, N)}$ is bounded in the norm topology on Y.*

PROOF. – [6, Theorems 11 and 13]. ■

The study of the existence of minimal subspaces turns out to be important to characterise Banach spaces which are isomorphic to dual spaces as shows the following result.

THEOREM 6. – *A Banach space X is isomorphic to a dual space if and only if X' contains a minimal subspace.*

PROOF. – [6, Theorem 17]. ■

Clearly a quasi-reflexive space X, (that is, $\dim X''/JX) < \infty$), is isomorphic to a dual space (briefly, is a dual space). Clark asks in [3] if a somewhat reflexive space, (that is, a Banach space in which each infinite dimensional closed subspace contains an infinite dimensional reflexive subspace), is a dual space. In order to prove that in general the answer to this question is negative we recall the following standard definition

DEFINITION 7. – *We say that a subclass $\mathcal{A}$ of the class of all Banach spaces has the three-space property if it satisfies the following condition : If A is a closed subspace of a Banach space X such that $X/A \in \mathcal{A}$ and $A \in \mathcal{A}$, then $X \in \mathcal{A}$.*

The following result is due to J. M. F. Castillo and M. González [2, Theorem 4.4.a.].

THEOREM 8. – *The class of somewhat reflexive spaces has the three-space property.*

EXAMPLE 9. – *There exists a somewhat reflexive space which is not isomorphic to a dual space.*

Let W be a Banach space such that $W'' = JW \oplus l_1$ [13] and let Y be a Banach space with copies of l_2 such that Y/l_2 is isomorphic to c_0, (see [14] for the existence of such Y). In [2, Theorem 3.7.b.] it is shown that $W \oplus Y'$ satisfies the following properties:

i) There is a Banach space, say X, such that $W \oplus Y'$ is isomorphic to X'.

ii) X contains a copy of l_2 with X/l_2 isomorphic to W'.

iii) X is not a dual space.

Since somewhat reflexivity is a three-space property (Theorem 8) and that if Z is a Banach space such that Z''/JZ is separable, then Z' is somewhat reflexive [3, Theorem 3.3], we conclude from ii) that X is somewhat reflexive.

However, it is not difficult to construct somewhat reflexive spaces isomorphic to a dual spaces. We cite two cases:

i) Let $\mathfrak{J}\mathfrak{T}$ be the James tree space which is obtained from the James space $\mathfrak{J}$ by replacing its index set (that is, the set of integers) by an infinite tree. Then $\mathfrak{J}\mathfrak{T}$ is somewhat reflexive and it is a dual space, [9, Chapter I.9].

ii) Let $X := (\sum \oplus \mathfrak{J})_2$. Since every closed infinite dimensional subspace of $\mathfrak{J}$ (respectively, of X') contains a subspace isomorphic to l_2 [9, Chapter I.9] (respectively, [17, Lemma 5.3]) we obtain that X is somewhat reflexive and X' does not contain isomorphic copy of l_1 which, by a virtue of a result due to G. Godofrey [10], implies that X is a dual space.

Section 4 investigates the connection between ultraweak compactness of $T \in L(X, Y)$ and minimal subspaces of Y'; in particular, using Dixmiers results on characteristics and Theorem 11 we obtain some results which extend Theorems 5 and 6 to ultraweakly compact operators. Finally, a sufficient condition is given for that an ultraweakly compact operator admits factorisations through Banach spaces isomorphic to a dual spaces.

2. – Characterisations.

LEMMA 10. – *Let N be a closed, $\sigma(Y', Y)$-dense subspace of Y' and K is a $\sigma(Y'', N)$-compact subset in Y''. Then we have:*

a) $K + N^{\perp}$ is $\sigma(Y'', N)$-compact.

b) $K + N^{\perp}$ is $\sigma(Y'', N)$-closed.

A subset E in Y is $\sigma(Y, N)$-compact if and only if JE is $\sigma(Y'', N)$-compact.

PROOF. – Let (u_α) be a net in $K + N^{\perp}$, $u_\alpha = k_\alpha + v_\alpha$, with $k_\alpha \in K$, $v_\alpha \in N^{\perp}$.

a) As K is $\sigma(Y'', N)$-compact, a subnet (k_β) of (k_α) converges in $\sigma(Y'', N)$ to some point $k \in K$. Therefore the subnet (u_β) of (u_α) converges to k with respect to $\sigma(Y'', N)$, so that the set $K + N^{\perp}$ is $\sigma(Y'', N)$-compact in Y''.

b) Let $u_\alpha \to u \in Y''$ in the $\sigma(Y'', N)$-topology. Then $u_\alpha(y') \to u(y')$ for all $y' \in N$. By a) there exists a subnet (u_β) of (u_α) such that $u_\beta(y') = k_\beta(y') \to k(y') = u(y')$ for all $y' \in N$. Hence $u - k \in N^{\perp}$ and so $u = k + (u - k) \in K + N^{\perp}$, showing that $K + N^{\perp}$ is $\sigma(Y'', N)$-closed in Y''.

The remaining part of the Lemma is clear. ∎

THEOREM 11. – *Let M be a closed, $\sigma(Y', Y)$-dense subspace in Y' with positive characteristic and let $T \in L(X, Y)$. Then the following properties are equivalent:*

a) $T \in M^{\perp} - WC(X, Y)$.

b) $R(T'') \subseteq JY \oplus M^{\perp}$.

c) JTB_X is relatively $\sigma(JY \oplus M^{\perp}, Y')$-compact.

d) T' is $\sigma(Y', JY \oplus M^{\perp}) - \sigma(X', X'')$ continuous.

e) The restriction of T' on M is $\sigma(M, Y) - \sigma(X', X'')$ continuous

PROOF. – a) $\Rightarrow$ b) Assume a) holds. Then $\overline{TB_X}^{\sigma(Y, M)}$ is $\sigma(Y, M)$-compact and by Lemma 10 it follows that $\overline{JTB_X}^{\sigma(Y'', M)} + M^{\perp}$ is $\sigma(Y'', M)$-closed. Now, since $B_{X''}$ is $\sigma(X'', X')$-compact in X'' (Alaoglus theorem), JB_X is $\sigma(X'', X')$-dense in $B_{X''}$ (Goldstine's theorem), T'' is $\sigma(X'', X') - \sigma(Y'', Y')$ continuous and the $\sigma(Y'', Y')$-closure of a set is contained in the $\sigma(Y'', M)$-closure of the set we obtain the chain of inclusions: $T'' B_{X''} = T'' \overline{JB_X}^{\sigma(X'', X')} \subseteq \overline{T'' JB_X}^{\sigma(Y'', Y')} = \overline{JTB_X}^{\sigma(Y'', Y')} \subseteq \overline{J\overline{TB_X}^{\sigma(Y, Y')}}^{\sigma(Y'', Y')} \subseteq \overline{J\overline{TB_X}^{\sigma(Y, M)}}^{\sigma(Y'', M)} \subseteq J\overline{TB_X}^{\sigma(Y, M)} + M^{\perp} \subseteq JY \oplus M^{\perp}$ as required.

b) $\Rightarrow$ a) Suppose that $R(T'') \subseteq JY \oplus M^{\perp}$ and let (x_α) be a net in B_X. Then

(Jx_α) is a net in $B_{X''}$ which is $\sigma(X'', X')$-compact and accordingly has a subnet, say (Jx_β), which is $\sigma(X'', X')$-convergent to some point $x'' \in B_{X''}$. Now, by the $\sigma(X'', X') - \sigma(Y'', Y')$ continuity of T'' we have $T''x'' = \sigma(Y'', Y') - \lim T''Jx_\beta$. By hyphotesis there exist $y \in Y$, $u \in M^\perp$ such that $T''x'' = Jy + u$ and consequently $\lim y'(Tx_\beta) = \lim JTx_\beta(y') = \lim T''Jx_\beta(y') = T''x''(y') = Jy(y') + u(y') = y'(y)$ for all $y' \in M$. Therefore (Tx_β) converges to y in the $\sigma(Y, M)$-topology on Y, showing that T is $M^\perp$-weakly compact.

b) $\Rightarrow$ c) Let (x_α) be a net in B_X. Then (Jx_α) is a net in $B_{X''}$ and there exists a subnet (x_β) of (x_α) such that $x'' = \sigma(X'', X') - \lim Jx_\beta$ for some $x'' \in B_{X''}$. Now, by assumption $T''x'' \in JY \oplus M^\perp$ and since T'' is $\sigma(X'', X') - \sigma(Y'', Y')$ continuous we obtain that $T''x''(y') = \lim JTx_\beta(y')$ for all $y' \in Y'$ which shows that JTB_X is relatively $\sigma(JY \oplus M^\perp, Y')$-compact, as required.

c) $\Rightarrow$ b) Suppose c) holds. Let $x'' \in B_{X''}$. Then, since $T''B_{X''}$ is contained in the $\sigma(Y'', Y')$-closure of JTB_X, there exists a net (x_α) in B_X such that $JTx_\alpha \to T''x''$ with respect to the $\sigma(Y'', Y')$-topology.

By hypothesis, (x_α) has a subnet (x_β) for which $y'' = \sigma(JY \oplus M^\perp, Y') - \lim JTx_\beta$ for some $y'' \in JY \oplus M^\perp$. Consequently $T''x'' = y''$. Hence $R(T'') \subseteq JY \oplus M^\perp$.

b) $\Rightarrow$ d) Let (y'_α) be a net in Y' converging to y' in the $\sigma(Y', JY \oplus M^\perp)$-topology on Y' and let $x'' \in X''$. Then, by assumption $T''x'' \in R(T'') \subseteq JY \oplus M^\perp$ and so $x''(T'y') = \lim x''(T'y'_\alpha)$, that is, $T'y' = \sigma(X', X'') - \lim T'y'_\alpha$.

d) $\Rightarrow$ e) Let (y'_α) be a net in M such that (y'_α) is $\sigma(M, Y)$-convergent to some $y' \in M$. Then $y' = \sigma(Y', JY \oplus M^\perp) - \lim y'_\alpha$, so that $T'y' = \sigma(X', X'') - \lim T'y'_\alpha$.

e) $\Rightarrow$ b) Let $T''x'' \in R(T'')$. We shall verify that $T''x''|_M$ is $\sigma(M, Y)$-continuous. For this, let (y'_α) be a net in M which is $\sigma(M, Y)$-convergent to some $y' \in M$. Then, by hypothesis $T'y' = \sigma(X', X'') - \lim T'y'_\alpha$ and hence $T''x''(y') = x''(T'y') = \lim x''(T'y'_\alpha) = \lim T''x''(y'_\alpha)$, as desired. Now, since M is $\sigma(Y', Y)$-dense in Y', there exists a linear functional on Y', say y'', such that $y''|_M = T''x''|_M$ and y'' is $\sigma(Y', Y)$-continuous. But, since a linear functional in Y'' belongs to JY if and only if it is $\sigma(Y', Y)$-continuous and $T''x'' - y''$ is an element in Y'' vanishing on M, we deduce that $T''x'' = y'' + (T''x'' - y'') \in JY \oplus M^\perp$. In consequence, $R(T'') \subseteq JY \oplus M^\perp$. ■

THEOREM 12. – *Let $T \in M^\perp - WC(X, Y)$. Let $(T'')_1$ denote the map $T'': X'' \to JY \oplus M^\perp$ and let U be the $\sigma(X''', X'')$-closure of $(T'')_1((M^\perp)')$. Then $T' \in ({}_\perp U)^\perp - WC(Y', X')$*

PROOF. – By Hahn-Banach theorem, we can identify $(JY \oplus M^\perp)'$ with $JY' \oplus (M^\perp)'$. We shall verify that $U := \overline{(T'')'_1((M^\perp)')}^{\sigma(X''', X'')}$ satisfies the following properties:

i) $U = ({}_\perp U)^\perp$.

ii) ${}_\perp U$ is a $\sigma(X'', X')$-dense subspace of X''.
iii) $r({}_\perp U) > 0$.
iv) T' is U-weakly compact.

The property i) follows immediately from Bipolar theorem.

Let $I : JY \oplus M^\perp \to Y''$ be the inclusion and J_1 denotes the canonical embedding of Y into $JY \oplus M^\perp$. Then from $J_Y T = T'' J_X$, $T'' = I(T'')_1$ and $J_Y = IJ_1$, it is seen that $(T'')_1 J_X = J_1 T$. Upon noting that $N(J_1')$ coincides with $(M^\perp)'$ and $N(J_X')$ is $\sigma(X''', X'')$-closed (as J_X' is $\sigma(X''', X'') - \sigma(X', X)$ continuous) it follows that $U \subseteq N(J_X') = R(J_X)^\perp$. This fact combined with $X''' = JX' \oplus (JX)^\perp$ [6, Theorem 15], $\delta(JX', U) \geq \delta(JX', (JX)^\perp)$ and Lemma 2 gives that ${}_\perp U$ is $\sigma(X'', X')$-dense in X'' and has positive characteristic. Hence ii) and iii) are true.

It only remains to prove iv). Since $T''' Y''' = (T'')_1'(JY \oplus M^\perp)' = (T'')_1'(JY' \oplus (M^\perp)') \subseteq (T'')_1' JY' + U \subseteq JX' \oplus U$, by Theorem 11 we can conclude that T' is $({}_\perp U)^\perp$-weakly compact, as desired. ■

The converse of this Theorem does not hold in general. Indeed, since $X''' = JX' \oplus (JX)^\perp$, the conjugate operator $T' \in L(Y', X')$ is always $(JX)^\perp$-weakly compact for any $T \in L(X, Y)$.

3. – Examples of ultraweakly compact operators.

We recall that $T \in L(X, Y)$ is said to be upper-semi-Fredholm, (briefly, $T \in SF_+(X, Y)$), if it has finite dimensional null space and closed range.

Follows immediately from [12, Proposition and Theorem 1] that if $T \in L(X, Y)$ is upper-semi-Fredholm and weakly compact, then X is reflexive. In order to obtain the analogue of this property for ultraweakly compact operators we shall need the following result.

THEOREM 13. – *Let A be a closed infinite dimensional subspace of X such that A is a dual space and X/A is reflexive. Then X is a dual space.*

PROOF. – See [2, Proposition 3.7.a.] ■

THEOREM 14. – *Let $T \in SF_+(X, Y) \cap UWC(X, Y)$. Then X is a dual space*

PROOF. – Assume that $T \in SF_+(X, Y)$. Then there exists a closed finite codimensional subspace A of X for which $(T|_A)^{-1}$ is continuous. Hence $I_A = (T|_A)^{-1}(T|_A)$. If $T \in UWC(X, Y)$, then it is clear that $T|_A$ is ultraweakly compact and so is I_A. Now, by Theorems 5, 6 and 11, A is isomorphic to a dual space and hence by above Theorem we have that X is a dual space, as required. ■

It is well-known that the classes of reflexive spaces and quasi-reflexive spaces have the three-space property. However, to be isomorphic to a dual space is not a three-space property. The construction of a counterexample in full details can be seen in [2, Theorem 3.7.b.].

As an application of Theorem 14 we obtain a result which gives conditions under which an ultraweakly compact operator is strictly singular. Recall that $T \in L(X, Y)$ is called strictly singular, abbreviated $T \in SS(X, Y)$, if it does not have a bounded inverse on any closed infinite dimensional subspace of X.

DEFINITION 15. – *We say that a Banach space X belongs to $\mathcal{VID}$ if no closed infinite dimensional subspace of X contains a closed infinite dimensional subspace isomorphic to a dual space.*

THEOREM 16. – *Let $T \in UWC(X, Y)$. Then $T \in SS(X, Y)$ if X or Y belongs to $\mathcal{VID}$.*

PROOF. – Suppose that T is not strictly singular, then there exists a closed infinite dimensional subspace A of X such that $T|_A$ is injective and open. Hence from Theorem 14, A is a dual space and so is $T(A)$, contradicting the assumption for X and Y. ■

Unfortunately not all the properties enjoyed by weakly compact operators are shared by ultraweakly compact operators. The following examples illustrate this fact.

It is clear that if X or Y is reflexive, then any $T \in L(X, Y)$ is weakly compact. However we have:

EXAMPLE 17. – *If Y is isomorphic to a dual space, then $T \in L(X, Y)$ is ultraweakly compact.*

It is a consequence of Theorems 5 and 11.

EXAMPLE 18. – *There exist Banach spaces X, Y and $T \in L(X, Y) \backslash UWC(X, Y)$ with X isomorphic to a dual space.*

To prove this we need only to remind that there is a surjective strictly singular operator $T : l_1 \rightarrow c_0$ [15, p. 75, 108]. Then $l_1$is a dual space (as $l'_\infty = Jl_1 \oplus (Jc_0)^\perp$) but T is not ultraweakly compact since if $T \in M^\perp - WC(l_1, c_0)$ for some subspace M of l_1 as in Definition 3, then by Theorem 11 we have that $R(T'') = l_\infty = Jc_0 \oplus M^\perp$ which contradicts the well-known fact that Jc_0 is not complemented in l_∞.

If $T \in L(X, Y)$ then T is weakly compact if and only if TB_X is $\sigma(Y, Y')$-sequentially compact. The following examples show that if M is a closed,

$\sigma(Y', Y)$-dense subspace of Y' with positive characteristic, then the statements «T is $M^{\perp}$-weakly compact» and «TB_X is $\sigma(Y, M)$-sequentially compact» are not equivalent.

EXAMPLE 19. – *There exist a Banach space X and a closed, $\sigma(X', X)$-dense subspace M of X′ with positive characteristic such that B_X is $\sigma(X, M)$-sequentially compact but not relatively $\sigma(X, M)$ -compact.*

Let X be the Banach space of all scalar-valued bounded functions on an uncountable set Γ, with countable support, equipped with the sup norm. Let M be the closed linear span of $\pi(\Gamma)$, where π is the canonical map defined by $\pi(\alpha)(x) := x(\alpha)$, $\alpha \in \Gamma$, $x \in X$. Then, by [8, Example 1], M is $\sigma(X', X)$ -dense, $r(M) > 0$, B_X is $\sigma(X, M)$-sequentially compact but not relatively $\sigma(X, M)$-compact.

EXAMPLE 20. – *There exist a Banach space X and a closed, $\sigma(X', X)$-dense subspace M of X′ with positive characteristic such that I_X is $M^{\perp}$-weakly compact but B_X is not $\sigma(X, M)$ -sequentially compact.*

Let X be the Banach space of all scalar-valued bounded functions on [0, 1], with the sup norm. Again let M be the closed linear span of $\pi([0, 1])$, where π is defined as in Example 19. Then M is $\sigma(X', X)$-dense in X' with positive characteristic. Moreover, B_X is $\sigma(X, M)$-compact but not $\sigma(X, M)$ -sequentially compact. A proof of these properties can be found in D. van Dulst [8, Example 2]. Thus $I_X \in M^{\perp} - WC(X) \backslash WC(X)$.

In contrast to Example 20 we should note that there exists a closed, $\sigma(l_\infty, l_1)$-dense subspace, say N, of l_∞ such that if $T \in L(l_1)$ then T is $N^{\perp}$-weakly compact if and only if T is compact.

Recall that if A is a closed subspace of a Banach space X a quasi-complement of A in X is a closed subspace $B \subset X$ such that $A \cap B = \{0\}$ and $A + B$ is dense in X.

EXAMPLE 21. – *Let N be a $\sigma(l_\infty, l_1)$-dense quasi-complement of Jc_0 in l_∞. Then $T \in L(l_1)$ is $N^{\perp}$-weakly compact if and only if T is compact.*

According to H. P. Rosenthal [18], there exists such a N and D. van Dulst [8, Corollary 1.3] proves that such a N has positive characteristic. Suppose that $T \in N^{\perp} - WC(l_1)$. Then TB_{l_1} is relatively $\sigma(l_1, N)$-compact and also $\sigma(l_1, N)$-sequentially compact since by virtue of [8, Corollary 1.2], if Z is a separable Banach space, then every closed subspace $M \subseteq Z'$ with positive characteristic has the Eberlein-Smulian property. Let (x_n) be a sequence in B_{l_1}. Then (x_n) has a subsequence (y_n) such that (Ty_n) is $\sigma(l_1, N)$-convergent. By the separability of c_0 and the Alaoglu theorem there exists a subsequence (z_n) of (y_n)

such that (Tz_n) is $\sigma(l_1, Jc_0)$-Cauchy, so that (Tz_n) is $\sigma(l_1, Jc_0 + N)$-Cauchy. Since $Jc_0 + N$ is dense in l_∞ we have that (Tz_n) is $\sigma(l_1, l_\infty)$-Cauchy. Now, upon observing that l_1 is weakly sequentially complete and the weak sequential convergence in l_1 coincides with the norm convergence we conclude that (Tz_n) is convergent. In consequence, T is compact.

4. – Ultraweakly compact operators and dual spaces.

THEOREM 22. – *Let $T \in L(X, Y)$ and let M be a closed, $\sigma(Y', Y)$-dense subspace of Y' such that M has positive characteristic and $M^\perp$ is contained in the closure of $R(T'')$. Then the following properties are equivalent:*

a) T is $M^\perp$-weakly compact but for no proper $\sigma(Y'', Y')$-closed subspace V of $M^\perp$ T is $({}_\perp V)^\perp$-weakly compact.

b) $\overline{R(T'')} = J\overline{R(T)} \oplus M^\perp$.

PROOF. – Suppose $a)$ holds. Then T is $M^\perp$-weakly compact and by Theorem 11, $R(T'') \subseteq JY \oplus M^\perp$. If $y'' = Jy + v \in R(T'') \subseteq N(T')^\perp$, $y \in Y$, $v \in M^\perp$, then $0 = y''(y') = y'(y) + v(y') = y'(y)$ for all $y' \in N(T') = R(T)^\perp$ (as $M^\perp \subseteq \overline{R(T'')} \subseteq N(T')^\perp$). Thus $y \in {}_\perp(R(T)^\perp) = \overline{R(T)}$. Therefore $R(T'') \subseteq J\overline{R(T)} \oplus M^\perp$. Now, as $\delta(J\overline{R(T)}, M^\perp) \geqslant \delta(JY, M^\perp)$ it follows from Lemmata 1 and 2 that $J\overline{R(T)} \oplus M^\perp$ is closed and hence $\overline{R(T'')} \subseteq J\overline{R(T)} \oplus M^\perp$. It is clear that $J\overline{R(T)}$ is contained in $\overline{R(T'')}$ and since $M^\perp \subseteq \overline{R(T'')}$ by hypothesis, we have the equality.

Assume that b) is true. Then by Theorem 11, $T \in M^\perp - WC(X, Y)$. Let V be a $\sigma(Y'', Y')$-closed subspace in Y''such that $V \subset M^\perp$ and $T \in ({}_\perp V)^\perp - WC(X, Y)$. Then $V = ({}_\perp V)^\perp$ (as V is $\sigma(Y'', Y')$-closed), ${}_\perp V$ is $\sigma(Y', Y)$ -dense with positive characteristic, $V \subset M^\perp \subseteq \overline{R(T'')}$ and T is V-weakly compact and so by the result just proved, $\overline{R(T'')} = J\overline{R(T)} \oplus V$ and since $\overline{R(T'')} = J\overline{R(T)} \oplus M^\perp$ by hypothesis with $V \subset M^\perp$, we deduce that $V = M^\perp$. ∎

As an application of Theorem 22 we are now going to characterise Banach spaces which are isomorphic to a dual spaces in terms of ultraweak compactness of operators.

THEOREM 23. – *A Banach space Z is isomorphic to a dual space if and only if there are X, Y Banach spaces and a range-closed $T \in L(X, Y)$ such that $R(T) = Z$ and T is $M^\perp$-weakly compact for some subspace M of Y' with $M^\perp \subseteq R(T'')$.*

PROOF. – If Z is a dual space, then by Theorems 5 and 6, there exists a minimal subspace N of Z' such that I_Z is $N^\perp$-weakly compact.

Conversely, let $T \in M^\perp - WC(X, Y)$ such that $R(T)$ is closed and $M^\perp$ is contained in $R(T'')$. As $R(T)$ is closed so is $R(T'')$ and by the first part of above Theorem we have that $R(T'') = JR(T) \oplus M^{\perp}$ and consequently $i''R(T)'' = R(T'') = JiR(T) \oplus M^{\perp}$, where i denotes the inclusion of $R(T)$ into Y. Therefore, $R(T)'' = JR(T) \oplus (i'')^{-1}(M^{\perp}) = JR(T) \oplus (i'M)^{\perp}$. But, as $M^{\perp} \subseteq R(T'') = R(i'')$ we obtain that ${}_{\perp}R(i'') = {}_{\perp}(N(i')^{\perp}) = N(i') \subseteq M$ and hence $N(i') + M$ is closed and so by [11, Lemma IV. 2.9], $i'(M)$ is closed. Again Theorem 5 assures that $i'M$ is a minimal subspace of $R(T)'$. ■

This Theorem is a generalisation of Theorem 6 and also of [19, Theorem 4.5].

Our next objective is to obtain a generalisation of Theorem 5 a) $\Leftrightarrow$ c) to ultraweakly compact operators. For this we shall need some auxiliar lemmata.

LEMMA 24. – *Let $T \in L(X, Y)$ and N a subspace of Y'. Then* $\delta(J\overline{R(T)}, N^{\perp}) = \sup\{\eta > 0 : \sup\{|y'(y)| : y' \in M \cap B_{Y'}\} \geq \eta\|y\|$ *for all* $y \in R(T)\}$.

PROOF. – The proof will not be included here since is long and it is very similar to that of [6, Theorem 9], with only minor changes. ■

LEMMA 25. – *Let $T \in M^{\perp} - WC(X, Y)$, E a subspace of Y such that $R(T'') \subseteq JE \oplus M^{\perp}$. Then the subset $G(E) := \{e \in E : Je + v \in T''B_{X''}$ for some $v \in M^{\perp}\}$ is $\sigma(Y, M)$-compact in Y.*

PROOF. – Let (e_α) be a net in $G(E)$. Then there exists a net (v_α) in $M^{\perp}$ such that $(Je_\alpha + v_\alpha)$ is a net in $T''B_{X''}$ which is $\sigma(Y'', Y')$-compact and also $\sigma(Y'', M)$-compact. Hence, a subnet $(Je_\beta + v_\beta)$ of $(Je_\alpha + v_\alpha)$ converges to some $u \in T''B_{X''}$ in the $\sigma(Y'', M)$-topology. If $u = Je + v$, $e \in E$, $v \in M^{\perp}$ we have that $e = \sigma(Y, M) - \lim e_\beta$ with $e \in G(E)$, showing that $G(E)$ is $\sigma(Y, M)$-compact in Y. ■

LEMMA 26. – *Let $T \in M^{\perp} - WC(X, Y)$ with $M^{\perp} \subseteq \overline{R(T'')}$. Then* $\sup\{\eta > 0 : \sup\{|y'(y)| : y' \in M \cap B_{Y'}\} \geq \eta\|y\|$ *for all* $y \in R(T)\} \leq \{\sup\{\|y\|/\|T\| : y \in \overline{TB_X}^{\sigma(Y, M)}\}\}^{-1} \leq 1$.

PROOF. – As in the first part of Theorem 22 we obtain that $\overline{R(T'')} = J\overline{R(T)} \oplus M^{\perp}$ and so the above Lemma assures that $G(\overline{R(T)})$ is $\sigma(Y, M)$-compact and hence $\sigma(Y, M)$ -closed. Consequently, $\overline{TB_X}^{\sigma(Y, M)} \subseteq G(\overline{R(T)}) \subseteq \overline{R(T)}$. Let $R := \sup\{\|y\|/\|T\| : y \in \overline{TB_X}^{\sigma(Y, M)}\}$. For $\varepsilon > 0$ there is $y \in \overline{TB_X}^{\sigma(Y, M)}$ such that $(R - \varepsilon)\|T\| \leq \|y\|$. Since $\overline{TB_X}^{\sigma(Y, M)} \subseteq \overline{R(T)}$ we may assume that $y \in R(T)$. Let $y' \in B_M$, $\lambda > 0$. Then since $y \in \overline{TB_X}^{\sigma(Y, M)}$ there is $z \in TB_X$ such that $|y'(z - y)| < \lambda$. Therefore $|y'(y)| \leq |y'(z)| + |y'(z - y)| \leq \|T\| + \lambda$ and thus $\sup\{|y'(y)| : y' \in B_M\} \leq \|y\|/(R - \varepsilon)$ and from this we obtain the first inequality and the second is obvious from the definition. The case $R = \infty$ clearly gives $\sup\{\eta > 0 : \sup\{|y'(y)| : y' \in M \cap B_{Y'}\} \geqslant \eta\|y\|$ for all $y \in R(T)\} = 0$. ∎

THEOREM 27. – *Let $T \in M^{\perp} - WC(X, Y)$ with $M^{\perp} \subseteq \overline{R(T'')}$. Then $\overline{TB_X}^{\sigma(Y, M)}$ is bounded in the norm topology on Y.*

PROOF. – $0 < \delta(JY, M^{\perp}) \leqslant \delta(J\overline{R(T)}, M^{\perp}) = \sup\{\eta > 0 : \sup\{|y'(y)| : y' \in M \cap B_{Y'}\} \geqslant \eta\|y\|$ for all $y \in R(T)\}$ by Lemmata 2 and 24. Then applying Lemma 26, $\sup\{\|y\|/\|T\| : y \in \overline{TB_X}^{\sigma(Y, M)}\}$ must be finite and the result is proved. ∎

THEOREM 28. – *Let N be a closed subspace of Y' with positive characteristic. Then $N^{\perp} \subseteq \overline{R(T'')}$ and T satisfies a) of Theorem 22 if and only if the following conditions hold for N:*

i) N is $\sigma(Y', Y)$-dense.

ii) $N^{\perp} \subseteq \overline{R(T'')}$.

iii) No proper closed subspace of N, say M, such that $\delta(J\overline{R(T)}, M^{\perp}) > 0$ satisfies the conditions i) and ii) above.

PROOF. – Suppose that $N^{\perp}$ is contained in the closure of $R(T'')$, $T \in N^{\perp} - WC(X, Y)$ but for no proper $\sigma(Y'', Y')$-closed subspace V of $N^{\perp}$ $T \in V - WC(X, Y)$. Then by the first part of Theorem 22, $\overline{R(T'')} = J\overline{R(T)} \oplus N^{\perp}$. Let M be a closed subspace of Y' verifying conditions i) and ii) and $M \subset N$. Then $N^{\perp} \subset M^{\perp} \subseteq \overline{R(T'')}$ and $\overline{R(T'')} = J\overline{R(T)} \oplus N^{\perp} \subseteq J\overline{R(T)} \oplus M^{\perp} \subseteq \overline{R(T'')}$, that is, T satisfies the property b) and equivalently the property a) in Theorem 22. Consequently $M = N$, so that iii) is true for N.

Conversely, if N is a closed subspace of Y' with positive characteristic satisfying conditions i)-iii), then $J\overline{R(T)} \oplus N^{\perp} \subseteq \overline{R(T'')}$. By Theorem 22 it is enough to prove the other inclusion. Suppose that there is $y'' \in \overline{R(T'')} \backslash J\overline{R(T)} \oplus N^{\perp}$ and let U be the subspace spanned by y''. Let $M := N \cap {}_{\perp}U$. Then M is a proper subspace of N. Since U is finite dimensional, $J\overline{R(T)} \oplus N^{\perp}$ is closed (as $\delta(J\overline{R(T)}, N^{\perp}) \geqslant \delta(JY, N^{\perp}) > 0$) and $(N \cap {}_{\perp}U)^{\perp} = N^{\perp} + ({}_{\perp}U)^{\perp} = N^{\perp} + U$ [4, Theorem III. 3.9], it follows that $J\overline{R(T)} \oplus N^{\perp} \oplus U = J\overline{R(T)} \oplus (N \cap {}_{\perp}U)^{\perp} := J\overline{R(T)} \oplus M^{\perp}$ is closed, so that by Lemma 1, $\delta(J\overline{R(T)}, M^{\perp}) > 0$ and it is clear that $M^{\perp} = N^{\perp} + U \subseteq \overline{R(T'')}$. Therefore by iii), M can not be $\sigma(Y', Y)$-dense. Let $y \in Y \backslash \{0\}$ such that $Jy \in JY \cap M^{\perp}$. Then $Jy = v + \lambda y''$, $v \in N^{\perp}$, $\lambda \in K \backslash \{0\}$ (as $JY \cap N^{\perp} = \{0\}$). Consequently, $y'' \in JY \oplus N^{\perp}$, showing that $\overline{R(T'')} \subseteq JY \oplus N^{\perp}$. Now, since $N^{\perp} \subseteq \overline{R(T'')}$ as in the first part of Theorem 22 we obtain that $\overline{R(T'')} \subseteq J\overline{R(T)} \oplus N^{\perp}$ and the proof is complete. ∎

It follows from Theorem 28 that Theorem 22 is a generalisation of Theorem 5 a) $\Leftrightarrow$ b). Thus it is seen from Theorems 11, 27 and 28 that we have Theorem 5.

To conclude, we shall obtain a result of factorisation for ultraweakly compact operators. For this end, we begin by presenting the factorisation construction of J. Davies, T. Figiel, W. Johnson and A. Pelczynski [5]. Given a Banach space Y, W a bounded convex symmetric non-empty subset of Y, we construct a Banach space Z and an operator $j : Z \to Y$ as follows. For $n \in N$, define $U_n := 2^n W + 2^{-n} B_Y$ which is an absorbing bounded convex symmetric set. Let $\|.\|_n$ be the Minkowski functional of U_n or equivalently, the unique norm such that $\{y \in Y : \|y\|_n < 1\} \subseteq U_n \subseteq \{y \in Y : \|y\|_n \leq 1\}$. We define, for $y \in Y$, $|||y||| := \left(\sum_{n=1}^{\infty} \|y\|_n^2\right)^{1/2}$, $Z := \{y \in Y : |||y||| < \infty\}$ and let j denote the identity embedding of Z into Y.

The factorisation construction may be used to factor any bounded operator between Banach spaces. Let $T \in L(X, Y)$, $W := TB_X$, then we have:

THEOREM 29. – *Given Banach spaces X and Y, an operator $T \in L(X, Y)$, the factorisation construction produces a Banach space Z and an operator $j : Z \to Y$ with the following properties:*

a) $TB_X \subseteq jB_Z$.

b) $(Z, |||.|||)$ *is a Banach space and j is continuous.*

c) j'' *is injective and* $(j'')^{-1} JY \subseteq JZ$.

d) $T = j(j^{-1}T)$ *with* $j^{-1}T \in L(X, Z)$ *and* $j \in L(Z, Y)$.

PROOF. – See [5, Lemma 1]. ■

From this result we deduce that j satisfies the properties:

e) $\overline{TB_X} \subseteq jB_Z$.

f) $j'' B_{Z''} \subseteq \overline{2^n J \overline{TB_X}^{\sigma(Y, Y')}}^{\sigma(Y'', Y')} + 2^{-n} B_{Y''}$.

Indeed, note that c) implies that jB_Z is closed [16, Corollary I.D.4] and so by a) it follows e).

From the definition of $\|.\|_n$ we have that $jB_Z \subseteq 2^n TB_X + 2^{-n} B_Y$. Applying J we obtain $JjB_Z = j'' JB_Z \subseteq 2^n JTB_X + 2^{-n} JB_Y \subseteq \overline{2^n J \overline{TB_X}^{\sigma(Y, Y')}}^{\sigma(Y'', Y')} + 2^{-n} \overline{JB_Y}^{\sigma(Y'', Y')}$. Then, since j'' is $\sigma(Z'', Z') - \sigma(Y'', Y')$ continuous, $\overline{JB_F}^{\sigma(F'', F')} = B_{F''}$ which is $\sigma(F'', F')$-compact for any Banach space F, we have the property f).

THEOREM 30. – *Let $T \in M^\perp - WC(X, Y)$ with $M^\perp \subseteq R(T'')$. Let Z and j be produced by the factorisation construction. Then $j'M$ is a minimal subspace of Z'.*

PROOF. – We first verify that $j'M$ is closed. Since $M^\perp \subseteq R(T'')$ it follows from d) that $M^\perp \subseteq R(T'') \subseteq R(j'') \subseteq N(j')^\perp$ and so, $M^\perp + N(j')^\perp$ is closed or equivalently $M + N(j')$ is closed [4, Theorem III.3.9] which implies that $j'M$ is a closed subspace of Z' by virtue of [11, Lemma IV.2.9].

Since T is $M^{\perp}$-weakly compact we have by Lemma 10 that $J\overline{TB_X}^{\sigma(Y, M)} + M^{\perp}$ is $\sigma(Y'', M)$-closed and so combining property f) with the fact that the $\sigma(Y'', Y')$-closure of a set is contained in the $\sigma(Y'', M)$-closure of the set, we obtain that $j''B_{Z''} \subseteq JY \oplus M^{\perp}$. In consequence, $M^{\perp} \subseteq R(j'') \subseteq JY \oplus M^{\perp}$ and thus [5, Lemma 4] assures that $Z'' = JZ \oplus (j'')^{-1}(M^{\perp}) = JZ \oplus (j'M)^{\perp}$ which according to Theorem 5 is equivalent to say that $j'M$ is a minimal subspace of Z'. ■

It is easy to see that in general the converse of this Theorem is false. Indeed, let $T \in L(l_1, c_0)$ be a surjective operator. Then it is clear that T factors through l_1 which is a dual space but T is not ultraweakly compact.

We also note that if T factors through a quasi-reflexive space if and only if there is a closed,$\sigma(Y', Y)$-dense subspace of Y', say M, such that $M^{\perp}$ is finite dimensional and T is $M^{\perp}$-weakly compact [1].

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