Bollettino
Unione Matematica Italiana

A. Porretta

Absorption effects for some elliptic equations with singularities

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **8-B** (2005), n.2, p. 369–395.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2005_8_8B_2_369_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2005.

Bollettino U. M. I.
(8) 8-B (2005), 369-395

Absorption Effects for Some Elliptic Equations with Singularities.

A. PORRETTA (*)

Sunto. – *In questa nota si presenta una breve rassegna di alcuni recenti risultati ottenuti su una classe di equazioni ellittiche con termini di assorbimento a crescita naturale e dati singolari. Si mettono in luce tipici fenomeni (stabilità, esistenza o nonesistenza, singolarità rimovibili, effetti di barriera) dovuti essenzialmente all'effetto regolarizzante dei termini di assorbimento che in alcuni casi può impedire la presenza o la diffusione di singolarità nell'equazione. Oltre all'esposizione di risultati già noti, si presenta una nuova applicazione al caso di crescita sottocritica per l'equazione modello (1.6), per la quale dimostriamo un risultato generale di esistenza con dato misura, nelle ipotesi ottimali che estendono la classica condizione di P. Benilan e H. Brezis* [4].

Summary. – *We give an expository review of recent results obtained for elliptic equations having natural growth terms of absorption type and singular data. As a new result, we provide an application to the case of lower order terms of subcritical growth, proving a general solvability result with measure data for a class of equations modeled on (1.6).*

1. – Introduction.

In this note we consider a class of elliptic equations in the form

$$A(u) + H(x, u, \nabla u) = f, \tag{1.1}$$

where A is a second order, possibly nonlinear, divergence form operator of the type introduced in [20], somehow modeled on the p-Laplace operator $-\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$. The lower order term $H(x, u, \nabla u)$, which could be seen as a perturbation of such operator A, will be assumed to satisfy two structure conditions, first of all the absorption hypothesis

$$\exists s_0 > 0 : H(x, s, \xi)\, s \geqslant 0 \qquad \forall s \in \boldsymbol{R} : |s| > s_0, \tag{1.2}$$

and secondly a so-called natural growth condition (see below), namely an en-

(*) Comunicazione presentata a Milano in occasione del XVII Congresso U.M.I.

ergy type growth with respect to the gradient. In order to simplify our exposition, we will mainly refer to the following simple model problem:

$$-\Delta u + H(x, u, \nabla u) = f \quad \text{in } \Omega , \tag{1.3}$$

where Ω is a bounded subset of $\boldsymbol{R}^N$ and H satisfies (1.2) and

$$g_2(s)|\xi|^2 - f_2(x) \leq |H(x, s, \xi)| \leq g_1(s)|\xi|^2 + f_1(x), \tag{1.4}$$

where g_1, g_2 are continuous functions, and $f_1, f_2 \in L^1(\Omega)$.

A particular motivation for dealing with natural growth terms as in (1.4) comes from the study of Euler equations of some functionals in the calculus of variations, as the following example:

$$J(v) = \frac{1}{2}\int_\Omega a(x, v)|\nabla v|^2\, dx - \int_\Omega fv\, dx . \tag{1.5}$$

In fact, we have (at least formally)

$$J'(v) = -\operatorname{div}(a(x, v)\nabla v) + \frac{1}{2}\left(\frac{\partial}{\partial v} a(x, v)\right)|\nabla v|^2 - f, \tag{1.5}$$

so that $J'(v)$ enters the class of operators in (1.1). This also shows the importance to consider problem (1.1) in its general form including nonlinear operators A. We also point out that equations like (1.3) appear very naturally in different contexts, as for instance in so-called viscous Hamilton Jacobi equations related to stochastic control problems. While these motivations account for the growth condition (1.4), the absorption assumption (1.2) is crucial for the type of questions we investigate and for the results obtained.

The main issue we address is the possibility that the Dirichlet problem for (1.3) admits a solution in case f is a measure on Ω. Indeed, due to (1.2), the lower order term may induce regularizing effects which prevent in some cases from development of singularities. The stability properties of the equation (and existence results) will then be affected by the behavior of the functions g_i in (1.4) and by the fact whether f charges or not sets of zero (harmonic) capacity. As a consequence, we apply this analysis to study the stability of minimizers of J and whether J admits minima (in any weak sense) corresponding to singular sources f, like the Dirac mass; in this case it turns out that, roughly speaking, J has a minimum (even in very weak sense) if and only if $a(x, s)$ is bounded from above and from below.

Moreover, these questions are related to results concerning removable singularities for the equation (1.3) (see [13 bis]), which point out even stronger regularizing effects having a local character. Indeed, in some situations, the development of singularities seems to be avoided by local barrier-type effects: this has led us to more recent researches (which we only sketch here, in the final section) concerning other classical absorption phenomena, as existence of local universal interior estimates and of solutions which blow-up at the boundary.

Let us recall that the type of questions we discuss has been deeply investigated, since the pioneering works by H. Brezis ([10], [11]), for semilinear equations, i.e. (1.3) where H does not depend on ∇u. We just refer the reader to the survey [37] and its references, and to the recent paper [13]. In that situation, however, an important role is played by monotonicity arguments (as if $H = H(s)$ is increasing), which in general can not be applied once the function H depends on the gradient. Actually, our study relies rather on compactness properties, and some heavy technical tools, like strong compactness for the truncations of solutions, seem to be unavoidable for dealing with gradient dependent terms.

In this paper we give an expository review of the results proved for (1.3), (1.5) and related questions, mainly based on references [24], [26], [29], [30]. However, in Section 2.2 we prove a new result, showing how having dealt with natural growth terms and recalling results for the semilinear case one can handle the general case of possibly subquadratic growth of H with respect to ∇u, in particular for the model problem

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^q = \mu & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.6}$$

where $0 \leqslant q \leqslant 2$ and μ is a bounded Radon measure. We prove then a general solvability result (i.e. existence of solutions of (1.6) for **any** measure μ) under the natural extension of the Benilan-Brezis condition known if $q = 0$.

2. – Measure data: existence and nonexistence of solutions.

2.1. *Natural growth terms.*

Let us start by considering the simple problem

$$\begin{cases} -\Delta u + H(x, u, \nabla u) = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.1}$$

where H satisfies the absorption condition

$$\exists s_0 > 0 : H(x, s, \xi)s \geqslant 0 \qquad \forall s \in \boldsymbol{R} : |s| > s_0. \tag{2.2}$$

We say that assumption (2.2) has a possibly regularizing effect since it induces an a priori estimate for solutions of (2.1) as

$$\|H(x, u, \nabla u)\|_{L^1(\Omega)} \leqslant C\|f\|_{L^1(\Omega)}. \tag{2.3}$$

A very interesting case appears if H has natural growth conditions as

$$g_2(s)\,|\xi|^2 - f_2(x) \leq |H(x,s,\xi)| \leq g_1(s)\,|\xi|^2 + f_1(x), \tag{2.4}$$

where g_1, g_2 are continuous functions, and $f_1, f_2 \in L^1(\Omega)$. In this situation, the function g_2 accounts for the possibly regularizing effect of H, and one can obtain from (2.3) further energy estimates which, in some cases, largely improve those already available for the non perturbed problem, i.e. when $H = 0$. As an example, assume that

$$\exists\, \theta \in [0, 1]: g_2(s) \geq |s|^{-\theta} \qquad \text{for } |s| > s_0 > 0\,. \tag{2.5}$$

Then (2.3) implies (see [7], [9], [31]):

$$\|u\|_{W^{1,\frac{N(2-\theta)}{N-\theta}}_{(\Omega)}} \leq C\|f\|_{L^1(\Omega)}. \tag{2.6}$$

Note that these gradient estimates, induced by the lower order term H, would not hold for the simple Laplace operator. In particular if $\theta = 0$, there are even estimates in $H_0^1(\Omega)$ depending only on the L^1-norm of the data, so that it is possible to find finite energy solutions even if the right hand side is not in the dual space $H^{-1}(\Omega)$.

In order to look for solutions of (2.1), these estimates are meant to be applied to sequences of approximating problems of the form

$$\begin{cases} -\Delta u_n + H(x, u_n, \nabla u_n) = f_n & \text{in } \Omega\,, \\ u_n = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.7}$$

where $\{f_n\}$ is a sequence of bounded functions converging to f. Any sequence of solutions $\{u_n\}$ of (2.7) (whose existence is ensured by (2.2), see e.g. [5]) will then satisfy uniform estimates; we are left then to study compactness and stability properties of such sequences of solutions trying to prove that a limit function u is a solution of (2.1). This program can be successfully performed when f belongs to $L^1(\Omega)$, or more generally to $L^1(\Omega) + H^{-1}(\Omega)$; it is proved in [31] (see also [9], [32]) that, under the assumption (2.2), and if H satisfies (2.4) for whatever g_1, g_2, then problem (2.1) has a solution. Moreover, as remarked above, this solution may have further regularity if H is more coercive at infinity, as in (2.5)-(2.6).

Let now μ be a general bounded Radon measure on Ω and consider the Dirichlet problem

$$\begin{cases} -\Delta u + H(x, u, \nabla u) = \mu & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.8}$$

When trying to extend the existence result to general measures μ, some new phenomena can be observed, and, due to the regularizing effect of the lower order term, existence of solutions may be lost if the measure is too concentrated. Since the estimates on the approximating solutions only depend on the L^1 norm of the sequence $\{f_n\}$ of approximating data, clearly these estimates continue to hold if f_n approximates (e.g. as a convolution) a measure with finite mass. However, the compactness and stability properties in the equation are strongly affected by the behavior of $H(x, s, \xi)$ for large s, and by the presence of measures possibly concentrated on thin sets, more precisely sets having zero capacity. Here the capacity is the standard notion of (harmonic) capacity defined in the energy space $H_0^1(\Omega)$ as

$$cap(A) = \inf\{\|\psi\|_{H_0^1(\Omega)},\ \psi \geqslant \chi_A \text{ a.e}\} \text{ if } A \text{ is open},$$

and

$$cap(E) = \inf\{cap(A),\ E \subset A,\ A \text{ open}\} \text{ for borelians } E .$$

As main examples, let us recall that compact $N-2$-dimensional manifolds have zero capacity in $\boldsymbol{R}^N$, in particular a point x_0 has zero capacity in $\boldsymbol{R}^N$, $N \geqslant 2$. A measure μ is said not to charge sets of zero capacity if $\mu(B) = 0$ whenever $cap(B) = 0$, for any Borelian set B. It has been proved in [8] that such measures can be written as the sum of two terms, one in $L^1(\Omega)$, one in $H^{-1}(\Omega)$. Thus, the existence result mentioned above applies to any measure μ which does not charge sets of zero capacity.

A full study in case of a general datum μ requires to know (see [15]) that any bounded measure μ in Ω admits a unique decomposition with respect to the capacity as

$$\mu = \mu_0 + \lambda , \tag{2.9}$$

where μ_0, λ are bounded measures such that λ is concentrated on a set $E \subset \Omega$ with $cap(E) = 0$ and μ_0 does not charge sets of zero capacity, which implies in view of [8] that

$$\mu_0 = f - \operatorname{div}(F), \qquad f \in L^1(\Omega), \quad F \in L^2(\Omega)^N. \tag{2.10}$$

Hereafter, we also assume that μ is nonnegative (which simplifies a few technical arguments) and shortly write $\mu \in \mathcal{M}_b^+(\Omega)$. Referring to the previous decomposition of μ and μ_0 in (2.9), (2.10), there exists a sequence μ_n of bounded functions such that

$$
(2.11)\qquad \begin{cases} \mu_n = \mu_{0n} + \lambda_n, \quad \mu_{0n} \geqslant 0,\ \lambda_n \geqslant 0,\ \mu_n \in L^\infty(\Omega) \\ \mu_{0n} = f_n - \operatorname{div}(F_n), \quad f_n \in L^\infty(\Omega), \quad F_n \in L^\infty(\Omega)^N, \\ f_n \to f \quad \text{strongly in } L^1(\Omega), \\ F_n \to F \quad \text{strongly in } L^2(\Omega)^N, \\ \int_\Omega \varphi \lambda_n \, dx \to \int_\Omega \varphi \, d\lambda \quad \forall \varphi \in C_b(\Omega), \end{cases}
$$

where $C_b(\Omega)$ denotes the space of bounded continuous functions in Ω. Such a sequence μ_n can be constructed using convolution and a suitable compactly supported approximation of μ. For fixed $n \in \boldsymbol{N}$, since $\mu_n \in L^\infty(\Omega)$, there exists a weak solution $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of the problem:

$$
(2.12)\qquad \begin{cases} -\Delta u_n + H(x, u_n, \nabla u_n) = \mu_n & \text{in } \Omega\,, \\ u_n = 0 & \text{on } \partial\Omega\,. \end{cases}
$$

The behavior of the approximating problems (2.12) was described in [28] and [24], yielding the following result. A crucial role is played by the truncation function $T_k(s) = \min(\max(s, -k), k)$.

THEOREM 2.1. – *Assume that H satisfies* (2.2) *and* (2.4), *and let $\mu \in \mathcal{M}_b^+(\Omega)$ and μ_n given by* (2.11). *Then there exists a subsequence of solutions of* (2.12), *still denoted u_n, and a function u $\left(\text{which belongs to } W_0^{1,q}(\Omega) \text{ for any } q < \frac{N}{N-1}\right)$ such that u_n almost everywhere converges to u in Ω, and*

$$
T_k(u_n) \to T_k(u) \quad \textit{strongly in } H_0^1(\Omega) \textit{ for any } k > 0\,.
$$

Moreover we have:

(i) if $\int_0^{+\infty} g_1(s)\, ds < +\infty$, then

$$
H(x, u_n, \nabla u_n) \to H(x, u, \nabla u) \quad \textit{strongly in } L^1(\Omega)\,,
$$

and u is a solution of (2.8).

(ii) if $\int_0^{+\infty} g_2(s)\, ds = +\infty$, then

$$
H(x, u_n, \nabla u_n) \to H(x, u, \nabla u) + \lambda \quad \textit{in the weak sense of measures},
$$

and u is a solution of

$$\begin{cases} -\Delta u + H(x, u, \nabla u) = \mu_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad \blacksquare$$

Let us stress that the conclusion of Theorem 2.1 is threefold: first of all, it gives a compactness result which holds true under a general growth condition (2.4), regardless of the functions g_1, g_2; in particular, it applies to possibly subquadratic growth of H with respect to the gradient. In other words, the absorption assumption (2.2) ensures, in a quite general situation, the existence of a limit function u; moreover, since the truncations are strongly compact in the energy space, the convergence of H depends on its behavior when $|u|$ is large. Then, (i) gives a sufficient condition for having solutions of (2.8), with **any** measure as right hand side. Thirdly, this condition is proved to be sharp by (ii), which says when and why we fail to find a solution with measures concentrated on sets of zero capacity; indeed, it happens when g_2 in (2.4) is not integrable at infinity since in that case these singular measures disappear in the limiting process. In particular, if μ is concentrated on a set of zero capacity (i.e. $\mu_0 = 0$), the approximating solutions u_n converge to zero in Ω, which could be described as a complete blow-down phenomenon.

In terms of stability, Theorem 2.1 characterizes the possible situations appearing under assumptions (2.2), (2.4): as a corollary, the model problem

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.13}$$

may be fully characterized in any formulation of stable solutions. In a nonlinear context, this can be done in the framework of renormalized solutions (see [24]); for the case of a linear operator, it is possible to consider the formulation by duality introduced by G. Stampacchia ([34]), also called notion of very weak solution. Namely, $u \in L^1(\Omega)$ is a very weak solution of (2.13) if

$$-\int_\Omega u\Delta\varphi\, dx + \int_\Omega g(u)\,|\nabla u|^2 \varphi\, dx = \int_\Omega \varphi\, d\mu, \quad \textit{for every } \varphi \in C_c^1(\overline{\Omega}) : \Delta\varphi \in L^\infty(\Omega),$$

where $C_c^1(\overline{\Omega})$ denotes the functions which are C^1in $\overline{\Omega}$ and zero on the boundary. Since a very weak solution is always a solution obtained by approximation, and it is also a renormalized solution, from [24] we obtain the following consequence for the case of Laplace operator.

THEOREM 2.2. – *Let $\mu \in \mathcal{M}_b^+(\Omega)$ and let g be a continuous function such that, for a positive constant s_0, $g(s)s \geq 0$ for $|s| \geq s_0$. Then problem* (2.13) *has a very weak solution if and only if one of the two following conditions hold:*

$\int_0^{+\infty} g(s)\,ds < +\infty$ or μ does not charge sets of zero (harmonic) capacity. ■

REMARK 2.3. – In the general case of problem

$$\begin{cases} -\operatorname{div}(a(x, u, \nabla u)) + H(x, u, \nabla u) = \mu & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.14}$$

same results as in Theorem 2.1 are proved to hold. One assumes that $a(x, s, \xi)$ satisfies the so-called Leray-Lions assumptions, i.e. $a(x, s, \xi) : \Omega \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ is a Carathéodory function which satisfies, for almost every x in Ω, for all $s \in \boldsymbol{R}$, and for all ξ, ξ' ($\xi \neq \xi'$) in $\boldsymbol{R}^N$:

$$a(x, s, \xi)\cdot\xi \geqslant \alpha|\xi|^p, \qquad \alpha > 0\,, \tag{2.15}$$

$$|a(x, s, \xi)| \leqslant \beta(k(x) + |s|^{p-1} + |\xi|^{p-1}), \qquad \beta > 0\,,\ k(x) \in L^{p'}(\Omega)\,, \tag{2.16}$$

$$(a(x, s, \xi) - a(x, s, \xi'))\cdot(\xi - \xi') > 0\,. \tag{2.17}$$

Moreover H satisfies (2.2) and

$$g_2(s)\, a(x, s, \xi)\cdot\xi - f_2(x) \leqslant |H(x, s, \xi)| \leqslant g_1(s)\, a(x, s, \xi)\cdot\xi + f_1(x)\,,$$

where g_1, g_2 are continuous functions, and $f_1, f_2 \in L^1(\Omega)$. The conclusions of Theorem 2.1, including (i) and (ii), still hold true (see [31] for even more general growth conditions on a). The interest in obtaining results for general nonlinear operators can be seen in Section 3 from applications to Calculus of Variations.

2.2. *General solvability for the subcritical growth.*

The alternative behavior given by (i) and (ii) in Theorem 2.1 distinguishes whether the lower order term H is «dominated» by the principal operator (the Laplacian for (2.8)) or whether H itself is the «leading term». In other words, if g is integrable, the first order perturbation H is controlled by the Laplacian and may be seen as an admissible perturbation, which preserves the surjectivity property on the space of bounded Radon measures. Let us recall that for the purely semilinear case

$$\begin{cases} -\Delta u + g(x, u) = \mu & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.18}$$

the condition on g under which there is existence for **any** measure μ was found by P. Benilan and H. Brezis ([4], see also [10], [16], [37]) and also called the weak singularity assumption on g: if $N > 2$, g should satisfy, beyond the absorption condition $g(x, s)\, s \geqslant 0$, the growth assumption

$$(2.19) \qquad |g(x,s)| \leqslant \tilde{g}(|s|), \quad \text{with } \tilde{g} \text{ nondecreasing and } \int_1^{+\infty} \tilde{g}(r)\, r^{-\frac{2(N-1)}{N-2}}\, dr < \infty .$$

Heuristically, this condition appears if one asks $\tilde{g}(\mathcal{G}) \in L^1(\Omega)$, where $\mathcal{G}$ is the fundamental solution for the Laplacian. When we consider general possibly first order perturbations as in (2.8), clearly (2.18) represents a limit case, where the lower order term is independent on the gradient, while (2.4) represents, in some sense, the opposite borderline case, having energy-like growth with respect to ∇u. In the next result, which to our knowledge is new even in the linear case, we show that joining the Benilan-Brezis result with Theorem 2.1 allows to deal with the complete problem whose simplest model is

$$(2.20) \qquad \begin{cases} -\Delta u + g(u)\, |\nabla u|^q = \mu & \text{in } \Omega\, , \\ u = 0 & \text{on } \partial\Omega\, , \end{cases}$$

with $0 \leqslant q \leqslant 2$. The natural assumption which makes the lower order term a «weak singularity» may again be heuristically found asking $g(\mathcal{G})\, |\nabla \mathcal{G}|^q \in L^1(\Omega)$, and reads as

$$\int_1^{+\infty} g(r)\, r^{-\frac{(2-q)(N-1)}{N-2}}\, dr < \infty \, .$$

Note that this condition is a good interpolation between (2.19) and $g \in L^1(\boldsymbol{R}^+)$, which are recovered in the two cases $q = 0$ and $q = 2$.

Since we need to apply the basic compactness result for the truncations as in Theorem 2.1, which is technically the hardest part, we will restrict ourselves to the case $\mu \geqslant 0$; however, we will set the result in a possibly nonlinear framework, including the p-laplace equation. The restriction $N > 2$ is only due to the fact that condition (2.19) should be suitably modified (with an exponential type growth, see [35]) if $N = 2$; this would modify also (2.22), although not the method of proof.

THEOREM 2.4. – *Let the vector field* $a : \Omega \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ *satisfy* (2.15)-(2.17). *Assume that* $H(x, s, \xi)$ *satisfies* (2.2) *and the growth condition*

$$(2.21) \quad |H(x, s, \xi)| \leqslant g(s)\, |\xi|^q + f_1(x), \qquad \forall s \in \boldsymbol{R},\ \forall \xi \in \boldsymbol{R}^N, \text{ a.e. } x \in \Omega\, ,$$

where $f_1 \in L^1(\Omega)$ *and* g *is a positive continuous function on* $\boldsymbol{R}$ *such that*

$g(s)\, s^{\frac{q(N-1)}{N-p}}$ is nondecreasing on $\boldsymbol{R}^+$ and

$$\int_1^{+\infty} g(s)\, s^{-\frac{(p-q)(N-1)}{N-p}}\, ds < +\infty\,. \tag{2.22}$$

Then for any nonnegative bounded measure μ there exists a (weak) solution of

$$\begin{cases} -\operatorname{div}(a(x, u, \nabla u)) + H(x, u, \nabla u) = \mu & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.23}$$

PROOF. – Let μ_n be an approximation of μ as in (2.11) and u_n a solution of

$$\begin{cases} -\operatorname{div}(a(x, u_n, \nabla u_n)) + H(x, u_n, \nabla u_n) = \mu_n & \text{in } \Omega\,, \\ u_n = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.24}$$

The absorption condition (2.2) allows to have the standard a priori estimates; indeed, choosing $T_k(u_n)\, e^{\lambda |T_{s_0}(u_n)|}$ as test function in (2.24) implies

$$\int_\Omega a(x, u_n, \nabla u_n)\, \nabla T_k(u_n)\, e^{\lambda |T_{s_0}(u_n)|}\, dx +$$
$$\lambda \int_\Omega a(x, u_n, \nabla u_n)\, \nabla T_{s_0}(u_n)\, |T_k(u_n)|\, e^{\lambda |T_{s_0}(u_n)|}\, dx +$$
$$\int_\Omega H(x, u_n, \nabla u_n)\, e^{\lambda |T_{s_0}(u_n)|}\, T_k(u_n)\, dx = \int_\Omega T_k(u_n)\, e^{\lambda |T_{s_0}(u_n)|}\, \mu_n\, dx\,.$$

Thanks to (2.2) and (2.21) we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\, \nabla T_k(u_n)\, e^{\lambda |T_{s_0}(u_n)|}\, dx +$$
$$\lambda \int_\Omega a(x, u_n, \nabla u_n)\, \nabla T_{s_0}(u_n)\, |T_k(u_n)|\, e^{\lambda |T_{s_0}(u_n)|}\, dx \leqslant$$
$$\int_{\{|u_n| < s_0\}} (g(u_n)\, |\nabla u_n|^q + f_1)\, e^{\lambda |T_{s_0}(u_n)|}\, |T_k(u_n)|\, dx + \int_\Omega T_k(u_n)\, e^{\lambda |T_{s_0}(u_n)|}\, \mu_n\, dx$$

hence

$$\int_\Omega a(x, u_n, \nabla u_n)\, \nabla T_k(u_n)\, e^{\lambda |T_{s_0}(u_n)|}\, dx +$$
$$\lambda \int_\Omega a(x, u_n, \nabla u_n)\, \nabla T_{s_0}(u_n)\, |T_k(u_n)|\, e^{\lambda |T_{s_0}(u_n)|}\, dx \leqslant$$
$$C_0 \int_{\{|u_n| < s_0\}} |\nabla u_n|^p\, e^{\lambda |T_{s_0}(u_n)|}\, |T_k(u_n)|\, dx + C_1 k e^{\lambda s_0} (1 + \|f_1\|_{L^1(\Omega)} + \|\mu_n\|_{L^1(\Omega)})\,.$$

Using (2.15) and choosing $\lambda > \frac{C_0}{\alpha}$ we obtain:

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p e^{\lambda |T_{s_0}(u_n)|}\, dx \leqslant C_1 k e^{\lambda s_0}(1 + \|f_1\|_{L^1(\Omega)} + \|\mu_n\|_{L^1(\Omega)}),$$

which implies

$$\int_\Omega |\nabla T_k(u_n)|^p\, dx \leqslant C(k+1),$$

for any $k > 0$. It is well known (see [3]) that this estimate implies that u_n is bounded in the Marcinkiewicz space $M^{\frac{N(p-1)}{N-p}}(\Omega)$, which means that

$$\text{meas}\,\{x \in \Omega : |u_n| > k\} \leqslant C k^{-\frac{N(p-1)}{N-p}}, \qquad \forall n \in \boldsymbol{N}. \tag{2.25}$$

Similarly $|\nabla u_n|$ is bounded in the Marcinkiewicz space $M^{\frac{N(p-1)}{N-1}}(\Omega)$. Thanks to these estimates, it can be proved as in Theorem 2.1 (see [31] for the general case) that there exists a function u such that $a(x, u, \nabla u) \in L^q(\Omega)^N$ for any $q < \frac{N}{N-1}$, the truncation $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$ and for a subsequence, not relabeled,

$$\begin{cases} u_n \to u \quad \text{a.e. in } \Omega, \\ T_k(u_n) \to T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega) \text{ for any } k > 0. \end{cases} \tag{2.26}$$

As a consequence of (2.26) and (2.25), we deduce that, again up to subsequences,

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega.$$

Let now h be a positive continuous function in $\boldsymbol{R}$ such that $\int_0^{+\infty} h(s)\, ds < +\infty$: choosing $\int_0^{u_n} h(s)\chi_{\{s>k\}}\, ds$ as test function in (2.24) with $k > s_0$ and using (2.2) we have

$$\int_{\{u_n > k\}} a(x, u_n, \nabla u_n)\, \nabla u_n\, h(u_n)\, dx \leqslant \|\mu_n\|_{L^1(\Omega)} \int_k^{+\infty} h(s)\, ds.$$

Therefore, (2.15) and the integrability of h imply

$$\lim_{k\to+\infty} \sup_n \int_{\{u_n > k\}} h(u_n)\, |\nabla u_n|^p\, dx = 0. \tag{2.27}$$

Thanks to (2.26) and (2.27) the sequence $h(u_n)|\nabla u_n|^p$ is then equi-integrable: indeed, for any subset $E \subset \Omega$

$$\int_E h(u_n)\, |\nabla u_n|^p\, dx \leqslant \int_E h(T_k(u_n))\, |\nabla T_k(u_n)|^p\, dx + \sup_n \int_{\{u_n > k\}} h(u_n)\, |\nabla u_n|^p,$$

hence the strong convergence of $T_k(u_n)$ gives

$$\lim_{\operatorname{meas}(E)\to 0} \sup_n \int_E h(u_n)\,|\nabla u_n|^p\,dx \le \sup_n \int_{\{u_n>k\}} h(u_n)\,|\nabla u_n|^p .$$

Letting k tend to infinity and using (2.27) we get the equi-integrability of $h(u_n)\,|\nabla u_n|^p$. Using the almost everywhere convergence and Vitali's theorem we conclude that

$$h(u_n)\,|\nabla u_n|^p \text{ strongly converges in } L^1(\Omega) \text{ for any } h\in C(\boldsymbol{R})^+ : \int_0^{+\infty} h(s)\,ds<\infty . \tag{2.28}$$

It is known from the work [4] that thanks to estimate (2.25) and again applying Vitali's theorem we also have

$$h(u_n) \text{ strongly converges in } L^1(\Omega) \quad \text{for any nondecreasing } h\in C(\boldsymbol{R})^+ : \int_1^{+\infty} h(s)\,s^{-\frac{p(N-1)}{N-p}}\,ds<\infty . \tag{2.29}$$

For the convenience of the reader, let us just recall that the main argument for getting (2.29) is again the equi-integrability condition

$$\lim_{k\to+\infty} \sup_n \int_{\{u_n>k\}} h(u_n)\,dx = 0 ,$$

which follows from the fact that, being h nondecreasing and using (2.25),

$$\int_{\{u_n>k\}} h(u_n)\,dx = \operatorname{meas}\{u_n>k\}\,h(k) + \int_k^{+\infty} \operatorname{meas}\{u_n>s\}\,dh(s) \le$$
$$\operatorname{meas}\{u_n>k\}\,h(k) + \int_k^{+\infty} C s^{-\frac{N(p-1)}{N-p}}\,dh(s) \le \tilde C \int_k^{+\infty} s^{-\frac{p(N-1)}{N-p}}\,h(s)\,ds .$$

This provides equi-integrability and then, by Vitali's theorem, implies (2.29). Now, we use Young's inequality with exponent $\frac{p}{q}$ to obtain

$$g(u_n)\,|\nabla u_n|^q \le c_1 g(u_n)(1+|u_n|)^{-\frac{(p-q)(N-1)}{N-p}}\,|\nabla u_n|^p + c_2 g(u_n)(1+|u_n|)^{\frac{q(N-1)}{N-p}} .$$

Thanks to (2.22), we can use (2.28) with $h(s)=g(s)(1+|s|)^{-\frac{(p-q)(N-1)}{N-p}}$ in order to obtain that $g(u_n)(1+|u_n|)^{-\frac{(p-q)(N-1)}{N-p}}\,|\nabla u_n|^p$ strongly converges in $L^1(\Omega)$. Moreover, since $g(s)\,s^{\frac{q(N-1)}{N-p}}$ is nondecreasing and (2.22) holds true, we can use (2.29) to deduce that $g(u_n)(1+|u_n|)^{\frac{q(N-1)}{N-p}}$ is also strongly convergent in $L^1(\Omega)$. Therefore, recalling the almost everywhere convergence of u_n and ∇u_n, we conclude that $g(u_n)\,|\nabla u_n|^q$ strongly converges in $L^1(\Omega)$ and then, from (2.21), that

$$H(x, u_n, \nabla u_n) \to H(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega).$$

We can pass then to the limit in (2.24) and conclude that u is a weak solution of (2.23). ■

REMARK 2.5. – The assumption that the function g should be such that $g(s)s^{\frac{q(N-1)}{N-p}}$ is nondecreasing is mainly technical and due to the zero order part, as in (2.19). However, note that $|H|$ should be *smaller* than such a function g, and that the integrability assumption (2.22) is largely consistent with this requirement. In other words, only pathological examples would escape this monotonicity condition. Note also that g may be allowed to be unbounded only if $q < \dfrac{N(p-1)}{N-1}$, which gives $\dfrac{(p-q)(N-1)}{N-p} > 1$ in (2.22). This is expected, since $\dfrac{N(p-1)}{N-1}$ is a well known critical value due to the fact that the a priori estimates only hold in $W_0^{1,r}(\Omega)$ for any $r < \dfrac{N(p-1)}{N-1}$ (this is also the regularity of the fundamental solution).

REMARK 2.6. – The optimality of the integral condition (2.22) comes from the fact that, if $\mathcal{G}$ is the fundamental solution of the Laplace equation in Ω, $g(\mathcal{G})|\nabla \mathcal{G}|^q \notin L^1(\Omega)$ if (2.22) is violated, so that (2.20) could not have a solution in this case if μ is the Dirac mass. For instance, consider the borderline case when $g(s)$ behaves like $s^{\frac{N-q(N-1)}{q(N-2)}}$ at infinity: having $g(u)|\nabla u|^q \in L^1(\Omega)$ would imply that $u \in W_0^{1,\frac{N}{N-1}}(\Omega)$ hence $\Delta u \in W^{-1,\frac{N}{N-1}}(\Omega)$ so that (2.20) could not hold if μ is the Dirac mass. However, let us stress that, while Theorem 2.4 gives optimal sufficient conditions for having a general solvability for (2.20), a full characterization of the problem (as in Theorem 2.2 for $q = 2$) has not yet been proved.

3. – Applications to Calculus of Variations.

The main motivation for considering growth conditions like (2.4) is the study of functionals in Calculus of Variations, as the following:

$$J(v) = \frac{1}{2}\int_\Omega a(x, v)|\nabla v|^2 dx - \int_\Omega v f dx .$$

where $a(x, s)$ satisfies a coerciveness condition

$$a(x, s) \geq \alpha > 0 \qquad \forall s \in \boldsymbol{R}, \text{ a.e. } x \in \Omega . \tag{3.1}$$

It is well known that, if $f \in L^2(\Omega)$, then (3.1) is enough to ensure the existence of a minimum of J. Assume also that $a(x, \cdot)$ is C^1 and let us denote

$$a'(x, s) := \left(\frac{\partial}{\partial s} a(x, s)\right).$$

In this situation, the absorption assumption becomes

$$\exists s_0 > 0 : s a'(x, s) \geqslant 0 \qquad \forall s : |s| \geqslant s_0, \tag{3.2}$$

which may yield further regularity for the minima, as is tipically the case when $a(x, \cdot)$ is convex.

Since under (3.1), (3.2), the geometry of the functional should lead to existence of minima, it is reasonable to try to miminize J even with less regular data f, for instance when the source term f is just $L^1(\Omega)$, or even a general measure μ, writing (in a formal way)

$$J(v) = \frac{1}{2} \int_\Omega a(x, v) |\nabla v|^2 \, dx - \int_\Omega v \, d\mu . \tag{3.3}$$

Of course, in such situation one should expect to find minima of infinite energy, since the functional will no more be coercive in the energy space $H_0^1(\Omega)$. As an example, the fundamental solution of the Laplace operator can still be seen as a minimum point, in some suitable sense, of the Dirichlet functional when the source is a Dirac mass. A natural approach, as in the case of Section 2, can be a stability, or a relaxation, method, that is the study of the behavior of minima of regularized problems, where μ is replaced by a suitable approximation μ_n. Defining

$$J_n(v) = \frac{1}{2} \int_\Omega a(x, v) |\nabla v|^2 \, dx - \int_\Omega \mu_n v \, dx , \tag{3.4}$$

one studies properties of sequences of minimizers $u_n \in \operatorname{argmin} J_n(v)$. Clearly, this is strictly related to the study of the Euler equation which was done in Section 2.1.

On the other hand, it is also possible to define suitable generalized notions of minima which allow to deal with the problem of minimizing J in presence of singular data. For data $\mu \in L^1(\Omega)$, the notion of T-minima has been introduced in [6] using properties of truncations, in connection with the notion of entropy solution of elliptic equations developed in [3]. Alternatively, one can use the notion of weak minima, introduced in [17], which seems to better fit the case of general measure data, as it is done in [25]. A weak minimum of J is, roughly speaking, a minimum point with respect to variations along smooth directions; more precisely, it is a function $u \in W_0^{1,1}(\Omega)$ such that

$$[a(x, u) |\nabla u|^2 - a(x, u + \varphi) |\nabla(u + \varphi)|^2] \in L^1(\Omega) \tag{3.5}$$

and which satisfies

$$\frac{1}{2}\int_\Omega [a(x,u)|\nabla u|^2 - a(x,u+\varphi)|\nabla(u+\varphi)|^2]\,dx + \int_\Omega \varphi\,d\mu \leq 0, \quad \forall\varphi \in C_c^\infty(\Omega). \tag{3.6}$$

Note that this definition makes sense since, although u should have infinite energy, i.e. $a(x, u)|\nabla u|^2 \notin L^1(\Omega)$, the difference in (3.5) (which is of lower order) is expected to be in $L^1(\Omega)$. For instance, the fundamental solution is a weak minimum of the Dirichlet functional with μ being the Dirac mass.

The technical tools developed for the equations as in (1.1) allow us (see [29]) to obtain results for functionals as in (3.3). First of all, it can be proved that, if $\mu \in L^1(\Omega) + H^{-1}(\Omega)$ (namely if μ does not charge sets of zero capacity), then the sequences of approximating minimizers are compact and converge to a «minimum» of J. More precisely, there exists a T-minimum for J. The absorbing assumption (3.2) also yields, in some cases, further regularity for these approximated minima; for example, if $a(x, s) \geq |s|^m$ (for $|s|$ large) with $m > 1$, then, even if $\mu \in L^1(\Omega)$, sequences of approximating minimizers are bounded in $H_0^1(\Omega)$, and there exist a T-minimum which belongs to $H_0^1(\Omega)$.

The possibility to find (weak) minima of J when the source μ is a general bounded measure is related to the results of Theorem 2.1. The assumption of integrability for the function g in Theorem 2.1 is now related to the boundedness of $a(x, s)$; roughly speaking, if $a(x, s)$ is bounded from above, then it is possible to find minima of J for **any** measure datum μ; if $a(x, s)$ is unbounded, then stability of minimizers (and existence of minima) holds only for measures not charging sets of zero capacity. A proper statement of these considerations is the following; however we need to set the assumptions on the derivative of $a(x, s)$ instead that directly on a. For the proof of next two results, see [29].

THEOREM 3.1. – *Let a satisfy assumptions* (3.1)-(3.2), *and let $\mu \in \mathcal{M}_b^+(\Omega)$ and μ_n be an approximation of μ in the sense* (2.11). *If u_n is a sequence of minimizers of $J_n(v)$, then there exist a function u $\left(\text{which belongs to } W_0^{1,q}(\Omega) \text{ for any } q < \frac{N}{N-1}\right)$ and a subsequence, still denoted u_n, such that u_n almost everywhere converges to u in Ω, and*

$$T_k(u_n) \to T_k(u) \quad \textit{strongly in } H_0^1(\Omega) \textit{ for any } k > 0 .$$

If moreover we have that

$$\sup_{x\in\Omega} |a'(x, s)| \in L^1(\boldsymbol{R}^+), \tag{3.7}$$

then u is a weak minimum of J. ■

Note that (3.7) implies that $a(x, s)$ is uniformly bounded from above; in the autonomous case, we have $a(x, s) = a(s)$, and assumption (3.7), in virtue also of

(3.2), simply becomes $\alpha \leq a(s) \leq \beta$ for two positive constants α, β. Next result says what can happen if a is unbounded and μ is singular.

THEOREM 3.2. – *Let $\mu \in m_b^+(\Omega)$, splitted as in* (2.9), *and let μ_n be an approximation of μ in the sense* (2.11). *Assume* (3.2), *and that there exist positive continuous functions $\alpha(s)$, $m(s)$ and constants β, γ, δ, such that:*

$$\text{(3.8)} \quad \begin{cases} 0<\beta<\alpha(s)\leq a(x,s)\leq \gamma\alpha(s)\,, \\ \dfrac{a'(x,s)}{\alpha(s)} \geq m(s) \ \forall s\geq\delta, \quad \text{with } m(s) \text{ bounded and } \displaystyle\int_\delta^{+\infty} m(s)\,ds = +\infty . \end{cases}$$

If u_n is a sequence of minimizers of J_n, then (up to subsequences) it converges a.e. to a T-minimum of the functional

$$J_0(v) = \frac{1}{2}\int_\Omega a(x,\, v)\,|\nabla v|^2 - \int_\Omega v\, d\mu_0 .$$

In particular, if $\mu_0 = 0$ and $s_0 = 0$ in (3.2), *then u_n converges to zero.* ■

As an example, consider the case that $a = a(s)$, $a'(s)s \geq 0$, and μ is a Dirac mass. Then the above results say, roughly speaking, that J has a weak minimum if and only if $a(s)$ is bounded, i.e. if the energy of the functional is equivalent to the Dirichlet energy; should $a(s)$ be unbounded at infinity, having a (possibly microscopic) regularizing effect with respect to the Dirichlet energy, then sequences of approximating minimizers blow-down completely. For the significant example of

$$J(v) = \int_\Omega (b(x) + |v|^m)\,|\nabla v|^2\,dx - \int_\Omega v\,d\mu\,,$$

where $\alpha \leq b(x) \leq \beta$ for positive constants α, β, we deduce that J has a weak minimum for any measure μ if $m = 0$, while if $m > 0$ and μ is concentrated on sets of zero capacity, then there is a loss of minima in the relaxation procedure. If μ is the Dirac mass, it can be also proved (independently from the stability argument) that J has a weak minimum if and only if $m = 0$.

4. – Connection with semilinear problems.

Let us come back to the model problem treated so far, that is the autonomous equation

$$\text{(4.1)} \quad \begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = \mu & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $g(s)\,s > 0$. It is known that problem (4.1) is, at least formally, equivalent to a semilinear problem; indeed, setting $w = \Phi(u)$, with $\Phi(s) = \int_0^s \exp(-G(t))\,dt$ and $G(s) = \int_0^s g(t)\,dt$, and setting

$$H(s) = \begin{cases} \exp(-G(\Phi^{-1}(s))), & \text{if } 0 \leq s < \Phi(\infty) \\ \exp(-G(\infty)), & \text{if } \Phi(\infty) \leq s, \end{cases} \tag{4.2}$$

problem (4.1) is transformed into the Dirichlet problem:

$$\begin{cases} -\Delta w = H(w)\,\mu & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

It is a natural question to understand the consequence that the results proved on (4.1) (which is entirely characterized by Theorem 2.2) have for (4.3). Moreover, the formulation of problem (4.3) is not clear «a priori», since when μ is a general bounded measure in Ω the product $H(w)\,\mu$ may not be well defined as soon as w is not continuous. On the other hand, at least when (4.3) is the transformed equation of (4.1), the function H given by (4.2) will satisfy two very important conditions:

$$H(s) > 0 \qquad \forall s \in \boldsymbol{R}^+, \tag{4.4}$$

and

$$\exists \lim_{s \to +\infty} H(s) =: H(\infty). \tag{4.5}$$

In [25] we analyze problem (4.3) under these two structure assumptions; using again, as in Theorem 2.1, a stability approach, we are led to suggest the following formulation for (4.3), inspired from ideas in [14]: splitting the measure μ as in (2.9), the product $H(w)\,\mu$ should be formally written as

$$H(w)\,\mu = H(w)\,\mu_0 + H(\infty)\,\lambda. \tag{4.6}$$

This interpretation, which is motivated from relaxation arguments, accounts for the fact that solutions are expected to blow up on the support of λ, the concentrated part of the measure. This actually happens if $H(\infty) > 0$, but if H tends to zero at infinity, a penalization effect would make this singular measure λ disappear. On the other hand the term $H(w)\,\mu_0$ will be well defined since μ_0 does not charge sets of zero capacity and w will admit a unique cap-quasi continuous representative. The following result, which we state here in its simplest form, is proved in [24] in a context of nonlinear operators.

THEOREM 4.1. – *Assume that H is a bounded continuous function on $\boldsymbol{R}$ satisfying* (4.4), (4.5). *Let μ be in $\mathcal{M}_b^+(\Omega)$, splitted as in (2.9), and let $\{\mu_n\}$ be an* *approximation of* μ *in the sense* (2.11). *If* w_n *is a sequence of solutions of*

$$\begin{cases} -\Delta w_n = H(w_n)\,\mu_n & \text{in } \Omega\,, \\ w_n = 0 & \text{on } \partial\Omega \end{cases} \tag{4.7}$$

then there exists $w \in W_0^{1,q}(\Omega)$ *for every* $q < \dfrac{N}{N-1}$ *such that* $T_k(w)$ *belongs to* $H_0^1(\Omega)$ *for every* $k>0$, $H(w)$ *belongs to* $L^\infty(\Omega, d\mu_0)$, *and for a subsequence* w_n, *not relabeled, we have*

$$\begin{cases} T_k(w_n) \to T_k(w) \quad \text{strongly in } H_0^1(\Omega) \text{ for every } k>0\,, \\ w_n \to w \quad \text{strongly in } W_0^{1,q}(\Omega) \text{ for every } q < \dfrac{N}{N-1}\,, \\ \lim\limits_{n\to+\infty} \displaystyle\int_\Omega \varphi H(w_n)\,\mu_{0n}\,dx = \int_\Omega \varphi H(w)\,d\mu_0 \text{ for every } \varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)\,, \\ \lim\limits_{n\to+\infty} \displaystyle\int_\Omega \varphi H(w_n)\,\lambda_n\,dx = H(\infty)\int_\Omega \varphi\,d\lambda \text{ for every } \varphi \in C_b(\Omega). \end{cases} \tag{4.8}$$

In particular, w *is a solution of* (4.3) *in the following sense:*

$$-\Delta w = H(w)\,\mu_0 + H(\infty)\lambda \quad \text{in } \mathcal{D}'(\Omega)\,. \quad \blacksquare \tag{4.9}$$

In view of Theorem 4.1, a new light is shed upon the results obtained in Section 2 on (4.1). Indeed, if H is given through (4.2), then $H(\infty)=0$ if and only if $\int_0^{+\infty} g(t)\,dt = +\infty$. In this case problem (4.1) has no solution if μ charges sets of zero capacity and, consistently, the concentrated measures disappear in (4.9) because $H(\infty)=0$: the regularizing effect of g in the absorption term of (4.1) corresponds to the fact that the term H in (4.9) penalizes the singular measures λ. By contrast, if $\int_0^{+\infty} g(t)\,dt < +\infty$, and equivalently $H(\infty)>0$, then both problems (4.1) and (4.3) admit solutions for any measure μ.

Moreover, if $g(s)\,s>0$ and H is as in (4.2) one can prove that (4.9) has a unique solution $w \in W_0^{1,q}$ $\forall q < \dfrac{N}{N-1}$, hence problem (4.1) has also a unique solution u.

5. – Removability of very singular approximations.

The results in Section 2 say that problem (2.8) may fail to have a solution for all measures data μ if g is not integrable at infinity and gives a motivation for this fact by stability arguments. The absorption effects of lower order terms may sometimes bring even stronger nonexistence results, under the form of removable singularities phenomena: for problem (2.8), this was proved by H. Brezis and L. Nirenberg (see [13 bis]).

THEOREM 5.1. – *Let K be a compact subset of Ω of zero (harmonic) capacity. Let u be a smooth solution of the problem*

$$-\Delta u + H(x, u, \nabla u) = f \quad \text{in } \Omega \backslash K, \tag{5.1}$$

where $f \in L^\infty(\Omega)$ and H satisfies

$$H(x, s, \xi)\, s \geq \gamma |\xi|^2, \qquad \gamma > 1, \tag{5.2}$$

for every $|s| \geq s_0 > 0$. Then u is a smooth solution of (5.1) *in the whole of Ω.*

Thus, if (5.2) holds true, sets of zero (harmonic) capacity are removable for problem (5.1), which implies that equation (2.8) does not admit *any* type of singularity on K. Note that, for having such strong nonexistence result, one requires assumption (5.1) which is stronger than $g \notin L^1(\boldsymbol{R})$. The result in Theorem 5.1 has suggested that similar features could also be observed in the stability approach, studying the effect of perturbations of the data which are possibly very singular (i.e. not necessarily bounded in L^1 as if converging to measures) but localized around sets of null capacity. We have then the following result, which can be found in more generality (nonlinear operators) in [26].

THEOREM 5.2. – *Let $K \subset \Omega$ be a compact set of zero (harmonic) capacity. Let f be a nonnegative function in $L^1(\Omega)$, and let f_n be a sequence of nonnegative $L^\infty(\Omega)$ functions which converges to f in $L^1_{\mathrm{loc}}(\Omega\backslash K)$, i.e.*

$$\lim_{n \to +\infty} \int_{\Omega \backslash I(K)} |f_n - f|\, dx = 0, \tag{5.3}$$

for any neighborhood $I(K)$ of K. Assume that H satisfies

$$H(x, s, \xi)\, \mathrm{sign}(s) \geq g(s)\, |\xi|^2, \tag{5.4}$$

where $g : \boldsymbol{R} \to \boldsymbol{R}^+$ is a continuous function such that

$$g(s) \geq 0, \quad g \in L^\infty(\boldsymbol{R}^+), \quad \exp(-G(s)) \in L^1(\boldsymbol{R}^+), \tag{5.5}$$

where $G(s) = \int_0^s g(t)\, dt$. Let u_n in $H^1_0(\Omega) \cap L^\infty(\Omega)$ be a solution of

$$\begin{cases} -\Delta u_n + H(x, u_n, \nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

Then u_n converges (up to subsequences) to a function $u \in W^{1,q}_0(\Omega)$ for every

$q < \dfrac{N}{N-1}$, *which solves (in the sense of distributions)*

$$\text{(5.7)} \qquad \begin{cases} -\Delta u + H(x, u, \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad \blacksquare$$

The result of Theorem 5.2 can be seen as an «exceptional» stability result for the solutions of problem (5.7): one can perturb the datum f with arbitrarily large (concentrating on K) functions but the solution u of problem (5.7) remains asymptotically stable. In particular, one can take $f = 0$ and $f_n = \varrho_n * D^k(\delta_{x_0})$, the convolution of derivatives of the Dirac mass. Under assumptions (5.4), (5.5), the approximating solutions converge to zero in the whole of Ω, so that this very strong perturbation is actually swept away by the regularizing effects of the equation.

REMARK 5.3. – Note that assumption (5.5) is slightly weaker than (5.2); moreover, it can be easily proved to be sharp for the model case (2.13). Indeed, if f_n is defined as

$$f_n(x) = \begin{cases} n^{\alpha} & \text{if } 0 \leqslant |x| \leqslant \dfrac{1}{n}, \\ 0 & \text{if } \dfrac{1}{n} < |x| \leqslant 1, \end{cases}$$

with $\alpha > N$, and if

$$\exp(-G(s)) \notin L^1(\boldsymbol{R}^+),$$

then the solutions of

$$\begin{cases} -\Delta u_n + g(u_n)\,|\nabla u_n|^2 = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

have a complete blow-up:

$$\lim_{n \to +\infty} u_n(x) = +\infty, \qquad \forall x \in \Omega. \qquad \blacksquare$$

The previous remark shows that the assumption that $\exp(-G(s)) \in L^1(\boldsymbol{R}^+)$ really plays the role of a borderline: if it is satisfied, any possible singularity arising around K does not propagate in the limiting process (5.6), while in the opposite case, as in Remark 5.3, too strong singularities can dramatically propagate in the whole of Ω. This reminds very much of a barrier phenomenon and appears very clearly in the model equation (2.13); actually, if $\exp(-G(s)) \in L^1(\boldsymbol{R}^+)$ holds true, it is even possible to construct solutions of (2.13) which blow-up on large sets, for instance on a smooth subdomain $\omega \subset \Omega$

with meas$(\omega) > 0$. Indeed, setting $M = \int_0^{+\infty} \exp(-G(s))\, ds$, the solution w of the obstacle problem

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \backslash \omega , \\ w \geqslant M\chi_\omega & \\ w = 0 & \text{on } \partial\Omega , \end{cases}$$

corresponds (at least formally), by putting $w = \int_0^u \exp(-G(s))\, ds$, to a solution u of the problem

$$\begin{cases} -\Delta u + g(u)\,|\nabla u|^2 = 0 & \text{in } \Omega \backslash \omega , \\ u = +\infty & \text{in } \omega , \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

A proper definition of solution of the previous problem can be given in some suitable way but is not our purpose here. We just use this simple remark as a motivation for studying more in detail the possibility to have local interior estimates for such type of equations and our interest for the construction of explosive solutions.

6. – Further developments: local estimates and explosive solutions.

We conclude by only mentioning some recent developments on related topics. Actually, the barrier-type effects observed in the previous study lead in a natural way to consider the problem of constructing solutions of

$$\begin{cases} -\Delta u + H(x, u, \nabla u) = f & \text{in } \Omega , \\ u = +\infty & \text{on } \partial\Omega . \end{cases} \tag{6.1}$$

The existence of such explosive solutions (also called large solutions) is clearly a nonlinear effect, usually generated by regularizing properties of the equation, as the existence of interior local estimates independent from the behavior of u at the boundary. This topic has a long history and is becoming more and more investigated in possibly very different situations. A review of the many references is beyond our purposes here. We just recall that the purely semilinear case

$$-\Delta u + h(u) = f \tag{6.2}$$

has been deeply investigated since the works [18], [27], where J. B. Keller and R. Osserman proved that local estimates hold for positive subsolutions of (6.2) if and only if $\int_a^{+\infty} (h(s)\, s)^{-1/2} ds < \infty$ (so-called Keller-Osserman condition).

Questions concerning explosive solutions have been then thoroughly studied (see e.g. [2], [21], [22], [23], [36]). The possibility to have local estimates in presence of gradient dependent lower order terms has also been object of research; we mainly refer to the fundamental results obtained in [19] in connection with stochastic control problems (the blow-up condition at the boundary is indeed related to a constraint required on the state variable). Existence and asymptotic behaviour of large solutions of equation

$$-\Delta u + h(u) = \pm |\nabla u|^q, \qquad q > 1\,,$$

were also proved in [1] mainly for the case $q \neq 2$. The case of natural growth terms, as in the model problem

$$-\Delta u + h(u) + g(u)\,|\nabla u|^2 = f \tag{6.3}$$

represents as usual a special case and may offer, as before, a clear picture for these absorption effects since there is always a semilinear (variational) structure behind equation (6.3). First of all, it should be observed that local estimates, as well as the existence of explosive solutions, cannot hold if $h = 0$; in fact, if $f = 0$ and $h(s) = 0$, arbitrarily large constants would be solutions of (6.3). Moreover, as already remarked in Section 5, the assumption

$$\int_0^{+\infty} \exp(-G(s))\, ds < +\infty\,, \tag{6.4}$$

plays the role of a borderline. Indeed, if we assume (6.4), then if u is a solution of

$$\begin{cases} -\Delta u + h(u) + g(u)\,|\nabla u|^2 \leqslant 0 & \text{in } \mathcal{D}'(\Omega)\,, \\ u \geqslant 0 & \text{in } \Omega\,, \end{cases} \tag{6.5}$$

the function $v = \int_u^{+\infty} \exp(-G(s))\, ds$ is a positive solution of the semilinear problem

$$-\Delta v \geqslant \varrho(v) \quad \text{in } \Omega\,, \tag{6.6}$$

where $\varrho(v) = h(u)\, e^{-G(u)}$. The existence of universal upper bounds for (6.5) is then related to the existence of positive lower bounds (hence of positive subsolutions) for (6.6). A very simple consequence is the following result on local estimates, where we denote by $\lambda_{1,\Omega}$ the first eigenvalue of the Laplacian (with Dirichlet boundary conditions) and by $\varphi_{1,\Omega}$ the corresponding positive first eigenfunction normalized so that $\max_{\overline{\Omega}} \varphi_{1,\Omega} = 1$. We also denote by $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. The proof of next result is in [30].

THEOREM 6.1. – *Assume* (6.4) *and that*

$$\lambda_{1,\Omega} < \liminf_{s \to +\infty} \frac{h(s)\exp(-G(s))}{\int_s^{+\infty} \exp(-G(t))\,dt}. \tag{6.7}$$

Then there exists a positive decreasing function $F(s)$, *with* $\lim_{s\to 0^+} F(s) = +\infty$, *such that any solution* $u \in H^1(\Omega) \cap L^\infty(\Omega)$ *of* (6.5) *satisfies*

$$u(x) \leqslant F(d_\Omega(x)) \quad \textit{for almost every } x \in \Omega\,.$$

In particular, if we set

$$\theta(s) := \frac{h(s)\exp(-G(s))}{\int_s^{+\infty} \exp(-G(t))\,dt}, \qquad \psi(s) = \int_s^{+\infty} \exp(-G(t))\,dt\,, \tag{6.8}$$

and if $\theta(s)$ *is increasing we have* (*a.e. in* Ω):

$$u(x) \leqslant \min\left\{\theta^{-1}\left(\frac{\lambda_{1,B_1(0)}}{d_\Omega(x)^2}\right),\ \psi^{-1}\big(\varphi_{1,\Omega}(x)\,\psi(\theta^{-1}(\lambda_{1,\Omega}))\big)\right\}. \tag{6.9}$$ ■

REMARK 6.2. – A new feature should be observed with respect to the semilinear problem (6.2); whereas the classical Keller-Osserman condition for (6.2) is independent on the domain Ω, the possibility to have local estimates for (6.5) may (in some cases) depend on Ω, through its first eigenvalue λ_1. This happens for instance if $g = 1$ and $h(s)$ is increasing and bounded. ■

The main application of the local estimates concerns the study of (6.1); let us consider here the model case

$$\begin{cases} -\Delta u + h(u) + g(u)\,|\nabla u|^2 = f & \text{in } \Omega\,, \\ \lim_{d_\Omega(x)\to 0} u(x) = +\infty\,, \end{cases} \tag{6.10}$$

where $f \in L^\infty(\Omega)$ and h and g are nondecreasing. The existence of a solution of (6.10) is a consequence of Theorem 6.1, which ensures that the sequence of solutions of

$$\begin{cases} -\Delta u_n + h(u_n) + g(u_n)\,|\nabla u_n|^2 = f & \text{in } \Omega\,, \\ u_n = n & \text{on } \partial\Omega \end{cases}$$

is bounded in $L^\infty_{\text{loc}}(\Omega)$. The monotonicity assumption on h and g also implies that u_n is an increasing sequence, so that it will converge, locally uniformly, to a function u which solves (6.10). A more delicate question is in general the uniqueness of solutions of (6.10), which is related to the possible asymptotic expansion of u near the boundary. Using the effect of the absorbing natural growth term, we can prove (see [30]) that if g is unbounded any explosive solution behaves, near the boundary, as the corresponding ODE's solution w solving

$$w'' = h(w) + g(w)\,|w'|^2, \qquad w(s) \to +\infty \ \text{ as } \ s \to 0\,.$$

More precisely, $u(x)$ satisfies:

$$u(x) = F^{-1}(d_\Omega(x)) + o(1) \quad \text{as} \ d_\Omega(x) \to 0\,, \tag{6.11}$$

$$\text{where } F(t) = \int_t^{+\infty} \frac{e^{-G(s)}}{\left(\int_{s_0}^{s} h(\xi)\, e^{-2G(\xi)}\, d\xi\right)^{1/2}}\, ds\,.$$

Since this asymptotics at the boundary is true for any explosive solution, and since the monotonicity of h and g yields a classical maximum principle inside Ω, one clearly deduces the uniqueness for (6.10) (see [30]).

THEOREM 6.3. – *Let $f \in L^\infty(\Omega)$. Assume that h is increasing, that g is nondecreasing and such that*

$$\lim_{s\to+\infty} h(s) = +\infty\,, \qquad \lim_{s\to+\infty} g(s) = +\infty\,, \tag{6.12}$$

and that there exists $L > 0$ such that $h(s)\, s \geq 0$, $g(s)s \geq 0$ for any s with $|s| > L$. Then there exists a unique solution u of (6.10). *Moreover $u \in C^1(\Omega)$ and satisfies* (6.11).

Note that, in particular, the previous result applies to the model problem

$$\begin{cases} -\Delta u + |u|^{\alpha-1}u + |u|^{\beta-1}u\,|\nabla u|^2 = f \quad \text{in } \Omega\,, \\ \lim\limits_{\operatorname{dist}(x,\,\partial\Omega)\to 0} u(x) = +\infty\,, \end{cases} \tag{6.13}$$

for any $\alpha, \beta > 0$, extending then results in [2], [22], [36], [19].

Other consequences of the local estimates of Theorem 6.1 also involve properties of the equation in the whole space $\boldsymbol{R}^N$, as existence, weak maximum principles or some Liouville type results for solutions of

$$-\Delta u + h(u) + g(u)\,|\nabla u|^2 = f \quad \text{in } \boldsymbol{R}^N. \tag{6.14}$$

Indeed, they are mainly consequences of the precise estimate (6.9) which, roughly speaking, gives that if $f \leqslant 0$ then $u \leqslant h^{-1}(0)$ for any $u \in H^1_{\rm loc}(\boldsymbol{R}^N)$ subsolution of (6.14). In a similar spirit, an interesting and still widely open question is whether uniqueness holds for (6.14) without condition at infinity, namely if for any $f \in L^\infty_{\rm loc}(\boldsymbol{R}^N)$ problem (6.14) has a unique solution $u \in H^1_{\rm loc}(\boldsymbol{R}^N) \cap L^\infty_{\rm loc}(\boldsymbol{R}^N)$, without prescribing any behavior at infinity, neither on f nor on u. For the semilinear equation (6.2) with $h(s) = |s|^{p-1}s$, $p > 1$, this result was proved by H. Brezis in [12]. A partial result for (6.14), in case of radial nonnegative function f, can be found in [33].

REFERENCES

[1] C. Bandle - E. Giarrusso, *Boundary blow-up for semilinear elliptic equations with nonlinear gradient terms*, Adv. Diff. Equat.. **1** (1996), 133-150.

[2] C. Bandle - M. Marcus, *Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behavior*, J. Anal. Math., **58** (1992), 9-24.

[3] P. Benilan - L. Boccardo - T. Gallouët - R. Gariepy - M. Pierre - J. L. Vázquez, *An L^1 theory of existence and uniqueness of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **22** (1995), 240-273.

[4] P. Benilan - H. Brezis, *Nonlinear problems related to the Thomas-Fermi equation*, J. Evolution Equations, **3** (2003), 673-770.

[5] A. Bensoussan - L. Boccardo - F. Murat, *On a nonlinear partial differential equation having natural growth terms and unbounded solutions*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **5** (1988), 347-364.

[6] L. Boccardo, *T-minima: an approach to minimization problems in L^1*, Contributions in honor of the memory of Ennio De Giorgi. Ricerche Mat., **49** (2000), 135-154.

[7] L. Boccardo - T. Gallouët, *Strongly nonlinear elliptic equations having natural growth terms and L^1 data*, Nonlinear Anal. T.M.A., **19** (1992), 573-579.

[8] L. Boccardo - T. Gallouët - L. Orsina, *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **13** (1996), 539-551.

[9] L. Boccardo - T. Gallouët - L. Orsina, *Existence and nonexistence of solutions for some nonlinear elliptic equations*, Journal d'Analyse Math., **73** (1997), 203-223.

[10] H. Brezis, *Nonlinear elliptic equations involving measures*, in *Variational Inequalities*, Cottle, Giannessi, Lions ed., Wiley, 1980, 53-73.

[11] H. Brezis, *Some Variational Problems of the Thomas-Fermi type*, in *Contributions to nonlinear partial differential equations (Madrid, 1981)*, 82-89, Res. Notes in Math. 89, Pitman, Boston Mass.-London, 1983.

[12] H. Brezis, *Semilinear equations in $\boldsymbol{R}^N$ without condition at infinity*, Appl. Math. Optim., **12**, no. 3 (1984), 271-282.

[13] H. Brezis - M. Marcus - A. Ponce, *Nonlinear elliptic equations with measures revisited*, to appear in Annals of Math. Studies, Princeton Univ. Press.

[13 bis] H. Brezis - L. Nirenberg, *Removable singularities for some nonlinear elliptic equations*, Top. Methods Nonlin. Anal., **9** (1997), 201-219.

[14] G. DAL MASO - F. MURAT - L. ORSINA - A. PRIGNET, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **28**, 4 (1999), 741-808.

[15] M. FUKUSHIMA - K. SATO - S. TANIGUCHI, *On the closable part of pre-Dirichlet forms and the fine support of the underlying measures*, Osaka J. Math., **28** (1991), 517-535.

[16] A. GMIRA - L. VERON, *Boundary singularities of solutions of some nonlinear elliptic equations*, Duke Math. J., **64** (1991), 271-324.

[17] T. IWANIEC - C. SBORDONE, *Weak minima of variational integrals*, J. Reine Angew. Math., **454** (1994), 143-161.

[18] J. B. KELLER, *On solutions of $\Delta u = f(u)$*, Comm. Pure Appl. Math., **10** (1957), 503-510.

[19] J.-M. LASRY - P.-L. LIONS, *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem*, Math. Ann., **283**, n. 4 (1989), 583-630.

[20] J. LERAY - J.-L. LIONS, *Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France, **93** (1965), 97-107.

[21] C. LOEWNER - L. NIRENBERG, *Partial differential equations invariant under conformal or projective transformations*, Contributions to analysis (a collection of papers dedicated to Lipman Bers), pp. 245-272, Academic Press, New York, 1974.

[22] M. MARCUS - L. VERON, *Uniqueness and asymptotic behaviour of solutions with boundary blow-up for a class of nonlinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **14** (1997), 237-274.

[23] M. MARCUS - L. VERON, *Existence and uniqueness results for large solutions of general nonlinear elliptic equations*, Journal of Evolution Equations, **3** (2003), 637-652.

[24] F. MURAT - A. PORRETTA, *Stability properties, existence and nonexistence of renormalized solutions for elliptic equations with measure data*, Comm. P.D.E., **27**, n. 11 & 12 (2002), 2267-2310.

[25] L. ORSINA, *Weak minima for some functionals and elliptic equations involving measures*, C. R. Acad. Sci. Paris **322**, Série I, (1996), 1151-1156.

[26] L. ORSINA - A. PORRETTA, *Strong stability results for nonlinear elliptic equations with respect to very singular perturbation of the data*, Comm. in Contemporary Math., **3** (2001), 259-285.

[27] R. OSSERMAN, *On the inequality $\Delta u \geq f(u)$*, Pacific J. Math., **7** (1957), 1641-1647.

[28] A. PORRETTA, *Some remarks on the regularity of solutions for a class of elliptic equations with measure data*, Houston Journ. of Math., **26** (2000), 183-213.

[29] A. PORRETTA, *Remarks on the existence or loss of minima of infinite energy*, preprint.

[30] A. PORRETTA, *Local estimates and large solutions for some elliptic equations with absorption*, Advances in Diff. Eq. 9, n. 3-4 (2004), 329-351.

[31] A. PORRETTA, *PHD Thesis*, Università di Roma «La Sapienza», 1999.

[32] A. PORRETTA, *Existence for elliptic equations in L^1 having lower order terms with natural growth*, Portugaliae Math., **57** (2000), 179-190.

[33] A. PORRETTA, *Some uniqueness results for elliptic equations without condition at infinity*, Commun. Contemporary Mathematics, **5**, n. 5 (2003), 1-13.

[34] G. STAMPACCHIA, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier, (Grenoble), **15**, n. 1 (1965), 189-258.
[35] J.L. VAZQUEZ, *On a semilinear equation in* $\boldsymbol{R}^2$ *involving bounded measures*, Proc. Royal Soc. Edinburgh, **95A** (1983), 181-202.
[36] L. VERON, *Semilinear elliptic equations with uniform blow-up on the boundary*, J. Analyse Math., **59** (1992), 231-250.
[37] L. VERON, *Elliptic equations involving measures*, to appear in Handbook for Partial Differential Equations, M. Chipot, P. Quittner Eds, Elsevier.

Dipartimento di Matematica, Università di Roma «Tor Vergata»
Via della Ricerca Scientifica 1, 00133 Roma, Italy
E-mail: porretta@mat.uniroma2.it

Pervenuta in Redazione
il 12 dicembre 2003