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Circumcenters in Real Normed Spaces (*).

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Sunto. – *Lo studio dei circocentri in tipi diferenti di triangoli appartenenti a spazi normati reali ci dà nuove carratterizzazioni degli spazi con prodotto scalare.*

Summary. – *The study of the circumcenters in different types of triangles in real normed spaces give us new characterizations of inner product spaces.*

1. – Introduction.

Our aim in this paper is to study circumcenters in real normed spaces and to derive from this some new characterizations of inner product spaces (i.p.s.) in the same line of [2] and [8]. Precisely, let $(E, \| \ \|)$ be a real normed space with dim $E \geq 2$ and consider the triangle of vertices x, y, z, Δxyz, where x, y, z belong to E, x, y linearly independent and z in the plane determined by x and y. Then, in ([1] p. 236-237), jointly with C. Alsina and P. Guijarro we defined the perpendicular bisectors

$$M(x,y) = \left\{ \frac{x+y}{2} + \lambda u(x,y) / \lambda \in \mathbb{R}\ u(x,y) = (\|y\|^2 - \rho'_+(x,y))x + (\|x\|^2 - \rho'_+(y,x))y \right\}$$

where $\rho'_\pm : E \times E \to \mathbb{R}$ are the functionals defined by

$$\rho'_\pm(x,y) = \lim_{t \to 0^\pm} \frac{\|x+ty\|^2 - \|x\|^2}{2t}.$$

The mappings $\rho'_\pm$ generalize (see [3 p. 16-17]) the concept of inner product due to the fact that, when the norm is derivable from an inner product, then $\rho'_\pm$ is precisely that inner product.

The basic properties of $\rho'_\pm$ that will be used frequently are the following

(i) $\rho'_\pm(x,x) = \|x\|^2$ and $|\rho'_\pm(x,y)| \leq \|x\| \, \|y\|$

(ii) $\rho'_-(x,y) \leq \rho'_+(x,y)$

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(iii) $\rho'_+(ax,y) = \rho'_+(x,ay) = a\rho'_+(x,y)$, $a \geq 0$
(iv) $\rho'_+(ax,y) = \rho'_+(x,ay) = a\rho'_-(x,y)$, $a \leq 0$
(v) $\rho'_+(x, ax+y) = a\|x\|^2 + \rho'_+(x,y)$, $a \geq 0$
(vi) $\rho'_+(x,y) = \rho'_+(y,x)$ for all x, y in E if and only if E is an i.p.s.

In an i.p.s., the perpendicular bisectors of the three sides of a triangle all pass through the circumcenter, which is the center of the circumscribed circle (see [4] p. 12-13 and [7] p. 92). Then, in a natural way we have the following

DEFINITION 1.1. – *The triangle Δxyz has a circumcenter C if and only if $M(x,y)$, $M(x,z)$ and $M(y,z)$ meet in the point C.*

Our aim in this paper will be to study first some properties of perpendicular bisector that characterize i.p.s. We will show also that, for some types of triangles, just the existence of the point C characterizes i.p.s. Moreover we will see how the study of circumcenter's properties in a real normed space may characterize the norm as derivable from an inner product.

2. – On some properties of perpendicular bisectors.

In [5], jointly with P. Guijarro we have studied some properties of the perpendicular bisectors and we have obtained some new characterizations of i.p.s.

If E is an i.p.s. and x, y are two vectors in E linearly independent, if we consider the perpendicular bisector $\dfrac{x+y}{2} + \lambda u(x,y)$, $u(x,y)$ is always orthogonal to $x-y$. Now we will show how this property characterizes i.p.s. whenever we consider in a real normed space $(E, \| \ \|)$ the following orthogonality relations (see [3]):

- $x \perp_\rho y$ if and only if $\rho'_+(x,y) = 0$
- $x \perp_p y$ (Pythagoras orthogonality) if and only if $\|x\|^2 + \|y\|^2 = \|x+y\|^2$
- $x \# y$ (James orthogonality) if and only if $\|x-y\| = \|x+y\|$
- $x \perp_B y$ (Birkhoff orthogonality) if and only if $\|x\| \leq \|x+ty\|$ for all t in $\mathbb{R}$.

PROPOSITION 2.1. – *Let $(E, \| \ \|)$ be a strictly convex real normed space with dim $E \geq 2$. Then the following conditions are equivalent*

(I) *E is an inner product space*
(II) *$x - y \perp_\rho u(x,y)$ for all x, y in E*
(III) *$x - y \perp_p u(x,y)$ for all x, y in E*
(IV) *$x - y \# u(x,y)$ for all x, y in E*
(V) *$x - y \perp_B u(x,y)$ for all x, y in E*

PROOF. – To prove this theorem, it is only necessary to check that properties (II), (III), (IV) and (V) imply (I).

If we assume condition (II) i.e. for all x, y in E

$$\rho'_+(x-y, u(x,y)) = 0, \tag{1}$$

changing y by $x-z$ and using (iv) and (v) we obtain for all x, z in E

$$\rho'_+(z, (\rho'_+(x-z,x) - \|x\|^2)z + A(z,x)x) = 0 \tag{2}$$

where $A(z,x) = \|x-z\|^2 + \rho'_-(x,z) - \rho'_+(x-z,x)$.

But

$$\lim_{\lambda\to 0^+} A(z,\lambda x) = \|z\|^2 > 0. \tag{3}$$

So, for $\lambda > 0$ in a zero neighborhood, $A(z,\lambda x) > 0$ and changing x by λx and using (iv) in (2), we have

$$(\rho'_+(\lambda x - z, x) - \lambda\|x\|^2)\|z\|^2 + A(z,\lambda x)\rho'_+(z,x) = 0.$$

Taking limit when λ tends to zero in last equation, using [6,(11)] and (3) we obtain $\rho'_+ = \rho'_-$.

On the other hand by the substitution $z = x-y$, $y = y$ in (1) by (v) and using $\rho'_+ = \rho'_-$ we have for all z, y in E

$$0 = (\|y\|^2 - \rho'_+(z+y,y))\|z\|^2 + (\|z+y\|^2 - \rho'_+(z+y,y) - \rho'_+(y,z))\rho'_+(z,y)$$

and for all $u = z$, $v = y$ unitary vectors linearly independent

$$\rho'_+(u+v,v)(1+\rho'_+(u,v)) = 1 + \|u+v\|^2\rho'_+(u,v) - \rho'_+(v,u)\rho'_+(u,v). \tag{4}$$

Interchanging u and v and using (i) and (v) in

$$\rho'_+(u+v,u) = \rho'_+(u+v,u+v-v) = \|u+v\|^2 - \rho'_-(u+v,v)$$

and $\rho'_+ = \rho'_-$ we obtain

$$\rho'_+(u+v,v)(1+\rho'_+(v,u)) = -1 + \|u+v\|^2 + \rho'_+(v,u)\rho'_+(u,v) \tag{5}$$

and equalizing $\rho'_+(u+v,v)$ in (4) and (5), getting out common factor $1 - \rho'_+(u,v)\rho'_+(v,u)$ (that it is different from zero because E is strictly convex and u, v are linearly independent) and dividing by the common factor we obtain

$$\|u+v\|^2 = 2 + \rho'_+(v,u) + \rho'_+(u,v),$$

and changing v by $-v$ and adding, we have $\|u+v\|^2 + \|u-v\|^2 = 4$ and, (see [3] p. 47 (6.1)) E is an i.p.s.

If condition (III) holds, then for all x, y in E

$$\|u(x,y)\|^2 + \|x-y\|^2 = \|u(x,y) + x - y\|^2,$$

and changing x by tx and y by ty for all $t > 0$, using $u(tx,ty) = t^3u(x,y)$ we have

$$t^4\|u(x,y)\|^2 + \|x-y\|^2 = \|t^2u(x,y) + x - y\|^2.$$

Then for all $\gamma > 0$, $\gamma^2\|u(x,y)\|^2 = \|\gamma u(x,y) + x - y\|^2 - \|x-y\|^2$, and dividing by 2γ and taking limit when γ tends to zero using the definition of ρ'_+ we obtain condition (II) and E is an i.p.s.

If we assume condition (IV) i.e. for all x, y in E

$$\|u(x,y) - x + y\| = \|u(x,y) + x - y\|,$$

changing x by tx and y by ty for all $t > 0$, $\|t^2u(x,y) - x + y\|^2 = \|t^2u(x,y) + x - y\|^2$ and, for all $\gamma > 0$ $\|\gamma u(x,y) - x + y\|^2 = \|\gamma u(x,y) + x - y\|^2$, and subtracting $\|x-y\|^2$ in the two members of last equality, dividing by 2γ and taking limit when γ tends to zero using the definition $\rho'_\pm$ we obtain

$$\rho'_+(x-y, u(x,y)) + \rho'_-(x-y, u(x,y)) = 0. \tag{6}$$

By the substitution $x - y = z$, $x = x$ in (6), using (v)

$$2(\rho'_+(x-z,x) - \|x\|^2)\|z\|^2 + \rho'_+(z, A(z,x)x) + \rho'_-(z, A(z,x)x) = 0,$$

where $A(z,x)$ is defined in (2). Changing x by λx, $\lambda > 0$, dividing by λ, taking limit when λ tends to zero, and using (3) and [6, (11)] we have $\rho'_+ = \rho'_-$. Then by (6) $\rho'_+(x-y, u(x,y)) = 0$ and therefore we have condition (II) and E is an i.p.s.

Finally, if condition (V) holds, then (see [3] p. 33 (iii)) for all x, y in E

$$\rho'_-(x-y, u(x,y)) \le 0 \le \rho'_+(x-y, u(x,y)). \tag{7}$$

If we make the substitution $z = x - y$, $x = x$ and after this $z = -ty$, $t > 0$ using (iv), $\rho'_+(x+ty, x) = \|x+ty\|^2 - t\rho'_-(x+ty, y)$ and dividing by t^2 we obtain

$$\rho'_-(y, A_t(x,y)x) \le (\|x\|^2 - \rho'_+(x+ty,x))\|y\|^2 \le \rho'_+(y, A_t(x,y)x) \tag{8}$$

where $A_t(x,y) = \rho'_+(x,y) - \rho'_-(x+ty,y)$.

Now, we consider two cases: $\rho'_+(x,y) = \rho'_-(x,y)$ or $\rho'_+(x,y) > \rho'_-(x,y)$. In this last case using [6, (11)], $\lim\limits_{t\to 0^+} A_t(x,y) > 0$ and for $t > 0$ in a neighborhood of zero $A_t(x,y) > 0$ and by (8) and (iii)

$$A_t(x,y)\rho'_-(y,x) \le (\|x\|^2 - \rho'_+(x+ty,x))\|y\|^2 \le A_t(x,y)\rho'_+(y,x),$$

so taking limit when t tends to zero, $\rho'_-(y,x) \le 0 \le \rho'_+(y,x)$.

Then for all x, y in E $\rho'_+(x,y) = \rho'_-(x,y)$ or $\rho'_-(y,x) \le 0 \le \rho'_+(y,x)$.

Now, for all $t > 0$ if we consider x and $x + ty$ we have, using (iii) and (v)

$$\rho'_+(x,y) = \rho'_-(x,y) \text{ or } \rho'_-(x+ty,x) \le 0 \le \rho'_+(x+ty,x).$$

If $\rho'_+(x,y) > \rho'_-(x,y)$, for all $t > 0$ $\rho'_-(x+ty,x) \le 0 \le \rho'_+(x+ty,x)$ and taking limit when t tends to zero we obtain the contradiction $\|x\| = 0$. Then $\rho'_+ = \rho'_-$ and by (7) we have $\rho'_+(x-y, u(x,y)) = 0$, i.e. condition (II) holds and E is an i.p.s. $\square$

Now we generalize other perpendicular bisector's property.

COROLLARY 2.2. – *Let* $(E, \|\ \|)$ *be a strictly convex real normed space with dim* $E \geq 2$. *Then* E *is an i.p.s. if and only if for all* x, y *in* E *linearly independents and for all* z *in* $M(x, y)$ *it is* $\|x - z\| = \|y - z\|$.

PROOF. – By hypothesis for all $\lambda > 0$

$$\left\|\frac{x+y}{2} + \lambda u(x,y) - x\right\| = \left\|\frac{x+y}{2} + \lambda u(x,y) - y\right\|,$$

i.e., $x - y \,\#\, 2\lambda u(x, y)$ and if $\lambda = \frac{1}{2}$ by our last proposition E is an i.p.s. □

3. – Circumcenters in real normed spaces.

First we will consider the class of triangles $\Delta xy(x + y)$ where we can show the following result:

THEOREM 3.1. – *Let* $(E, \|\ \|)$ *be a strictly convex real normed space with dim* $E \geq 2$. *Then E is an i.p.s. if and only if there exists the circumcenter of* $\Delta xy(x + y)$ *for all* x, y *linearly independent vectors in E.*

PROOF. – If we consider $\Delta xy(x + y)$, using the linearly independence of x and y it is a straightforward computation to prove that the three straight lines $M(x, y)$, $M(x, x + y)$ and $M(y, x + y)$ meet in a point if and only if the following equation holds

$$\begin{aligned}(9)\quad (\|y\|^2 - \rho'_+(x,y))(\|x\|^2 - \rho'_+(x+y,x))(\|y+x\|^2 - \|y\|^2 - \rho'_+(y,x))\\ = (\|y\|^2 - \rho'_+(x+y,y))[(\|y+x\|^2 - \|x\|^2 - \rho'_+(x,y))(\|x\|^2 - \rho'_+(y,x)) +\\ + 2(\|x\|^2 - \rho'_+(x+y,x))(\|x\|^2 - \rho'_+(y,x) - \|y\|^2 + \rho'_+(x,y)].\end{aligned}$$

If we change x by λx ($\lambda > 0$), using (iii), simplifying λ, dividing by 2λ the term $\|y + \lambda x\|^2 - \|y\|^2 - \lambda\rho'_+(y, x)$ in the left part of last equality and dividing also by 2λ the term $\|y\|^2 - \rho'_+(\lambda x + y, y)$ in the right part of last equality, using the equality $\rho'_+(\lambda x + y, y) = \|\lambda x + y\|^2 - \lambda\rho'_-(\lambda x + y, x)$, and finally taking limit when λ tends to zero, by [6, (11)] and the definition of ρ'_+ we obtain $\rho'_+(y, x) = \rho'_-(y, x)$ or $\rho'_+(y, x) = 0$. Changing x by $-x$ we have $\rho'_+(y, x) = \rho'_-(y, x)$ or $\rho'_-(y, x) = 0$, and combining the four possibles cases we obtain $\rho'_+ = \rho'_-$.

Moreover by the substitution $x = u$, $y = v - u$ in (9), using (v), $\rho'_+ = \rho'_-$, operating and grouping in a suitable way we have

$$-\|u\|^2 - \|v\|^2 + \|u - v\|^2 + 2\rho'_+(v, u) = 0$$

or

$$(10)\quad \rho'_+(v-u,u)(\|u\|^2+\|v\|^2-\rho'_+(u,v)-\rho'_+(v,u))$$
$$=\|u-v\|^2(-\|u\|^2+\rho'_+(v,u))+\|u\|^2\|v\|^2-\rho'_+(u,v)\rho'_+(v,u)$$

By symmetry and using $\rho'_+=\rho'_-$ and $\rho'_+(u-v,v)=-\|v-u\|^2-\rho'_+(v-u,u)$ we have

$$-\|u\|^2-\|v\|^2+\|u-v\|^2+2\rho'_+(u,v)=0$$

or

$$(11)\quad \rho'_+(v-u,u)(\|u\|^2+\|v\|^2-\rho'_+(u,v)-\rho'_+(v,u))$$
$$=\|u-v\|^2(-\|u\|^2+\rho'_+(v,u))-\|u\|^2\|v\|^2+\rho'_+(v,u)\rho'_+(u,v)$$

If (10) and (11) hold then $\rho'_+(u,v)\rho'_+(v,u)=\|u\|^2\|v\|^2$, but by hypothesis E is strictly convex and we obtain a contradiction.

Then, for all u,v in E $\dfrac{\|u\|^2+\|v\|^2-\|u-v\|^2}{2}\in\{\rho'_+(u,v),\rho'_+(v,u)\}$ and, changing v by $-v$ and using $\rho'_+=\rho'_-$, $\dfrac{\|u+v\|^2-\|u\|^2-\|v\|^2}{2}\in\{\rho'_+(u,v),\rho'_+(v,u)\}$.

Then for all $t>0$ the function defined by $f(t)=\dfrac{\|u+tv\|^2-\|u\|^2-t^2\|v\|^2}{2t}$ is continuous in $(0,+\infty)$, $f(t)\in\{\rho'_+(u,v),\rho'_+(v,u)\}$ for all $t>0$ and $\lim_{t\to 0^+}f(t)=\rho'_+(u,v)$. Then $\rho'_+(u,v)=\rho'_+(v,u)$ for all u,v in E and E is an inner product space (see [3], p. 17 (2.2)). □

Note that Theorem 3.1 covers a general case because $\Delta xy(x+y)$ is equivalent to the triangle determined by x and y (i.e. sides x,y and $x-y$).

Following Precupanu [6 p. 161] a norm $\|\ \|$ is smooth if $x\to\|x\|^2$ is Gâteaux differentiable, i.e. $\rho'_+=\rho'_-$.

Then, with a similar proof of last theorem we have the following

COROLLARY 3.2. – *Let $(E,\|\ \|)$ be a strictly convex real normed space with dim $E\geq 2$ such that there exist the circumcenter of $\Delta xy(ax+y)$ for all x,y in E linearly independent and a in $\mathbb{R}$ fixed. Then the norm $\|\ \|$ is smooth.*

By a straightforward computation, we can prove that the intersection of the three perpendicular bisectors of Δxyz give us the three points

$$
\text{(12)} \qquad \begin{cases} C_1 = \dfrac{x+z}{2} + \dfrac{A(z_2-1) - Bz_1}{2(CB + DBz_1 - ADz_2)}(Cx + Dz) \\[2ex] C_2 = \dfrac{x+y}{2} + \dfrac{E(1-z_1) + Fz_2}{2(BFz_1 - AE - AFz_2)}(Ax + By) \\[2ex] C_3 = \dfrac{y+z}{2} + \dfrac{D(z_1+z_2) + C}{2(-CE - CFz_2 - DEz_1)}(Ey + Fz) \end{cases}
$$

where x and y are linearly independent vectors in E, $z = z_1x + z_2y$ (z_1, z_2 in $\mathbb{R}$), $A = \|y\|^2 - \rho'_+(x,y)$, $B = \|x\|^2 - \rho'_+(y,x)$ $C = \|z\|^2 - \rho'_+(x,z)$ $D = \|x\|^2 - \rho'_+(z,x)$, $E = \|z\|^2 - \rho'_+(y,z)$, $F = \|y\|^2 - \rho'_+(z,y)$ and, by Theorem 3.1, in a real normed space, the three points are, in general, differents.

DEFINITION 3.3. – *The points C_1, C_2, C_3 are called the* **circumcenter points** *of Δxyz.*

4. – New characterization of inner product spaces.

We consider some classical properties concerning the circumcenter of a triangle and we translate these properties into a real normed space considering the circumcenter C_1 (we have the same results for C_2 and C_3), and we will obtain new characterizations of inner product spaces.

THEOREM 4.1. – *Let $(E, \| \ \|)$ be a strictly convex real normed space with dim $E \geq 2$. Then E is an inner product space if and only if the circumcenter C_1 of Δxyz is the origin, if x, y, z are vectors in E with $\|x\| = \|y\| = \|z\|$.*

PROOF. – The direct part is a well-known result. Reciprocally if the circumcenter C_1 of Δxyz for all x, y, z in E such that $\|x\| = \|y\| = \|z\|$ is the origin, in particular for x and y unitary linearly independent vectors in E, $z = z_1x + z_2y$ z_1, z_2 in $\mathbb{R}$, $z_2 \neq 0$ and z unitary vector in E, we have, using the expression (12) and the linearly independence of x and y, the system

$$
1 + z_1 + \frac{A(z_2-1) - Bz_1}{CB + DBz_1 - ADz_2}(C + Dz_1) = 0
$$

$$
z_2 + \frac{A(z_2-1) - Bz_1}{CB + DBz_1 - ADz_2} - Dz_2 = 0
$$

where A, B, C and D are defined in the previous page and it is very easy to see that $C = D$ and therefore $\rho'_+(x,z) = \rho'_+(z,x)$ for all x, z in E with $\|x\| = \|z\| = 1$ and then (see [3], p. 18 (2.5)] E is an inner product space. $\square$

THEOREM 4.2. – *Let $(E, \| \ \|)$ be a strictly convex real normed space with dim $E \geq 2$. Then E is an inner product space if and only if the circumcenter C_1 of $\Delta xy(\lambda x + y)$ is $(x+y)\dfrac{\|x\|^2\lambda + \rho'_+(x,y) + \rho'_+(y,x)}{2\|x\|^2 + \rho'_+(x,y) + \rho'_+(y,x)}$ whenever x, y are vectors in E linearly independents with $\|x\| = \|y\|$ and λ belongs to $\mathbb{R}$.*

PROOF. – The direct part of the statement is just a verification. Reciprocally, consider $\Delta xy(\lambda x + y)$, where x and y are unitary vectors and assume $C_1 = (x+y)\dfrac{\|x\|^2\lambda + \rho'_+(x,y) + \rho'_+(y,x)}{2\|x\|^2 + \rho'_+(x,y) + \rho'_+(y,x)}$. Then, using expression (12) of C_1 (where $z_1 = \lambda$ and $z_2 = 1$) and the linear independence of x and y we have:

$$\begin{aligned}\frac{1+\lambda}{2} - \frac{B\lambda}{2(CB + DB\lambda - AD)}(C + D\lambda) &= \frac{\lambda + \rho'_+(x,y) + \rho'_+(y,x)}{2 + \rho'_+(x,y) + \rho'_+(y,x)} \\ &= \frac{1}{2} - \frac{B\lambda}{2(CB + DB\lambda - AD)}\,D.\end{aligned}$$

And, using last equations, with a long computation we can prove that

$$\rho'_+(x,y) = \rho'_+(y,x) \quad \text{or} \quad (\rho'_+(x,y) + \rho'_+(y,x) = 2(1-\lambda) \text{ and } 1 = \rho'_+(\lambda x + y, x))$$

Now, we claim that $|\rho'_+(x,y)| = |\rho'_+(y,x)|$ for all x, y unitary vectors in E.

Let x and y be two unitary linearly independent vectors in E.

If $\lambda = 1$ the result is evident, then for $\lambda \neq 1$ we have:

If

$$\rho'_+(x,y) + \rho'_+(y,x) = 2(1-\lambda) \text{ and } 1 = \rho'_+(\lambda x + y, x), \tag{13}$$

we consider the vectors x and $\dfrac{\lambda x + y}{\|\lambda x + y\|}$, then we have two cases:

$$\rho'_+(\lambda x + y, x) = \rho'_+(x, \lambda x + y) \text{ or } \rho'_+(x, \lambda x + y) + \rho'_+(\lambda x + y, x) = 2(1-\lambda)\|\lambda x + y\|$$
$$\text{and } 1 = \rho'_+\left(\lambda x + \frac{\lambda x + y}{\|\lambda x + y\|}, x\right)$$

In the first case by (v) and (13), $1 = \lambda + \rho'_+(x,y)$ and $\rho'_+(x,y) = 1 - \lambda = \rho'_+(y,x)$.

In the second case we consider the vectors x and $\dfrac{\lambda x + \frac{\lambda x + y}{\|\lambda x + y\|}}{\left\|\lambda x + \frac{\lambda x + y}{\|\lambda x + y\|}\right\|}$ and repeat the process.

Summarizing, if we consider the sequence defined by $b_1 = \lambda x + y$ and $b_n = \lambda x + \dfrac{b_{n-1}}{\|b_{n-1}}\|$ for all $n \geq 2$ we have to bear in mind two possibilities.

Possibility 1. There exist n in $\mathbb{N}$ such that $\rho'_+(x,y)+\rho'_+(y,x)=2(1-\lambda)$, $1=\rho'_+(b_1,x)$, $\rho'_+(x,b_k)+1=2(1-\lambda)\|b_k\|$, $\quad 1=\rho'_+(b_{k+1},x)$ for $1\le k\le n-1$ and $\rho'_+(b_n,x)=\rho'_+(x,b_n)$.

We will prove, by induction, that we can infer $\rho'_+(x,y)=\rho'_+(y,x)=1-\lambda$.

For $n=1$ we have proved that it is true. If it is true for $n-1$ we want to prove that it is true for n.

Using (v), $1=\rho'_+(x,b_n)=\rho'_+\left(x,\lambda x+\frac{b_{n-1}}{\|b_{n-1}\|}\right)=\lambda+\frac{\rho'_+(x,b_{n-1})}{\|b_{n-1}\|}$ and $\|b_{n-1}\|=\frac{\rho'_+(x,b_{n-1})}{1-\lambda}$.

So $\rho'_+(x,b_{n-1})+1=2\rho'_+(x,b_{n-1})$ and $\rho'_+(x,b_{n-1})=1=\rho'_+(b_{n-1},x)$ and by induction hypothesis $\rho'_+(x,y)=\rho'_+(y,x)=1-\lambda$.

Possibility 2. For all n in $\mathbb{N}$

$$\rho'_+(x,b_n)+1=2(1-\lambda)\|b_n\| \tag{14}$$

and

$$\rho'_+(x,y)+\rho'_+(y,x)=2(1-\lambda),\quad 1=\rho'_+(\lambda x+y,x) \tag{15}$$

Now, if we consider $\Delta yx(\lambda y+x)$, and we apply the hypothesis to this triangle, in a similar way to the results obtained before, we have $|\rho'_+(x,y)|=|\rho'_+(y,x)|$ or

Possibility 2'. For all n in $\mathbb{N}$

$$\rho'_+(y,c_n)+1=2(1-\lambda)\|c_n\| \tag{16}$$

and

$$\rho'_+(x,y)+\rho'_+(y,x)=2(1-\lambda),\quad 1=\rho'_+(\lambda y+x,y)$$

where $c_1=\lambda y+x$ and $c_n=\lambda y+\frac{c_{n-1}}{\|c_{n-1}\|}$ for all $n\ge 2$.

In this case, using (14) and (16) for $n=1$ we obtain

$$\rho'_+(x,y)=2(1-\lambda)\|\lambda x+y\|-\lambda-1 \text{ and } \rho'_+(y,x)=2(1-\lambda)\|\lambda y+x\|-\lambda-1.$$

Substituting in (15)

$$2(1-\lambda)=2(1-\lambda)(\|\lambda x+y\|+\|\lambda y+x\|)-2\lambda-2$$

and

$$\|\lambda x+y\|+\|\lambda y+x\|=\frac{2}{1-\lambda}.$$

Then, using $0\le\|\lambda x+y\|+\|\lambda y+x\|\le 2|\lambda|+2$, and by (15), using (i) and the fact that x,y are unitary vectors $2(1-\lambda)\le 2$ and $\lambda>0$ we have $0\le\frac{2}{1-\lambda}\le 2\lambda+2$ and $1-\lambda>0$, then $2\le 2(\lambda+1)(1-\lambda)$, $1\le 1-\lambda^2$, $\lambda=0$ and we obtain a contradiction. So, this case it is impossible. Then $|\rho'_+(x,y)|=|\rho'_+(y,x)|$ for all x,y in E unitary vectors and by ([3] (2.5) p. 18) E is an i.p.s.

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