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Hideki Takuwa

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A Note of Uniqueness on the Cauchy Problem for Schrödinger or Heat Equations with Degenerate Elliptic Principal Parts.

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Sunto. – *In questo articolo studiamo la locale unicità nel problema di Cauchy per equazioni di Schrödinger o del calore con parte principale non negativa. Otteniamo l'unicità compatta sotto la condizione di una forma debole di pseudo convessità. Questo si collega ai risultati noti in ipotesi di pseudo convessità conormale ottenuti da Tataru, Hörmander, Robbiano-Zuily e L. T'Joen. Il nostro metodo si basa su di un tipo di trasformazione integrale ed una forma debole di stime di Carleman per operatori ellittici degeneri.*

Summary. – *We study the local uniqueness in the Cauchy problem for Schrödinger or heat equations whose principal parts are nonnegative. We show the compact uniqueness under a weak form of pseudo convexity. This makes up for the known results under the conormal pseudo convexity given by Tataru, Hörmander, Robbiano-Zuily and L. T'Joen. Our method is based on a kind of integral transform and a weak form of Carleman estimate for degenerate elliptic operators.*

1. – Introduction and the main results.

In this paper we consider the local uniqueness in the Cauchy problem for Schrödinger or heat type operators. We consider the following type operators

$$P(t, x, \partial_t, \partial_x) = \tilde{D}_t - \sum_{j,k=1}^{n} \partial_{x_j}(a^{jk}(x)\,\partial_{x_k}) + \sum_{l=1}^{n} b_l(x)\,\partial_{x_l} + c(t, x), \tag{1}$$

where $\tilde{D}_t$ is $-i\partial_t$ or ∂_t, a^{jk} are real valued C^1 functions with $a^{jk} = a^{kj}$, b_l are real valued L^∞_{loc} functions, and c is a complex valued L^∞_{loc} function.

Let S be an oriented C^2 hypersurface $S = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n;\ \phi(x) = \phi(x_0)\}$ $(\nabla_x \phi(x_0) \neq 0)$ which is noncharacteristic for $P(t, x, \partial_t, \partial_x)$. Since the problem is of local nature, we may assume $x_0 = 0$.

We shall denote by p_2 the principal symbol of $P(t, x, \partial_t, \partial_x)$, $p_2(x, \xi) =$

$\sum_{j,k=1}^{n} a^{jk}(x)\,\xi_j\xi_k$, and by H_{p_2} its Hamiltonian, $H_{p_2} = \sum_{l=1}^{n}\left(\frac{\partial p_2}{\partial \xi_l}\frac{\partial}{\partial x_l} - \frac{\partial p_2}{\partial x_l}\frac{\partial}{\partial \xi_l}\right)$.

Let us introduce now the main assumptions.

$$p_2(x,\xi) \geqslant 0, \quad \text{for } \xi \in \mathbb{R}^n,\ x \in \{x \in \mathbb{R}^n;\ \phi(x) \leqslant \phi(0)\}, \tag{2}$$

(3) $(t, x) \mapsto c(t, x)$ is C^∞ function in V_0, and there exists a positive r_0 such that c can be extended as the function $c(z, x)$ holomorphic in $|z| \leqslant r_0$,

One can find positive constants C_1, C_2 such that

$$H_{p_2}^2\phi(x,\xi) + C_1 p_2(x,\xi) \geqslant C_2\left|\sum_{l=1}^{n} b_l(x)\,\xi_l\right|^2, \tag{4}$$
$$\text{for } \xi \in \mathbb{R}^n,\ x \in \{x \in \mathbb{R}^n;\ \phi(x) \leqslant \phi(0)\}.$$

The assumption (4) is essentially a weak form of pseudo convexity. Our uniqueness result is the following one (compact uniqueness).

THEOREM 1. – *Let $P(t, x, \partial_t, \partial_x)$ be a differential operator of type* (1) *in a neighborhood Ω of the origin in $\mathbb{R}^{n+1}$ satisfying* (2), (3) *and* (4). *Assume that V is a sufficiently small neighborhood of the origin in $\mathbb{R}^{n+1}$ and $u \in H^1_{\text{loc}}(V)$ satisfies $Pu = 0$ in V and $\phi(x) \leqslant \phi(0)$ in* $\operatorname{supp} u$. *Moreover* $\operatorname{supp} u \cap \{(t, x) \in V;\ \phi(x) = \phi(0)\}$ *is compact in $\{(t, x) \in V;\ \phi(x) = \phi(0)\}$. Then there exists a neighborhood W of $0 \in \mathbb{R}^{n+1}$ where $u = 0$.*

Let us remark that, using the result of [10] on estimation of some commutator, it would be possible to weaken the analyticity assumption (3) to some Gevrey class. However we will not pursue this idea here.

As usual, Theorem 1 will be proved by means of a Carleman estimate that we describe now. From the fact that S is noncharacteristic, we may assume $\phi_{x_n}(0) \neq 0$, where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. We introduce new variables (s, y', y_n) given by

$$s = t, \qquad y' = x', \qquad y_n = \phi(0) - \phi(x).$$

Next we make another change of variables to get rid of cross terms between y' and y_n. Since $\sum a^{jk}(x, y)\,\xi_j\xi_k \geqslant 0$, we have $\left|\sum \frac{\partial a^{jk}}{\partial x_k}\xi_j\right| \leqslant C\sum a^{jk}\xi_j\xi_k$. Under these transformations, our hypothesis (2), (3) and (4) remain invariant (see L. Nirenberg [6], p. 215).

Therefore we have only to consider the problem for the following operator and assumptions.

Let $P(t, x, y, \partial_t, \partial_x, \partial_y)$ be the operator on $\mathbb{R}_t \times \mathbb{R}_x \times \mathbb{R}_y^{n-1}$ given by

$$P(t, x, y, \partial_t, \partial_x, \partial_y) = \frac{1}{a(x, y)} \widetilde{D}_t - \partial_x^2 - \sum_{j, k=1}^{n-1} \partial_{y_j}(a^{jk}(x, y)\, \partial_{y_k}) + \sum_{l=1}^{n-1} b_l(x, y)\, \partial_{y_l} + b_n(x, y)\, \partial_x + c(t, x, y), \tag{5}$$

where $a(x, y)$ and $a^{jk}(x, y)$ are real valued C^1 functions with $a(0, 0) \neq 0$, $a^{jk} = a^{kj}$, $b_l(x, y)$ are real valued L^∞_{loc} functions, and $c(t, x, y)$ is a complex valued L^∞_{loc} function.

The assumption (2) is equivalent to

$$\sum_{j, k=1}^{n-1} a^{jk}(x, y)\, \eta_j \eta_k \geqslant 0\,, \qquad \text{for } x \geqslant 0,\ \eta \in \mathbb{R}^{n-1}. \tag{6}$$

The assumption (3) is not changed ($c(t, x, y)$ is analytic in t) and $\operatorname{supp} u \subset \{(t, x, y);\ x \geqslant 0\}$. The assumption (4) becomes

$$\sum_{j, k=1}^{n-1} \left(\frac{\partial(a^{jk}(x, y))}{\partial x} + C_1 a^{jk}(x, y) \right) \eta_j \eta_k \geqslant C_2 \left| \sum_{l=1}^{n-1} b_l(x, y)\, \eta_l \right|^2, \qquad \text{for } x \geqslant 0,\ \eta \in \mathbb{R}^{n-1}. \tag{7}$$

Then we can state the following result.

THEOREM 2. – *Let $P(t, x, y, \partial_t, \partial_x, \partial_y)$ be a differential operator of type* (5) *satisfying* (3), (6) *and* (7). *Then there exist a small $\kappa > 0$, a large τ_0 and a positive constant C such that with $\psi(x) = (x - 6\kappa)^2 - (6\kappa)^2$ we have*

$$\int e^{\tau\psi(x)} \left(\frac{1}{2} \tau^3 \kappa^2 \, | e^{-\frac{1}{2\tau}|D_t|^2} u |^2 + \frac{\tau}{2} \, | e^{-\frac{1}{2\tau}|D_t|^2} u_x |^2 \right) dt\, dx\, dy \leqslant C \int e^{\tau\psi(x)} \, | e^{-\frac{1}{2\tau}|D_t|^2} Pu |^2 \, dt\, dx\, dy + C e^{-\frac{\tau\kappa^2}{2}} \int |u|^2 \, dt\, dx\, dy\,, \tag{8}$$

for all $\tau > \tau_0$ and all $u \in C_0^\infty(B_{\frac{\kappa}{4}})$ with $\operatorname{supp} u \in \{(t, x, y);\ x \geqslant 0\}$.

Let us now compare our result with the other works on this subject. First of all, the weak pseudo-convexity condition (4) has been considered by Nirenberg [6] in his work on degenerate elliptic operators with C^∞ coefficients (see also Colombini-Del Santo-Zuily [1]). When the coefficients are smooth and moreover analytic with respect to some variables, Tataru introduced new pseudo-convexity and principal normality conditions taking in accounts the partial analyticity and he proved uniqueness results assuming these conditions in [11]. He developed the idea given by Robbiano in [7]. However his analyticity assumption was of global nature and his work has been extended by Hörmander and Robbiano-Zuily under local analyticity assumptions. Later on, L. T'Joen [5] extended the above results to quasi-homogeneous operators.

In the case of Schrödinger operator, of type (1), if the coefficients are analytic with respect to x, then true uniqueness follows from the work of L. T'Joen [5]. For operators of type (1) (heat or Schrödinger), if p_2 is elliptic and c is analytic in t, then true uniqueness follows from Hörmander's or Robbiano-Zuily's work.

Therefore our aim is to weaken the ellipticity of p_2 to a degenerate elliptic condition, assuming a Levi type condition on the lower order terms.

Our plan in this paper is as follows. In Section 2 we give a Carleman-type estimate for a degenerate elliptic operator proved by Nirenberg in [6]. In Section 3 we review the properties of FBI transform as Hörmander in [4] in order to make use of the analyticity of coefficients. Finally in Section 4 we can prove Theorem 1 and Theorem 2 by using the estimates given in Section 2 and Section 3.

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2. – Carleman estimate for a degenerate elliptic second order operator.

In this section we shall prove a Carleman-type estimate for a degenerate elliptic second order operator following Nirenberg [6]. Before stating the lemma we introduce some notations. We set

$$A(x, y, \partial_x, \partial_y) = \partial_x^2 + \sum_{j,k=1}^{n-1} \partial_{y_j}(a^{jk}(x, y)\, \partial_{y_k}), \tag{9}$$

$$R_1(x, y, \partial_x, \partial_y) = \sum_{l=1}^{n-1} b_l(x, y)\, \partial_{y_l} + b_n(x, y)\, \partial_x. \tag{10}$$

The Carleman estimate for a degenerate elliptic second order operator is the following one (see Theorem and Section 3 in [6]).

THEOREM 3 [6]. – *Let Ω be a neighborhood of the origin in $\mathbb{R}^{n+1}$. One can find $C > 0$, $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ there exists $\tau(\delta) > 0$ such that,* *with* $\psi(x) = (x-\delta)^2 - \delta^2$, *and* $d\mu = dt\,dx\,dy$ *we have*

$$\int e^{\tau\psi(x)}\left(\tau^3(x-\delta)^2\,|v|^2 + \frac{\tau}{2}\,|v_x|^2\right)d\mu \leqslant \tag{11}$$

$$C\int e^{\tau\psi(x)}\,|(-A(x,\,y,\,\partial_x,\,\partial_y) + R_1(x,\,y,\,\partial_x,\,\partial_y))\,v|^2\,d\mu\,,$$

for all $\tau > \tau(\delta)$ *and* $v \in C^\infty(\Omega)$ *such that* $\operatorname{supp} v \subset \{(t,\,x,\,y) \in \mathbb{R}^{n+1};\ |t| \leqslant \delta,$ $0 \leqslant x \leqslant \frac{\delta}{2},\ |y| \leqslant \delta\}$.

For the sake of completeness we give the proof of this results.

PROOF OF THEOREM 3. – Let us set $w = e^{\frac{\tau}{2}\psi(x)}\,v$.
Then

$$v_x = \partial_x(e^{-\frac{\tau}{2}\psi(x)}w) = e^{-\frac{\tau}{2}\psi(x)}\left(\partial_x - \frac{\tau}{2}\psi_x(x)\right)w$$

$$= e^{-\frac{\tau}{2}\psi(x)}(\partial_x - \tau(x-\delta))\,w\,.$$

$$v_{xx} = e^{-\frac{\tau}{2}\psi(x)}(\partial_x^2 - 2\tau(x-\delta)\,\partial_x + \tau^2(x-\delta)^2 - \tau)\,w\,.$$

Thus

$$e^{\frac{\tau}{2}\psi(x)}A(x,\,y,\,\partial_x,\,\partial_y)\,v =$$

$$w_{xx} - 2\tau(x-\delta)\,w_x + (\tau^2(x-\delta)^2 - \tau)\,w + \sum_{j,\,k=1}^{n-1}\partial_{y_j}(a^{jk}\,\partial_{y_k}w) \equiv Qw\,.$$

We write $Q = Q_1 + Q_2$, where

$$Q_1 = \partial_x^2 + \tau^2(x-\delta)^2 + \sum_{j,\,k=1}^{n-1}\partial_{y_j}(a^{jk}\,\partial_{y_k})\,,$$

$$Q_2 = -2\tau(x-\delta)\,\partial_x - \tau\,.$$

Then

$$\|Qw\|^2 = \langle (Q_1+\tau)\,w + (Q_2-\tau)\,w, (Q_1+\tau)\,w + (Q_2-\tau)\,w\rangle =$$

$$\|(Q_1+\tau)\,w\|^2 + 2\operatorname{Re}\langle (Q_1+\tau)\,w, (Q_2-\tau)\,w\rangle + \|(Q_2-\tau)\,w\|^2 \geqslant$$

$$4\operatorname{Re}\langle (Q_1+\tau)\,w, (Q_2-\tau)\,w\rangle.$$

On the other hand

$$2\,\mathrm{Re}\langle w_x, (x-\delta)\,w\rangle = -\langle w, w\rangle,$$

$$2\,\mathrm{Re}\langle w_{xx}, (x-\delta)\,w_x\rangle = -\langle w_x, w_x\rangle,$$

$$2\,\mathrm{Re}\langle (x-\delta)^2 w, (x-\delta)\,w_x\rangle = -\,3\langle (x-\delta)^2 w, w\rangle,$$

and

$$2\,\mathrm{Re}\left\langle \sum_{j,\,k=1}^{n-1} \partial_{y_j}(a^{jk} w_{y_k}), (x-\delta)\,w_x\right\rangle =$$

$$\sum_{j,\,k=1}^{n-1}\left(\langle a^{jk} w_{y_k}, w_{y_j}\rangle + \left\langle (x-\delta)\,\frac{\partial a^{jk}}{\partial x} w_{y_k}, w_{y_j}\right\rangle\right).$$

By combining above inequalities, we have

$$\begin{aligned}
\|Qw\|^2 \geqslant{}& -8\tau\,\mathrm{Re}\langle w_{xx}, (x-\delta)\,w_x\rangle - 8\tau\,\mathrm{Re}\langle w_{xx}, w\rangle \\
& -8\tau^3\mathrm{Re}\langle (x-\delta)^2 w, (x-\delta)\,w_x\rangle - 8\tau^3\mathrm{Re}\langle (x-\delta)^2 w, w\rangle \\
& -8\tau\,\mathrm{Re}\left\langle \sum_{j,\,k=1}^{n-1} \partial_{y_j}(a^{jk} w_{y_k}), (x-\delta)\,w_x\right\rangle - 8\tau\,\mathrm{Re}\left\langle \sum_{j,\,k=1}^{n-1} \partial_{y_j}(a^{jk} w_{y_k}), w\right\rangle \\
& -8\tau^2\mathrm{Re}\langle w, (x-\delta)\,w_x\rangle - 8\tau^2\langle w, w\rangle \\
={}& 12\tau\langle w_x, w_x\rangle + 4\langle (\tau^3(x-\delta)^2 - \tau^2)\,w, w\rangle \\
& +4\tau\left(\sum_{j,\,k=1}^{n-1}\left(\langle a^{jk} w_{y_k}, w_{y_j}\rangle - \left\langle (x-\delta)\,\frac{\partial a^{jk}}{\partial x} w_{y_k}, w_{y_j}\right\rangle\right)\right) \\
\geqslant{}& 10\tau\langle w_x, w_x\rangle + \tau\langle w_x - \tau(x-\delta)\,w, w_x - \tau(x-\delta)\,w\rangle \\
& +\langle (2\tau^3(x-\delta)^2 - 4\tau^2)\,w, w\rangle \\
& +4\tau\left(\sum_{j,\,k=1}^{n-1}\langle a^{jk} w_{y_k}, w_{y_j}\rangle - \left\langle (x-\delta)\,\frac{\partial a^{jk}}{\partial x} w_{y_k}, w_{y_j}\right\rangle\right) \\
\geqslant{}& \int e^{\tau\psi(x)}\Bigg[\tau|v_x|^2 + \tau^3(x-\delta)^2\,|v|^2 \\
& +4\tau\sum_{j,\,k=1}^{n-1}\left(a^{jk} v_{y_k}\overline{v_{y_j}} - (x-\delta)\,\frac{\partial a^{jk}}{\partial x} v_{y_k}\overline{v_{y_j}}\right)\Bigg]\,d\mu\,.
\end{aligned}$$

In above calculation we use the triangle inequality

$$2\langle w_x, w_x\rangle \geqslant \langle w_x - \tau(x-\delta)\, w,\, w_x - \tau(x-\delta)\, w\rangle - 2\langle \tau(x-\delta)\, w,\, \tau(x-\delta)\, w\rangle,$$

and the fact that $\tau^3(x-\delta)^2 - 4\tau^2 \geqslant 0$ if τ large enough, since $|x| \leqslant \frac{\delta}{2}$ on the support of v.

It follows that

$$\int e^{\tau\psi(x)}\left[\tau^3(x-\delta)^2\,|v|^2 + \tau|v_x|^2 + 4\tau\sum_{j,k=1}^{n-1}\left(a^{jk} - (x-\delta)\frac{\partial a^{jk}}{\partial x}\right) v_{y_k}\overline{v_{y_j}}\right] d\mu \leqslant$$

$$\int e^{\tau\psi(x)}\,|-A(x, y, \partial_x, \partial_y)v|^2\, d\mu\,.$$

On the other hand

$$\int e^{\tau\psi(x)}\,|-A(x, y, \partial_x, \partial_y)\, v|^2\, d\mu \leqslant$$

$$2\int e^{\tau\psi(x)}\,|-A(x, y, \partial_x, \partial_y)\, v + R_1(x, y, \partial_x, \partial_y)\, v|^2\, d\mu +$$

$$4\int e^{\tau\psi(x)}\left(\left|\sum_{l=1}^{n-1} b_l(x, y)\, v_{y_l}\right|^2 + |b_n(x, y)\, v_x|^2\right) d\mu\,.$$

Then

$$\int e^{\tau\psi(x)}\left[\tau^3(x-\delta)^2\,|v|^2 + \left(\tau - 4\sup_{x,y}|b_n(x, y)|^2\right)|v_x|^2 +\right.$$

$$\left. 4\tau\sum_{j,k=1}^{n-1}\left(a^{jk} - (x-\delta)\frac{\partial a^{jk}}{\partial x}\right) v_{y_k}\overline{v_{y_j}} - 4\left|\sum_{l=1}^{n-1} b_l(x, y)\, v_{y_l}\right|^2\right] d\mu \leqslant$$

$$2\int e^{\tau\psi(x)}\,|-A(x, y, \partial_x, \partial_y)\, v + R_1(x, y, \partial_x, \partial_y)\, v|^2\, d\mu\,.$$

By the assumption (7), for large τ

$$4\tau\sum_{j,k=1}^{n-1}\left(a^{jk} - (x-\delta)\frac{\partial a^{jk}}{\partial x}\right) v_{y_k}\overline{v_{y_j}} - 4\left|\sum_{l=1}^{n-1} b_l(x, y)\, v_{y_l}\right|^2 \geqslant$$

$$\sum_{j,k=1}^{n-1}\left[4\left(\tau - \frac{C_1}{C_2}\right) a^{jk} + 4\left(-\tau(x-\delta) - \frac{1}{C_2}\right)\frac{\partial a^{jk}}{\partial x}\right] v_{y_k}\overline{v_{y_j}} \geqslant 0\,.$$

The proof of Lemma 1 is completed by making τ large enough and $\delta > 0$ small enough. ■

3. – Fundamental properties for an integral transform.

In this section we review the properties of an integral transform studied by L. Hörmander [4] and D. Tataru [11]. We consider the following operator acting on $L^2(\mathbb{R})$,

$$e^{-\frac{1}{2\tau}|D_t|^2}u(t)=\int_{\mathbb{R}}\left(\frac{\tau}{2\pi}\right)^{\frac{1}{2}}e^{-\frac{\tau}{2}(t-s)^2}u(s)\,ds\,,\qquad t\in\mathbb{R}\,.$$

A simple computation yields

$$e^{-\frac{1}{2\tau}|D_t|^2}(t^l u)=\left(t+i\frac{D_t}{\tau}\right)^l e^{-\frac{1}{2\tau}|D_t|^2}u\,.$$

We introduce the new operator associated with $t+iD_t/\tau$. We fix a cutoff function $\chi\in C_0^\infty(\mathbb{R})$ such that $0\leqslant\chi\leqslant 1$, $\chi(t)=1$ when $|t|\leqslant 1$, and $\chi(t)=0$ when $|t|\geqslant 2$. For $\kappa>0$ we set

$$X_\kappa(t,D_t)=\chi\left(\frac{t}{\kappa}\right)t+i\chi\left(\frac{D_t}{\kappa\tau}\right)\frac{D_t}{\tau}\,.$$

In what follows L^2 will mean $L^2(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ will denotes the Schwartz space

Thanks to the next lemma we can regard $\widetilde{D}_t$ as a lower order term.

LEMMA 1. – (*i*) *For any* $\kappa>0$, $\tau>0$, *we have*

$$\left\|\chi\left(\frac{D_t}{\kappa\tau}\right)\frac{D_t}{\tau}W\right\|_{L^2}\leqslant 2\kappa\|W\|_{L^2},\qquad W\in\mathcal{S}(\mathbb{R})\,.$$

(*ii*) *For any* $\kappa>0$ *there exists* $\tau_0(\kappa)>0$ *such that for* $\tau>\tau_0(\kappa)$ *we have*

$$\left\|\left(1-\chi\left(\frac{D_t}{\kappa\tau}\right)\right)\frac{D_t}{\tau}e^{-\frac{1}{2\tau}|D_t|^2}w\right\|_{L^2}\leqslant e^{-\frac{\tau\kappa^2}{4}}\|w\|_{L^2},\qquad w\in\mathcal{S}(\mathbb{R})\,. \tag{13}$$

PROOF OF LEMMA. – 1. – We denote by ξ_0 the dual variable of t in this section.

By the definition of χ we have

$$\chi\left(\frac{\xi_0}{\kappa\tau}\right)\frac{|\xi_0|}{k\tau}\leqslant 2\,.$$

Then the estimate (12) follows from Parseval's formula.

Let us prove (13). We have only to prove

$$\left(1-\chi\left(\frac{\xi_0}{\kappa\tau}\right)\right)\frac{|\xi_0|}{\tau}e^{-\frac{1}{2\tau}|\xi_0|^2}\leqslant e^{-\frac{\tau\kappa^2}{4}}.$$

On the support of $1-\chi\left(\frac{\xi_0}{\kappa\tau}\right)$ we have $|\xi_0|\geqslant\kappa\tau$. This implies

$$\frac{1}{2\tau}|\xi_0|^2-\frac{\tau\kappa^2}{4}\geqslant\frac{\kappa}{2}|\xi_0|-\frac{\kappa}{4}|\xi_0|=\frac{\kappa}{4}|\xi_0|\geqslant\frac{\tau\kappa^2}{4}.$$

Then there exists a positive constant $\tau_0(\kappa)>0$ such that we have

$$\left(1-\chi\left(\frac{\xi_0}{\kappa\tau}\right)\right)\frac{|\xi_0|}{\tau}e^{-\left(\frac{1}{2\tau}|\xi_0|^2-\frac{\tau\kappa^2}{4}\right)}\leqslant\left(1-\chi\left(\frac{\xi_0}{\kappa\tau}\right)\right)|\xi_0|e^{-\frac{\kappa}{8}|\xi_0|}\,\frac{1}{\tau}e^{-\frac{\tau\kappa^2}{8}}\leqslant 1,$$

for $\tau\geqslant\tau_0(\kappa)$.

The proof is completed. ■

This lemma implies

$$\text{(14)}\qquad \|\widetilde{D}_t e^{-\frac{1}{2\tau}|D_t|^2}w\|_{L^2}\leqslant 2\kappa\tau\|e^{-\frac{1}{2\tau}|D_t|^2}w\|_{L^2}+\tau e^{-\frac{\tau\kappa^2}{4}}\|w\|_{L^2},\qquad w\in\mathcal{S}(\mathbb{R}).$$

The next lemma will be fundamental in estimating the commutator between the operator $e^{-\frac{1}{2\tau}|D_t|^2}$ and the coefficients of our operator.

LEMMA 2. – *(i) For $\kappa>0$ and $l\in\mathbb{N}$ we have*

$$\text{(15)}\qquad \left\|\chi\left(\frac{t}{\kappa}\right)t^lW\right\|_{L^2}\leqslant(2\kappa)^l\|W\|_{L^2},\qquad W\in\mathcal{S}(\mathbb{R}).$$

(ii) For $\kappa>0$, there exists a $\tau_0(\kappa)>0$ such that for $\tau>\tau_0(\kappa)$ we have

$$\text{(16)}\qquad \left\|\left(1-\chi\left(\frac{t}{\kappa}\right)\right)t^le^{-\frac{1}{2\tau}|D_t|^2}w\right\|_{L^2}\leqslant C_le^{-\frac{\tau\kappa^2}{4}}\|w\|_{L^2},\qquad w\in C_0^\infty(B_{\frac{\kappa}{4}}).$$

PROOF OF LEMMA 2. – The proof of (15) is straightforward. Let us prove (16). On the support of $1-\chi\left(\frac{t}{\kappa}\right)$ we have $|t|\geqslant\kappa$. For $w\in C_0^\infty(B_{\frac{\kappa}{4}})$

$$\text{(17)}\qquad e^{-\frac{1}{2\tau}|D_t|^2}w=\int_{|s|<\frac{\kappa}{4}}e^{-\frac{\tau}{2}(t-s)^2}w(s)\,ds.$$

For $|t|\geqslant\kappa$ and $|s|\leqslant\frac{\kappa}{4}$ we have $|t-s|\geqslant|t|-|s|\geqslant|t|-\frac{\kappa}{4}\geqslant\frac{3}{4}|t|$.

Therefore

$$|e^{-\frac{1}{2\tau}|D_t|^2}w| \leqslant \left(\int\limits_{|s|<\frac{\kappa}{4}} e^{-\tau(t-s)^2}ds\right)^{\frac{1}{2}} \|w\|_{L^2} \leqslant$$

$$\left(e^{-\tau\frac{9}{16}|t|^2}\int\limits_{|s|<\frac{\kappa}{4}} ds\right)^{\frac{1}{2}} \|w\|_{L^2} \leqslant e^{-\frac{\tau}{2}\frac{9}{16}|t|^2}\left(\frac{\kappa}{2}\right)^{\frac{1}{2}} \|w\|_{L^2}.$$

It follows that

$$\left(\int\left|\left(1-\chi\left(\frac{t}{\kappa}\right)\right)t^l e^{-\frac{1}{2\tau}|D_t|^2}w\right|^2 dt\right)^{\frac{1}{2}} \leqslant \left(\frac{\kappa}{2}\right)^{\frac{1}{2}} \|w\|_{L^2}\left(\int\limits_{|t|\geqslant\kappa} t^{2l}e^{-\tau\frac{9}{16}|t|^2}dt\right)^{\frac{1}{2}} \leqslant$$

$$\left(\frac{\kappa}{2}\right)^{\frac{1}{2}} \|w\|_{L^2}\left(\int\limits_{|t|\geqslant\kappa} t^{2l}e^{-\tau\frac{1}{16}|t|^2}dt\right)^{\frac{1}{2}} e^{-\frac{\tau\kappa^2}{4}} \leqslant C_\kappa e^{-\frac{\tau\kappa^2}{4}} \|w\|_{L^2}.$$

The proof of Lemma 2 is complete. ■

The following lemma is due to L. Hörmander (see [4] Lemma 3.5.).

LEMMA 3. – (*i*) *Let* $f(z)$ *be an analytic function in* $\Delta = \{z \in \mathbb{C};\ |z| < 5\kappa\}$, *and* $L = 5\sup_{\Delta}|f(z)|$. *Then*

(18) $f(X_\kappa(t, D_t))$ *can be defined by the power series expansion of* $f(t)$.

(*ii*) *For* $\kappa > 0$, *there exists a* $\tau_0(\kappa) > 0$ *such that for* $\tau > \tau_0(\kappa)$ *we have*

(19) $$\int |e^{-\frac{1}{2\tau}|D_t|^2}(f(t)\,u) - f(X_\kappa(t, D_t))\, e^{-\frac{1}{2\tau}|D_t|^2}u|^2 dt \leqslant$$

$$L^2 e^{-\frac{\tau\kappa^2}{2}}\int |u|^2 dt, \qquad u \in C_0^\infty(B_{\frac{\kappa}{4}}),$$

and

(20) $$\int |f(X_\kappa(t, D_t))\, w|^2 dt \leqslant L^2 \int |w|^2 dt, \qquad w \in \mathcal{S}(\mathbb{R}).$$

4. – Proof of the main theorems.

The lemmas of Section 3 will be applied to the function $t \mapsto u(t, x, y)$.

Let B_r denote the ball of radius r in $\mathbb{R}^{n+1}$. We shall first cut off the support of functions.

If $u \in C_0^\infty(B_{\kappa/4})$, and $v_1 = e^{-\frac{1}{2\tau}|D_t|^2} u$, then $\operatorname{supp} v_1 \subset \{(t, x, y) \in \mathbb{R}^{n+1}; |(x, y)| \leq \kappa/4\}$. We introduce $V(t, x, y)$ by

$$v(t, x, y) = \chi\left(\frac{t}{\kappa}\right) v_1(t, x, y).$$

Then the support of v is contained in $B_{9\kappa/2}$.

Since

$$v_1 - v = (1 - \chi(t/\kappa))\, e^{-\frac{1}{2\tau}|D_t|^2} u\,,$$

it follows from Lemma 2 that for $u \in B_{\kappa/4}$ and large τ,

$$\int\!\!\int e^{\tau\psi(x)}\Big(\int |v_1|^2\, dt\Big)\, dx\, dy \leq 2 \int\!\!\int e^{\tau\psi(x)}\Big(\int |v|^2\, dt\Big)\, dx\, dy + e^{-\frac{\tau\kappa^2}{2}} \int\!\!\int e^{\tau\psi(x)}\Big(\int |u|^2\, dt\Big)\, dx\, dy\,. \tag{21}$$

We shall prove Theorem 2.

Let $u \in C_0^\infty(B_{\frac{\kappa}{4}})$. By choosing $\delta = 6\kappa$, we can apply Theorem 3 for $v = \chi(t/\kappa)\, e^{-\frac{1}{2\tau}|D_t|^2} u$.

$$\int e^{\tau\psi(x)}\left[\tau^3 (x - 6\kappa)^2 \left|\chi\left(\frac{t}{\kappa}\right) e^{-\frac{1}{2\tau}|D_t|^2} u\right|^2 + \frac{\tau}{2}\left|\partial_x\left(\chi\left(\frac{t}{\kappa}\right) e^{-\frac{1}{2\tau}|D_t|^2} u\right)\right|^2\right] d\mu \leq$$

$$C \int e^{\tau\psi(x)} \left|(-A(x, y, \partial_x, \partial_y) + R_1(x, y, \partial_x, \partial_y))\,\chi\left(\frac{t}{\kappa}\right) e^{-\frac{1}{2\tau}|D_t|^2} u\right|^2 d\mu\,.$$

Since $\chi(t/\kappa)$ and $e^{-\frac{1}{2\tau}|D_t|^2}$ are commutable with $-A(x, y, \partial_x, \partial_y) + R_1(x, y, \partial_x, \partial_y)$, we have

$$\int e^{\tau\psi(x)}\Big(\tau^3 (x - 6\kappa)^2 |v|^2 + \frac{\tau}{2} |v_x|^2\Big)\, d\mu \leq$$

$$C \int e^{\tau\psi(x)} \left| e^{-\frac{1}{2\tau}|D_t|^2} (-A(x, y, \partial_x, \partial_y) + R_1(x, y, \partial_x, \partial_y))\, u\right|^2 d\mu\,.$$

On the other hand

$$|e^{-\frac{1}{2\tau}|D_t|^2}(-A(x,y,\partial_x,\partial_y)+R_1(x,y,\partial_x,\partial_y))u|^2 \leqslant$$

$$2|e^{-\frac{1}{2\tau}|D_t|^2}(\widetilde{D}_t - A + R_1 + c)u|^2 + 4(|e^{-\frac{1}{2\tau}|D_t|^2}\widetilde{D}_t u|^2 + |e^{-\frac{1}{2\tau}|D_t|^2} cu|^2).$$

We get

$$\int e^{\tau\psi(x)}\Big(\tau^3(x-6\kappa)^2|v|^2 + \frac{\tau}{2}|v_x|^2\Big)\,d\mu \leqslant$$

$$C\Big[\int e^{\tau\psi(x)}|e^{-\frac{1}{2\tau}|D_t|^2}Pu|^2 d\mu + \int e^{\tau\psi(x)}|e^{-\frac{1}{2\tau}|D_t|^2}\widetilde{D}_t u|^2 d\mu + \int e^{\tau\psi(x)}|e^{-\frac{1}{2\tau}|D_t|^2}cu|^2 d\mu\Big].$$

By (14) we have

$$\int e^{\tau\psi(x)}|e^{-\frac{1}{2\tau}|D_t|^2}\widetilde{D}_t u|^2 d\mu = \int e^{\tau\psi(x)}|D_t e^{-\frac{1}{2\tau}|D_t|^2}u|^2 d\mu =$$

$$\int\int e^{\tau\psi(x)}\Big(\int |D_t e^{-\frac{1}{2\tau}|D_t|^2}u|^2 dt\Big)\,dx\,dy \leqslant C(2\kappa\tau)^2\int\int e^{\tau\psi(x)}\Big(\int |e^{-\frac{1}{2\tau}|D_t|^2}u|^2 dt\Big)\,dx\,dy +$$

$$C_\kappa e^{-\frac{\tau\kappa^2}{3}}\int\int\int e^{\tau\psi(x)}|u|^2\,dt\,dx\,dy\,,$$

and by Lemma 3 we have,

$$\int e^{\tau\psi(x)}|e^{-\frac{1}{2\tau}|D_t|^2}c(t,x,y)u|^2 d\mu \leqslant$$

$$\int\int e^{\tau\psi(x)}\Big[\int |c(X_\kappa(t,D_t),x,y)\,e^{-\frac{1}{2\tau}|D_t|^2}u|^2 dt + L^2 e^{-\frac{\tau\kappa^2}{2}}\int |u|^2 dt\Big]\,dx\,dy \leqslant$$

$$L^2\int\int e^{\tau\psi(x)}\Big[\int |e^{-\frac{1}{2\tau}|D_t|^2}u|^2 dt + e^{-\frac{\tau\kappa^2}{2}}\int |u|^2 dt\Big]\,dx\,dy\,.$$

By combining these inequalities, we get

$$\int e^{\tau\psi(x)}\Big(\tau^3(x-6\kappa)^2|v|^2 + \frac{\tau}{2}|v_x|^2\Big)\,d\mu \leqslant C\int e^{\tau\psi(x)}|e^{-\frac{1}{2\tau}|D_t|^2}Pu|^2 d\mu +$$

$$(C(2\kappa\tau)^2 + L^2)\int e^{\tau\psi(x)}|e^{-\frac{1}{2\tau}|D_t|^2}u|^2 d\mu + C(L,\kappa)\,e^{-\frac{\tau\kappa^2}{2}}\int e^{\tau\psi(x)}|u|^2 d\mu\,.$$

Then, since we have $|x-6\kappa| \geqslant \frac{3}{2}\kappa$ on the support of v, using the inequality (21) between v_1 and v, and taking τ large enough, we get

$$\int e^{\tau\psi(x)}\left(\tau^3 \kappa^2 |v_1|^2 + \frac{\tau}{2}|v_{1x}|^2\right) d\mu \leqslant$$

$$C\int e^{\tau\psi(x)} |e^{-\frac{1}{2\tau}|D_t|^2} Pu|^2 d\mu + C_\kappa e^{-\frac{\tau\kappa^2}{2}} \int e^{\tau\psi(x)} |u|^2 d\mu .$$

Therefore Theorem 2 is proved. ■

In order to prove our uniqueness result, Theorem 1, we shall make use of the following lemma.

LEMMA 4. – *Let $u \in L^2(\mathbb{R}^{n+1})$ with compact support, and $\phi \in C^\infty$. If there exist $\tau_0 > 0$ and $C > 0$ such that*

$$\int |e^{-\frac{1}{2\tau}|D_t|^2} e^{\tau\phi} u|^2 \, dt\, dx\, dy \leqslant C, \qquad \tau > \tau_0, \tag{22}$$

then $u = 0$ on $\{(t, x, y) \in \mathbb{R}^{n+1};\ \phi(t, x, y) > 0\}$.

The proof of Lemma 4 can be seen in [11] and [4] (see Proposition 2.1 in [11] and Proposition 4.1 in [4]).

By approximation the estimate in Theorem 2 is extended to all $u \in H^1$ with $Pu \in L^2$, and $\operatorname{supp} u \subset B_{\kappa/8}$.

Let h be in $C^\infty(\mathbb{R})$ with $0 \leqslant h \leqslant 1$ such that $h(x) = 1$ when $x \leqslant \varepsilon$, and $h(x) = 0$ when $x \geqslant 2\varepsilon$, where ε is a small enough positive constant.

Since $\operatorname{supp} u \cap \{(t, x, y);\ x = 0\}$ is compact, we have $hu \in H^1$, $\operatorname{supp}(hu) \subset B_{\kappa/8}$.

So we can apply Theorem 2.

$$\int e^{\tau\psi(x)} |e^{-\frac{1}{2\tau}|D_t|^2} P(hu)\ |^2 dt\, dx\, dy \leqslant \int e^{\tau\psi(x)} |P(hu)\ |^2 dt\, dx\, dy \leqslant$$

$$C\int e^{\tau\psi(x)} |[P, h]\, u|^2 dt\, dx\, dy + C\int e^{\tau\psi(x)} |(hPu)|^2 dt\, dx\, dy .$$

On the support of $[P, h]\, u$, there exists a $\tilde{\delta} > 0$ such that $\psi(x) \leqslant -\tilde{\delta} < 0$. The second term in the above inequality is equal to zero by using $Pu = 0$ near the origin. Therefore there exist $\delta > 0$ and τ_0 such that

$$\int e^{\tau(\psi(x)+\delta)} |e^{-\frac{1}{2\tau}|D_t|^2} (hu)|^2 dt\, dx\, dy \leqslant C, \qquad \tau > \tau_0. \tag{23}$$

By Lemma 4, $hu = 0$ on $\{(t, x, y) \in \mathbb{R}^{n+1};\ \psi(x) + \delta > 0\}$.

This implies $u = 0$ near the origin. Therefore the proof of Theorem 1 is completed. ■

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Department of Applied Mathematics and Physics, Graduate School of Informatics
Kyoto University, Kyoto 606-8501, Japan

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