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L^p-Boundedness for Pseudodifferential Operators with non Smooth Symbols and Applications.

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Sunto. – *Utilizzando una formulazione generalizzata della caratterizzazione per corone diadiche degli spazi di Sobolev, nel presente lavoro si dimostra la continuità L^p per operatori pseudodifferenziali il cui simbolo $a(x, \xi)$ non è infinitamente differenziabile rispetto alla variabile x, mentre le sue derivate rispetto alla variabile ξ decadono con ordine ϱ, con $0 < \varrho \leq 1$. Viene poi provata una proprietà di algebra per una classe di spazi di Sobolev pesati, che ben si applica allo studio della regolarità delle soluzioni di equazioni semi lineari multi-quasi-ellittiche.*

Summary. – *Starting from a general formulation of the characterization by dyadic crowns of Sobolev spaces, the authors give a result of L^p continuity for pseudodifferential operators whose symbol $a(x, \xi)$ is non smooth with respect to x and whose derivatives with respect to ξ have a decay of order ϱ with $0 < \varrho \leq 1$. The algebra property for some classes of weighted Sobolev spaces is proved and an application to multi - quasi - elliptic semilinear equations is given.*

1. – Introduction.

The study of the local solvability and regularity of the solutions of general nonlinear partial differential equations immediately leads to two basic problems: the algebra properties of some spaces of distributions, for example the Sobolev or Hölder spaces, and the study of linear partial differential operators with non smooth coefficients.

In the literature of the last twenty years we find two main approaches to such problems: the paradifferential calculus introduced by J.M. Bony [3], 1981, and the theory of pseudodifferential operators with non smooth symbols. More precisely in this second outlook M. Beals and M.C. Reeds in [1], 1984, check the L^2 continuity and the symbolic calculus for pseudodifferential operators with symbols $a(x, \xi)$ smooth with respect to ξ and whose Sobolev norm $\|\cdot\|_{H^s}$ with respect to the x variable satisfies suitable estimates, at least for great s.

J. Marschall, [17], [18], 1987-88, devotes many efforts in proving the L^p

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properties of pseudodifferential operators with non smooth symbols. He considers the symbols in the classes $H^{r,p}S^m_{\varrho,\delta}$ characterized by the estimate:

$$\|\partial^\alpha_\xi a(x,\xi)\|_{H^{r,p}} < c_\alpha (1+|\xi|)^{m-\varrho|\alpha|+\delta|\beta|}. \tag{1}$$

For $r>0$ suitably large he obtains good results of continuity when $0\leq\delta<\varrho=1$. But when ϱ becomes strictly less than 1, Marschall himself must improve the assumptions on the symbols in $H^{r,p}S^m_{\varrho,\delta}$ in order to obtain some Sobolev continuity results, which at any rate are considerably cut down; namely the L^p continuity of the operators with symbol in $H^{r,p}S^0_{\varrho,0}$, $p\neq 2$, cannot be assured, also for great r.

In the present paper we prove the L^p continuity, $p\neq 2$, for a significant subclass of the pseudodifferential operators whose symbols satisfy (1), with $\delta=0$, $0<\varrho<1$.

For the sake of generality we work in the frame of the *weighted* Sobolev spaces $H^{s,p}_\Lambda$ and the *weighted* symbols $H^{r,p}_\Lambda S^m_\Lambda$, where $\Lambda(\xi)$ is a positive weight function suitably defined, which takes the place of the usual euclidean norm in $\mathbb{R}^n_\xi$.

Our main result may be now resumed as follows: for any $a(x,\xi)$ in a suitable subspace of $H^{r,p}_\Lambda S^m_\Lambda$, $1<p<\infty$, $m\in\mathbb{R}$ and r large, we can show that

$$a(x,D): H^{s+m,p}_\Lambda \to H^{s,p}_\Lambda \ \text{ continuously for } \ 0\leq s\leq r. \tag{2}$$

The work is essentially based on three main tools:

1) the Lizorkin-Marcinckiewicz lemma on continuity of Fourier multipliers, [16], 1963;

2) the characterization of weighted Sobolev and Besov spaces by means of non-homogeneous partitions of unity, given in Triebel [25]-[29], 1977-1979;

3) the decomposition of $a(x,\xi)$ in expansions of elementary symbols obtained by following a technique of Coifman and Meyer [6], 1978.

The paper is planned as follows: in § 2 we introduce the weight functions $\Lambda(\xi)$ together with their main properties.

In § 3, 4, 5 the weighted Sobolev spaces are defined and their basic properties are studied in the more general outlook of Besov and Triebel function spaces, introduced by means of the non-homogeneous partitions of unity above quoted.

In the next § 6 we define the symbols with limited smoothness and at the same time we consider their expansions in elementary symbols.

Finally in § 7 we prove the main result of continuity which we have already summed up in (2).

As trivial corollary we can then say that the weighted Sobolev spaces $H_\Lambda^{s,p}$, for $1 < p < \infty$ and suitably large s, are function algebras.

In the last § 8 a result of local regularity in $H_\Lambda^{s,p}$ for semilinear equations whose linear part is *multi-quasi-elliptic* is given.

2. – Weight functions.

DEFINITION 2.1. – *Let us say that a positive function $\Lambda(\xi) \in C^\infty(\mathbb{R}^n)$ is a weight function if it fulfills the following assumptions:*

1) *there exist two constants $\mu_0 \geq 1$ and $C > 1$ such that*

$$\Lambda(\xi) \geq \frac{1}{C}(1 + |\xi|)^{\mu_0}, \qquad \xi \in \mathbb{R}^n; \tag{3}$$

2) *for every multi-index $\gamma \in \mathbb{Z}_+^n$ there exists a suitable positive constant C_γ such that*

$$\prod_{j=1}^n (1 + \xi_j^2)^{\gamma_j/2} |\partial^\gamma \Lambda(\xi)| \leq C_\gamma \Lambda(\xi), \quad \xi \in \mathbb{R}^n; \tag{4}$$

3) *for some $C > 1$ we have*

$$\Lambda(t\xi) \leq C\Lambda(\xi), \quad t, \xi \in \mathbb{R}^n, \quad \max_{1 \leq j \leq n} |t_j| \leq 1, \tag{5}$$

where $t\xi := (t_1\xi_1, \ldots, t_n\xi_n)$.

For similar definitions of weight function the reader can see [27] and [11].

Examples

1) The standard *elliptic weight function* of order $m \in \mathbb{N}$

$$P_m(\xi) = \left(1 + \sum_{j=1}^n \xi_j^{2m}\right)^{1/2}.$$

It is asymptotically equivalent to the *homogeneous weight* $\langle\xi\rangle^m$, where here and in the following we set $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$.

2) The *quasi-elliptic weight function* defined by

$$P_M(\xi) = \left(1 + \sum_{j=1}^n \xi_j^{2m_j}\right)^{\frac{1}{2}}, \quad M = (m_1, \ldots, m_n) \in \mathbb{N}^n, \quad \min_j m_j \geq 1.$$

$P_M(\xi)$ is asymptotically equivalent to the *quasi-homogeneous weight* $[\xi]_M$, which is the unique positive number satisfying the condition $\sum_{j=1}^n t^{-2/m_j} \xi_j^2 = 1$, $[0]_M = 0$.

3) The following examples are provided by Triebel [27]:

$$\prod_{j=1}^{n}(1+\xi_j^2)^{s_j/2}, \quad s=(s_1,\dots,s_n)\in\mathbb{R}_+^n, \quad \min_{1\le j\le n} s_j \geq 1\,;$$

$$\langle\xi\rangle^s[\log(2+\langle\xi\rangle)]^t, \quad s,t\geq 1\,.$$

A more significant example of weight function will be moreover described in details in § 8.

REMARK 1. – For any weight function $\Lambda(\xi)$ we can always find two constants C, μ_1 both greater than 1 such that:

$$\Lambda(\xi)\leq C(1+|\xi|)^{\mu_1}, \quad \xi\in\mathbb{R}^n \tag{6}$$

(see [27] Lemma 2.1/2 (ii)).

As a straightforward consequence of (6) and (4) it follows that for any $\gamma\in\mathbb{Z}_+^n$ there is a positive C_γ such that:

$$|\partial^\gamma\Lambda(\xi)|\leq C_\gamma(1+|\xi|)^{\mu_1}, \quad \xi\in\mathbb{R}^n.$$

Notice that, using the Faà di Bruno formula, we also obtain from (4)

$$\prod_{j=1}^{n}(1+\xi_j^2)^{\gamma_j/2}|\partial^\gamma(\Lambda(\xi)^m)|\leq C'_\gamma\Lambda(\xi)^m, \quad \xi\in\mathbb{R}^n,\ \gamma\in\mathbb{Z}_+^n,$$

for every $m\in\mathbb{R}$ and then, for $r:=\max(m\mu_0, m\mu_1)$ we have:

$$|\partial^\gamma(\Lambda(\xi)^m)|\leq C''_\gamma(1+|\xi|)^r, \quad \xi\in\mathbb{R}^n,\ \gamma\in\mathbb{Z}_+^n.$$

3. – Weighted Sobolev spaces.

Hereafter we will write $\mathcal{F}_{x\to\xi}u(\xi)=\widehat{u}(\xi)$ for the Fourier transform of a rapidly decreasing function $u(x)\in\mathcal{S}(\mathbb{R}^n)$ (or a tempered distribution $u\in\mathcal{S}'(\mathbb{R}^n)$) and $\mathcal{F}^{-1}_{\xi\to x}u(x)$ for its inverse Fourier transform.

For a given function $a(\xi)$ we set $a(D)\,u(x):=\mathcal{F}^{-1}_{\xi\to x}(a(\xi)\,\widehat{u}(\xi))(x)=(\mathcal{F}^{-1}_{\xi\to x}a*u)(x)$, for any $u(x)\in\mathcal{S}(\mathbb{R}^n)$ (or $u\in\mathcal{S}'(\mathbb{R}^n)$), provided that all the expressions involved make sense.

For any weight function $\Lambda(\xi)$ we can define a scale of weighted Sobolev spaces as follows.

DEFINITION 3.1. – *For $s\in\mathbb{R}$ and $1<p<\infty$, $H^{s,p}_\Lambda$ is the space of all the tempered distributions $u\in\mathcal{S}'(\mathbb{R}^n)$ such that $\Lambda(D)^s u\in L^p(\mathbb{R}^n)$.*

REMARK 2. – For any $s\in\mathbb{R}$ and $1<p<\infty$, $H^{s,p}_\Lambda$ is a Banach space with respect to the norm $\|u\|_{s,p,\Lambda}:=\|\Lambda(D)^s u\|_p$; when $p=2$ it is in particular a Hilbert space if equipped with inner product $(u, v)_{s, \Lambda} := (\Lambda(D)^s u, \Lambda(D)^s v)_2$, u, $v \in H^{s,2}_\Lambda$.

Hereafter we write H^s_Λ for $H^{s,2}_\Lambda$, $s \in \mathbb{R}$.

The spaces H^s_Λ are particular cases of the Bony-Chemin Sobolev spaces introduced in [4]; see also [11].

In order to study the relations between the weighted Sobolev spaces $H^{s,p}_\Lambda$, $H^{t,p}_\Lambda$ for different s and t, we use the following result due to Lizorkin and Marcinkiewicz (see [16] and [20]).

LEMMA 3.1. – *Let $m(\xi)$ be a continuous function together with its derivatives $\partial^\gamma m(\xi)$ for any γ in the set $\mathbb{K} := \{0, 1\}^n$ of the multi-indices with all the components equal to either 0 or 1. If there exists a constant $B > 0$ such that*

$$|\xi^\gamma \partial^\gamma m(\xi)| \leq B, \quad \xi \in \mathbb{R}^n, \ \gamma \in \mathbb{K}, \tag{7}$$

then for every $1 < p < \infty$ we can find a constant $A_p > 0$, only depending on p, B and the dimension n, such that:

$$\|m(D)\, u\|_p \leq A_p \|u\|_p, \tag{8}$$

for any $u \in \mathcal{S}(\mathbb{R}^n)$.

REMARK 3. – The estimate (8) can immediately be extended by density arguments to any function $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$.

PROPOSITION 3.1. – *For $s, t \in \mathbb{R}$, $s < t$, and $1 < p < \infty$ the inclusions $\mathcal{S}(\mathbb{R}^n) \subset H^{t,p}_\Lambda \subset H^{s,p}_\Lambda \subset \mathcal{S}'(\mathbb{R}^n)$ hold with continuous embedding. Moreover $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{s,p}_\Lambda$.*

PROOF. – The inclusions $\mathcal{S}(\mathbb{R}^n) \subset H^{s,p}_\Lambda \subset \mathcal{S}'(\mathbb{R}^n)$ are trivial consequences of Definition 3.1, Remark 1 and the well-known continuous inclusions $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.

For the remaining embedding, it suffices to write for $u \in \mathcal{S}'(\mathbb{R}^n)$

$$\Lambda(D)^s u = \Lambda(D)^{s-t}(\Lambda(D)^t u) = \Lambda(D)^{s-t} v\,,$$

where $v := \Lambda(D)^t u$ and then to observe that $\Lambda(\xi)^{s-t}$ fulfills estimate (7), since it satisfies (4) and $s - t$ is negative.

Thus by Lemma 3.1 for any $u \in H^{t,p}_\Lambda$ we obtain $u \in H^{s,p}_\Lambda$ and moreover

$$\|u\|_{s,p,\Lambda} \leq C \|u\|_{t,p,\Lambda}$$

with some constant $C > 0$ independent of u, as $v = \Lambda(D)^t u \in L^p(\mathbb{R}^n)$.

In order to prove the last statement, let u be any distribution in $H^{s,p}_\Lambda$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, we can find a sequence of functions $v_\nu \in \mathcal{S}(\mathbb{R}^n)$ converging to $v := \Lambda(D)^s u$ in $L^p(\mathbb{R}^n)$. Let us define $u_\nu := \Lambda(D)^{-s} v_\nu$, $\nu = 1, 2, \ldots$; it follows that $u_\nu \in \mathcal{S}(\mathbb{R}^n)$ for any ν and the sequence $\{u_\nu\}$ converges to u in $H_\Lambda^{s,p}$. ■

REMARK 4. – All definitions and results of this section could be stated for a weight function $\Lambda(\xi) \geq c > 0$ which satisfies only the assumption 2) of Definition 2.1.

4. – Partition of unity.

For more details on the partition of unity described below, the reader can see Triebel [27].

For the sake of brevity, hereafter we write $\mathbb{E} := \{-1, 1\}^n$ for the set of the n-ples $\lambda = (\lambda_1, \ldots, \lambda_n)$ whose components are all equal to either -1 or 1. Let $H > 1$ be a fixed constant; for any $h \in \mathbb{Z}_+^n$ and $\lambda \in \mathbb{E}$ we define:

$$P_{h,\lambda}^{(H)} := \left\{ \xi \in \mathbb{R}^n : \frac{1}{H} 2^{h_j} \eta_{h_j} \leq \lambda_j \xi_j \leq H 2^{h_j + 1} \quad j = 1, 2, \ldots, n \right\},$$

with $\eta_h = -1$ if $h = 0$ and $\eta_h = 1$ if $h > 0$.

We will call non-homogeneous decomposition of $\mathbb{R}^n$ the family of n-cubes: $\{P_{h,\lambda}^{(H)}\} = \{P_{h,\lambda}^{(H)}\}_{\substack{h \in \mathbb{Z}_+^n \\ \lambda \in \mathbb{E}}}$.

PROPOSITION 4.1. – *Let $\{P_{h,\lambda}^{(H)}\}$ be a non-homogeneous decomposition of $\mathbb{R}^n$. Then for a suitable integer $N_0(H) > 0$, only depending on H, we have $P_{h,\lambda}^{(H)} \cap P_{k,\varepsilon}^{(H)} = \emptyset$ when $|h_j - k_j| > N_0(H)$ for some $j = 1, 2, \ldots, n$ and $\lambda, \varepsilon \in \mathbb{E}$.*

PROOF. – Let us recall that two n-cubes $P_{h,\lambda}^{(H)}$ and $P_{k,\varepsilon}^{(H)}$ are disjoint when at least for one direction the corresponding sides do not overlap.

For a fixed $1 \leq j \leq n$ the corresponding sides L_{λ_j, h_j} and L_{ε_j, k_j} of $P_{h,\lambda}^{(H)}$ and $P_{k,\varepsilon}^{(H)}$ are described as it follows

$$\begin{aligned} L_{\lambda_j, h_j} = L_{\lambda_j, h_j}^{(H)} &:= \left\{ \xi \in \mathbb{R};\ \frac{1}{H} 2^{h_j} \eta_{h_j} \leq \lambda_j \xi_j \leq H 2^{h_j + 1} \right\} \\ L_{\varepsilon_j, k_j} = L_{\varepsilon_j, k_j}^{(H)} &:= \left\{ \xi \in \mathbb{R},\ \frac{1}{H} 2^{k_j} \eta_{k_j} \leq \varepsilon_j \xi_j \leq H 2^{k_j + 1} \right\} \end{aligned} \tag{9}$$

where η_{h_j} and η_{k_j} are defined as before.

Assuming that h_j and k_j are both strictly positive the inequalities which characterize (9) reduce to

$$\frac{1}{H}2^{h_j} \leqslant |\xi_j| \leqslant H2^{h_j+1} \qquad \frac{1}{H}2^{k_j} \leqslant |\xi_j| \leqslant H2^{k_j+1}$$

respectively.

So L_{λ_j, h_j} and L_{ε_j, k_j} are disjoint when $H2^{k_j+1} < \frac{1}{H}2^{h_j}$ or $H2^{h_j+1} < \frac{1}{H}2^{k_j}$, that is when $|h_j - k_j| > N_0(H)$ with $N_0(H) := 1 + 2\log_2(H)$.

We can also check that the same condition $|h_j - k_j| > N_0(H)$ assures $L_{\lambda_j, h_j} \cap L_{\varepsilon_j, k_j} = \emptyset$ when $h_j = 0$. This ends the proof. ■

DEFINITION 4.1. – *For a fixed $H > 1$, $\Phi^{(H)}$ is the set of all the sequences $\{\varphi_{h,\lambda}(\xi)\} = \{\varphi_{h,\lambda}(\xi)\}_{\substack{h \in \mathbb{Z}_+^n \\ \lambda \in \mathbb{E}}}$ of functions $\varphi_{h,\lambda}(\xi) \in C_0^\infty(\mathbb{R}^n)$ which satisfy the following:*

1) *For any $h \in \mathbb{Z}_+^n$ and $\lambda \in \mathbb{E}$, supp $\varphi_{h,\lambda} \subset P_{h,\lambda}^{(H)}$;*

2) *For every multi-index $\alpha \in \mathbb{Z}_+^n$ there exists a constant $C_\alpha > 0$ such that:*

$$|\partial_\xi^\alpha \varphi_{h,\lambda}(\xi)| \leqslant C_\alpha 2^{-h\cdot\alpha}, \qquad \xi \in \mathbb{R}^n, \quad h \in \mathbb{Z}_+^n, \quad \lambda \in \mathbb{E}\,; \tag{10}$$

3) *For any $\xi \in \mathbb{R}^n$, $\sum_{h \in \mathbb{Z}_+^n,\, \lambda \in \mathbb{E}} \varphi_{h,\lambda}(\xi) = 1$.*

We also set $\Phi := \bigcup_{H>1} \Phi^{(H)}$.

REMARK 5. – From Proposition 4.1 it follows that the sum in 3) reduces to a finite number of terms, say $N_1(H)$, independent of ξ.

In order to see that Φ is not empty, see Triebel [27], let us consider a function $\psi(\xi) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leqslant \psi(\xi) \leqslant 1$, $\psi(\xi) = 1$ for $\xi \in Q_0 := \{\xi \in \mathbb{R}^n\colon |\xi_j| \leqslant \frac{1}{2}, j = 1, \ldots, n\}$ and supp $\psi \subset Q_1 := \{\xi \in \mathbb{R}^n\colon |\xi_j| \leqslant \frac{K}{2}, j = 1, \ldots, n\}$, for some $1 < K < 3$.

For any $h \in \mathbb{Z}_+^n$ and $\lambda \in \mathbb{E}$ we define:

$$\psi_{h,\lambda}(\xi) := \psi(2^{-h_1 - \theta_{h_1}}(\xi_1 - \lambda_1 c_{h_1}), \ldots, 2^{-h_n - \theta_{h_n}}(\xi_n - \lambda_n c_{h_n})),$$

where

$$\theta_h = \begin{cases} 1 & \text{if } h = 0 \\ 0 & \text{if } h > 0 \end{cases} \quad \text{and} \quad c_h := \begin{cases} 1 & \text{if } h = 0 \\ \frac{2}{3}2^h & \text{if } h > 0\,. \end{cases} \tag{11}$$

It is easy to see that the system $\{\psi_{h,\lambda}(\xi)\}$ satisfies (10); moreover it holds that supp $\psi_{h,\lambda} \subset \tilde{P}^{(K)}_{h,\lambda} := \{\xi \in \mathbb{R}^n : \lambda_j \xi_j \in J_{K,h_j},\ j = 1, \dots, n\}$ with

$$J_{K,h_j} := \begin{cases} [1-K,\ 1+K] & \text{if } h_j = 0 \\ \left[\dfrac{3-K}{2} 2^{h_j},\ \dfrac{3+K}{4} 2^{h_j+1}\right] & \text{if } h_j > 0\,. \end{cases}$$

Furthermore it holds that $1 \leq \sum_{h,\lambda} \psi_{h,\lambda}(\xi) \leq N_1(H)C$, $\xi \in \mathbb{R}^n$.

Thus setting $\varphi_{h,\lambda}(\xi) := \dfrac{\psi_{h,\lambda}(\xi)}{\sum_{k,\varepsilon} \psi_{k,\varepsilon}(\xi)}$, for any $h \in \mathbb{Z}^n_+$ and $\lambda \in \mathbb{E}$, it turns out that $\{\varphi_{h,\lambda}(\xi)\}$ belongs to $\Phi^{(H)}$ if we choose a constant $H > 1$ so that $\dfrac{2}{3-K} < H < \dfrac{1}{K-1}$; this is always possible provided that $1 < K < \dfrac{5}{3}$.

5. – Besov and Triebel spaces.

Using the above defined non-homogeneous partition of unity, we can now define two scales of non-homogeneous spaces of Besov and Triebel type related to a weight function $\Lambda(\xi)$.

For any $1 \leq q \leq \infty$ we denote as usual by ℓ^q the space of the sequences of complex numbers $\{c_j\}_{j=1}^{\infty} = \{c_j\}$ such that $\sum_{j=1}^{\infty} |c_j|^q < \infty$ ($\sup_j |c_j| < \infty$ for $q = \infty$); ℓ^q is a Banach space with respect to the norm $\|\{c_j\}\|_{\ell^q} := \left(\sum_{j=1}^{\infty} |c_j|^q\right)^{1/q}$ ($\|\{c_j\}\|_{\ell^q} := \sup_j |c_j|$ for $q = \infty$).

For any $1 \leq p, q \leq \infty$, we define $\ell^q(L^p(\mathbb{R}^n)) = \ell^q(L^p)$ as the space of the sequences $\{f_j\}_{j=1}^{\infty} = \{f_j\}$ of functions $f_j \in L^p(\mathbb{R}^n)$ such that $\{\|f_j\|_p\}$ belongs to ℓ^q.

$\ell^q(L^p)$ realizes to be a Banach space with respect to the norm $\|\{f_j\}\|_{\ell^q(L^p)} := \|\{\|f_j\|_p\}\|_{\ell^q}$. Lastly, we denote by $L^p(\mathbb{R}^n; \ell^q) = L^p(\ell^q)$ the space of the L^p functions taking values in the space ℓ^q; namely the general element of $L^p(\ell^q)$ is a sequence $\{f_j\}$ of measurable functions $f_j = f_j(x)$ in $\mathbb{R}^n$ such that the real valued function $x \mapsto \|\{f_j(x)\}\|_{\ell^q}$ belongs to $L^p(\mathbb{R}^n)$.

When equipped with the norm $\|\{f_j\}\|_{L^p(\ell^q)} := \|\|\{f_j(.)\}\|_{\ell^q}\|_p$, $L^p(\ell^q)$ is a Banach space. For more details the reader can see Triebel [27].

DEFINITION 5.1. – *Let $\Lambda(\xi)$ be a weight function, $1 < p < \infty$, $s \in \mathbb{R}$ and $\{\varphi_{h,\lambda}(\xi)\} \in \Phi^{(H)}$, for some $H > 1$. We define*:

(i) $B^{s,\Lambda}_{p,q} := \{u \in \mathcal{S}'(\mathbb{R}^n) : \{\Lambda(c^{(H)}_{h,\lambda})^s u_{h,\lambda}\} \in \ell^q(L^p)\}$, $1 \leq q \leq \infty$;

(ii) $F^{s,\Lambda}_{p,q} := \left\{u \in \mathcal{S}'(\mathbb{R}^n) : \{\Lambda(c^{(H)}_{h,\lambda})^s u_{h,\lambda}\} \in L^p(\ell^q)\right\}$, $1 < q < \infty$,

where $u_{h,\lambda}(x) := \varphi_{h,\lambda}(D)u(x)$, $x \in \mathbb{R}^n$, and $c^{(H)}_{h,\lambda}$ is the center of the n-cube $P^{(H)}_{h,\lambda}$, for any $h \in \mathbb{Z}^n_+$ and $\lambda \in \mathbb{E}$.

$B^{s,\Lambda}_{p,q}$ and $F^{s,\Lambda}_{p,q}$ are Banach spaces with respect to the norms

$$\|u\|_{B^{s,\Lambda}_{p,q}} := \|\{\Lambda(c^{(H)}_{h,\lambda})^s u_{h,\lambda}\}\|_{\ell^q(L^p)} = \Big(\sum_{h,\lambda}\|\Lambda(c^{(H)}_{h,\lambda})^s u_{h,\lambda}\|^q_p\Big)^{1/q}, \tag{12}$$

for $1 \leq q < \infty$ (modification for $q = \infty$) and

$$\|u\|_{F^{s,\Lambda}_{p,q}} := \|\{\Lambda(c^{(H)}_{h,\lambda})^s u_{h,\lambda}\}\|_{L^p(\ell^q)} = \Big\|\Big(\sum_{h,\lambda}|\Lambda(c^{(H)}_{h,\lambda})^s u_{h,\lambda}|^q\Big)^{1/q}\Big\|_p. \tag{13}$$

At a first glance the norms (12), (13) obviously depend on the system $\{\varphi_{h,\lambda}(\xi)\}$; anyway it may be shown that for different choices of $\{\varphi_{h,\lambda}(\xi)\} \in \Phi$ they are equivalent, see Triebel [27]. Therefore the spaces $B^{s,\Lambda}_{p,q}$, $F^{s,\Lambda}_{p,q}$ themselves do not depend on the system $\{\varphi_{h,\lambda}(\xi)\}$.

We have now all the tools to characterize the weighted Sobolev spaces $H^{s,p}_\Lambda$, introduced in § 3, in terms of a non-homogeneous decomposition of $\mathbb{R}^n$; namely

PROPOSITION 5.1. – *Let $\Lambda(\xi)$ be a weight function, $1 < p < \infty$ and $s \in \mathbb{R}$. Then*

$$H^{s,p}_\Lambda = F^{s,\Lambda}_{p,2}.$$

More precisely there are two constants $c_1, c_2 > 0$ such that for all $u \in \mathcal{S}'(\mathbb{R}^n)$:

$$c_1\|u\|_{F^{s,\Lambda}_{p,2}} \leq \|u\|_{H^{s,p}_\Lambda} \leq c_2\|u\|_{F^{s,\Lambda}_{p,2}}.$$

For the proof the reader can see Triebel [27], where the weighted Sobolev, Besov and Triebel spaces are defined in the context of a wider class of weight functions which only satisfy (4).

In what follows we give a number of properties of weighted Sobolev, Besov and Triebel spaces that will be used in the next sections.

PROPOSITION 5.2. – *Let $\Lambda(\xi)$ be a weight function, $1 < p < \infty$ and $s \in \mathbb{R}$. Then the following inclusions hold with continuous embeddings*

1) *If $1 < q < \infty$, then*

$$B^{s,\Lambda}_{p,\min(p,q)} \subset F^{s,\Lambda}_{p,q} \subset B^{s,\Lambda}_{p,\max(p,q)}. \tag{14}$$

2)

$$\mathcal{S}(\mathbb{R}^n) \subset B^{s,\Lambda}_{p,q_1} \subset B^{s,\Lambda}_{p,q_2} \subset \mathcal{S}'(\mathbb{R}^n) \quad \textit{if } 1 \leq q_1 < q_2 \leq \infty; \tag{15}$$

$$\mathcal{S}(\mathbb{R}^n) \subset F^{s,\Lambda}_{p,q_1} \subset F^{s,\Lambda}_{p,q_2} \subset \mathcal{S}'(\mathbb{R}^n) \quad \text{if } 1 < q_1 < q_2 < \infty. \tag{16}$$

3) *For any* $\varepsilon > 0$, *then*:

(17) $$B^{s+\varepsilon,\Lambda}_{p,q_1} \subset B^{s,\Lambda}_{p,q_2}, \quad \textit{if } 1 \leqslant q_1, q_2 \leqslant \infty;$$

(18) $$F^{s+\varepsilon,\Lambda}_{p,q_1} \subset F^{s,\Lambda}_{p,q_2}, \quad \textit{if } 1 < q_1, q_2 < \infty.$$

PROOF. – The inclusions (14)-(16) are proved in Triebel [27].

To prove inclusions (17) and (18) it suffices to observe that for any sequence $\{b_{h,\lambda}\}$ of positive numbers the following estimates hold for all $1 \leqslant q_1, q_2 < \infty$, $H > 1$ and $A_{q_2,\varepsilon} := \Big(\sum_{h,\lambda} \Lambda(c^{(H)}_{h,\lambda})^{-\varepsilon q_2}\Big)^{1/q_2}$:

(19) $$\Big(\sum_{h,\lambda} \Lambda(c^{(H)}_{h,\lambda})^{sq_2} b^{q_2}_{h,\lambda}\Big)^{1/q_2} \leqslant A_{q_2,\varepsilon} \sup_{h,\lambda} \Lambda(c^{(H)}_{h,\lambda})^{s+\varepsilon} b_{h,\lambda} \leqslant$$

$$A_{q_2,\varepsilon} C_{q_1} \Big(\sum_{h,\lambda} \Lambda(c^{(H)}_{h,\lambda})^{(s+\varepsilon)q_1} b^{q_1}_{h,\lambda}\Big)^{1/q_1},$$

.

since $\|.\|_{\ell^\infty} \leqslant C\|.\|_{\ell^{q_1}}$ and $A_{q_2,\varepsilon} < \infty$ (obvious modifications for $q_2 = \infty$). In order to show the convergence of the above expansion, notice that the center $c^{(H)}_{h,\lambda}$ of the cube $P^{(H)}_{h,\lambda}$ has coordinates $c^{(H)}_{h,\lambda} = (\lambda_1 c_{1,H} 2^{h_1}; \dots; \lambda_n c_{n,H} 2^{h_n})$ and the numbers $c_{j,H}$, $1 \leqslant j \leqslant n$ are equal to either $H + \dfrac{1}{2H}$ or $H - \dfrac{1}{2H}$.

From (3) it follows that there exists a number $C_H > 0$ such that:

$$\Lambda(c^{(H)}_{h,\lambda}) \geqslant C_H \Big(1 + \sum_{l=1}^{n} 2^{h_l}\Big)^{\mu_0}, \qquad h \in \mathbb{Z}^n_+,\ \lambda \in \mathbb{E}.$$

Thus we have

$$\sum_{h,\lambda} \Lambda(c^{(H)}_{h,\lambda})^{-\varepsilon q_2} \leqslant C'_{H,\varepsilon,q_2} \sum_{\lambda} \sum_{h} \frac{1}{\Big(1 + \sum_{l=1}^{n} 2^{h_l}\Big)^{\mu_0 \varepsilon q_2}} \leqslant$$

$$C'_{H,\varepsilon,q_2} 2^n \sum_{h_1=0}^{\infty} \frac{1}{(2^{h_1})^{\mu_0 \varepsilon q_2}} \cdots \sum_{h_n=0}^{\infty} \frac{1}{(2^{h_n})^{\mu_0 \varepsilon q_2}}$$

and $\sum_{h_j=0}^{\infty} \dfrac{1}{(2^{h_j})^{\mu_0 \varepsilon q_2}} < \infty$.

We immediately get inclusion (17) by setting $b_{h,\lambda} = \|u_{h,\lambda}\|_p$, while inclusion (18) follows by setting $b_{h,\lambda} = |u_{h,\lambda}(x)|$ and taking the L^p norm of the first and last sides of (19). ∎

REMARK 6. – The characterization of Sobolev spaces given by Proposition 5.1, jointly with the continuous embeddings in (14), gives

$$B^{s,\Lambda}_{p,\min(p,2)} \subset H^s_{p,\Lambda} \subset B^{s,\Lambda}_{p,\max(p,2)}, \quad \text{with continuous embeddings}.$$

PROPOSITION 5.3. – *Let $\Lambda(\xi)$ be a weight function, $s \in \mathbb{R}$, $1 < p_1 < p_2 < \infty$ and $1 \leq q \leq \infty$. Then*

$$B^{s + \frac{n}{\mu_0}\left(\frac{1}{p_1} - \frac{1}{p_2}\right), \Lambda}_{p_1, q} \subset B^{s, \Lambda}_{p_2, q},$$

holds with continuous embedding.

To prove this proposition, we have to slightly modify the Nikol'skij inequalities, given in Triebel [24].

LEMMA 5.1. – *Let α be a multi-index, $1 \leq p \leq q \leq \infty$ and $H > 1$. Then there exists a constant $C_\alpha > 0$, only depending on α, p, q, n and H such that*

$$\|D^\alpha f\|_q \leq C_\alpha 2^{h \cdot \alpha + \left(\frac{1}{p} - \frac{1}{q}\right)|h|} \|f\|_p, \tag{20}$$

for every function $f \in L^p(\mathbb{R}^n)$ such that supp $\widehat{f} \subset P^{(H)}_{h,\lambda}$, $h \in \mathbb{Z}^n_+$ and $\lambda \in \mathbb{E}$.

PROOF. – For sake of simplicity, suppose that $h_j > 0$ for all j.

Let us define the function $g_h(x)$ by:

$$g_h(x) := f(2^{-h_1} x_1, \ldots, 2^{-h_n} x_n), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

From well-known properties of the Fourier transform, it follows that

$$\widehat{g_h}(\xi) = 2^{|h|} \widehat{f}(2^{h_1} \xi_1, \ldots, 2^{h_n} \xi_n), \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$$

so that supp $\widehat{g_h} \subset P^{(H)} := [-2H, 2H]^n$.

From the Nikol'skij inequalities we know there exists a constant C_α, only depending on α, p, q, n and the compact $P^{(H)}$ such that:

$$\|D^\alpha g_h\|_q \leq C_\alpha \|g_h\|_p. \tag{21}$$

But $D^\alpha g_h(x) = 2^{-h \cdot \alpha}(D^\alpha f)(2^{-h_1} x_1, \ldots, 2^{-h_n} x_n)$, whence $\|g_h\|_p = 2^{|h|\frac{1}{p}} \|f\|_p$ and $\|D^\alpha g_h\|_q = 2^{-h \cdot \alpha} 2^{|h|\frac{1}{q}} \|D^\alpha f\|_q$.

Thus inequality (20) follows from (21) by replacing the previous expressions for $\|D^\alpha g_h\|_q$ and $\|g_h\|_p$. ■

PROOF. – (of Proposition 5.3)

Let u be a distribution in $B^{s + \frac{n}{\mu_0}\left(\frac{1}{p_1} - \frac{1}{p_2}\right), \Lambda}_{p_1, q}$; then

$$\|u\|^q_{B^{s + \frac{n}{\mu_0}\left(\frac{1}{p_1} - \frac{1}{p_2}\right), \Lambda}_{p_1, q}} := \sum_{h, \lambda} \Lambda(c^{(H)}_{h,\lambda})^{\left[s + \frac{n}{\mu_0}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right]q} \|u_{h,\lambda}\|^q_{p_1} < \infty.$$

In view of Lemma 5.1 there exists a constant $C > 0$, independent of u, h and λ, such that:

$$\|u_{h,\lambda}\|_{p_2} \leq C 2^{|h|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|u_{h,\lambda}\|_{p_1}. \tag{22}$$

On the other hand (3) yields that $2^{\mu_0 h_j} \leq \left(1 + |c_{h,\lambda}^{(H)}|\right)^{\mu_0} \leq C_H \Lambda(c_{h,\lambda}^{(H)})$ and then

$$2^{|h|} \leq C_{H,n} \Lambda(c_{h,\lambda}^{(H)})^{\frac{n}{\mu_0}}, \quad h \in \mathbb{Z}_+^n, \ \lambda \in \mathbb{E}. \tag{23}$$

The above estimate, jointly with (22), implies that

$$\sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{sq} \|u_{h,\lambda}\|_{p_2}^q \leq C_1 \sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{\left[s + \frac{n}{\mu_0}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right] q} \|u_{h,\lambda}\|_{p_1}^q. \qquad \blacksquare$$

COROLLARY 5.1. – *Let $\Lambda(\xi)$ be a weight function, $s \in \mathbb{R}$, $1 < p_1 < p < p_2 < \infty$ and $\delta_1, \delta_2 > 0$. Then the following inclusions hold with continuous embedding*

$$B_{p_1, p}^{s + \frac{n}{\mu_0}\left(\frac{1}{p_1} - \frac{1}{p}\right) + \delta_1, \Lambda} \subset H_\Lambda^{s,p} \subset B_{p_2, p}^{s - \frac{n}{\mu_0}\left(\frac{1}{p} - \frac{1}{p_2}\right) - \delta_2, \Lambda}. \tag{24}$$

PROOF. – By the propositions 5.3 (with p instead of both p_2 and q, $s + \delta_1$ instead of s), 5.1 and 5.2 we obtain the continuous embeddings $B_{p_1, p}^{s + \frac{n}{\mu_0}\left(\frac{1}{p_1} - \frac{1}{p}\right) + \delta_1, \Lambda} \subset B_{p,p}^{s+\delta_1, \Lambda} \subset B_{p, \min(p,2)}^{s, \Lambda} \subset F_{p,2}^{s,\Lambda} = H_\Lambda^{s,p}$; this proves the left inclusion.

Similarly we have $H_\Lambda^{s,p} \subset B_{p, \max(p,2)}^{s,\Lambda} \subset B_{p,p}^{s-\delta_2, \Lambda} \subset B_{p_2, p}^{s - \frac{n}{\mu_0}\left(\frac{1}{p} - \frac{1}{p_2}\right) - \delta_2, \Lambda}$ which shows the right inclusion. $\blacksquare$

PROPOSITION 5.4. – *Let us consider $1 < p < \infty$, $s \in \mathbb{R}$ and $H > 1$. Then we can find a constant $M = M_{p,n,s,H} > 0$ such that for every $u \in \mathcal{S}'(\mathbb{R}^n)$:*

$$\|u_{h,\lambda}\|_\infty \leq M \|u\|_{H_\Lambda^{s,p}} \Lambda(c_{h,\lambda}^{(H)})^{-s} 2^{\frac{|h|}{p}}, \quad h \in \mathbb{Z}_+^n, \quad \lambda \in \mathbb{E}, \tag{25}$$

where $u_{h,\lambda}(x) := \varphi_{h,\lambda}(D)\, u(x)$ and $\{\varphi_{h,\lambda}\}$ is any system in $\Phi^{(H)}$.

PROOF. – From Proposition 5.2 it follows that $H_\Lambda^{s,p} \subset B_{p,\max(p,2)}^{s,\Lambda} \subset B_{p,\infty}^{s,\Lambda}$ with continuous embeddings; this just means that, given a system $\{\varphi_{h,\lambda}\} \in \Phi^{(H)}$, any $u \in H_\Lambda^{s,p}$ fulfills the following estimates:

$$\|u_{h,\lambda}\|_p \leq M' \|u\|_{H_\Lambda^{s,p}} \Lambda(c_{h,\lambda}^{(H)})^{-s}, \quad h \in \mathbb{Z}_+^n, \ \lambda \in \mathbb{E},$$

with some positive M' depending on p, H, s, the dimension n and independent of u. To get estimates (25), it suffices now to apply Lemma 5.1 with $q = \infty$, $\alpha = 0$ and $f(x) = u_{h,\lambda}(x)$.

These estimates are clearly trivial when the distribution u does not belong to $H_\Lambda^{s,p}$. $\blacksquare$

REMARK 7. – In view of inequality (23) we easily obtain from estimate (25) the following:

$$\sup_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{s-\frac{n}{\mu_0 p}} \|u_{h,\lambda}\|_\infty \leq M\|u\|_{H_\Lambda^{s,p}}, \tag{26}$$

for every $u \in \mathcal{S}'(\mathbb{R}^n)$.

Notice also that the inequality

$$\left(\sum_{h,\lambda} \|\Lambda(c_{h,\lambda}^{(H)})^{s-\frac{n}{\mu_0}\left(\frac{1}{p}-\frac{1}{p_2}\right)-\delta_2} u_{h,\lambda}\|_{p_2}^p\right)^{\frac{1}{p}} \leq M\|u\|_{H_\Lambda^{s,p}}, \tag{27}$$

arising from the right inclusion in (24), can be extended from $p_2 < \infty$ to $p_2 = \infty$ as a consequence of (26) and the arguments used in the proof of point 3) of Proposition 5.2.

More precisely, it suffices to argue on the first inequality in (19), with $s - \dfrac{n}{\mu_0 p} - \delta_2$ instead of s, $\varepsilon = \delta_2$, $q_2 = p$, $b_{h,\lambda} = \|u_{h,\lambda}\|_\infty$, jointly with estimate (26).

The following is essentially the Hilbert space version of Lemma 3.1 (see Triebel [27], Stein [20] and Lizorkin [16] for a proof in the context of a generic Hilbert space $\mathcal{H}$).

THEOREM 5.1. – *Let $m_{j,l}(\xi)$ $(j, l = 1, 2, \ldots)$ be some n times continuously differentiable functions defined in $\mathbb{R}^n \setminus A$, where $A := \{\xi \in \mathbb{R}^n : \prod_{j=1}^n \xi_j = 0\}$. Assume that there exists a constant $B > 0$ such that:*

$$|\xi^\gamma| \left(\sum_{j,l=1}^\infty |D^\gamma m_{j,l}(\xi)|^2\right)^{\frac{1}{2}} \leq B, \qquad \xi \in \mathbb{R}^n \setminus A, \tag{28}$$

for all the multi-indices $\gamma \in \mathbb{Z}_+^n$ with $\gamma_j \in \{0, 1\}$ $(j = 1, \ldots, n)$.

If $1 < p < \infty$ then there exists a positive number c, only depending on p and the dimension n, such that for all the sequences $\{f_j\}_{j=1}^\infty$ of functions $f_j(x) \in \mathcal{S}(\mathbb{R}^n)$, satisfying $f_j(x) = 0$ with exception of a finite number of j,

$$\left\| \left\{ \mathcal{F}_{\xi \to x}^{-1} \left(\sum_{l=1}^\infty m_{j,l}(\xi) \hat{f}_l(\xi) \right)(x) \right\}_j \right\|_{L^p(\ell^2)} \leq cB \|\{f_j(x)\}\|_{L^p(\ell^2)} \tag{29}$$

holds.

6. – Elementary symbols.

Through this Section X is a Banach space with norm $\|\,.\,\|$ and $\Lambda(\xi)$ is a weight function according to Definition 2.1.

DEFINITION 6.1. – *We say that a measurable function* $a(x, \xi)$ *on* $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ *belongs to the symbol class* XM^m_Λ, *where* $m \in \mathbb{R}$, *if for any multi-index* $\gamma \in \mathbb{Z}^n_+$ *there exists a positive constant* C_γ *such that the following estimates hold*

$$(30)\qquad \begin{aligned} &\prod_{j=1}^{n} (1+\xi_j^2)^{\frac{\gamma_j}{2}} \, |\partial_\xi^\gamma a(x, \xi)| \le C_\gamma \Lambda(\xi)^m, \quad x, \xi \in \mathbb{R}^n; \\ &\prod_{j=1}^{n} (1+\xi_j^2)^{\frac{\gamma_j}{2}} \, \|\partial_\xi^\gamma a(\cdot, \xi)\| \le C_\gamma \Lambda(\xi)^m, \quad \xi \in \mathbb{R}^n. \end{aligned}$$

REMARK 8. – Since $\Lambda(\xi)$ satisfies (3) and (6), setting $\gamma = 0$ in (30) we see that $a(x, \xi)$ has at most a polynomial growth in (x, ξ) and so it belongs to $\mathcal{S}'(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$.

From (30) we also obtain that, for any $\xi \in \mathbb{R}^n$, $\partial_\xi^\gamma a(., \xi) \in L^\infty(\mathbb{R}^n) \cap X$.

DEFINITION 6.2. – *We say that a measurable function* $a(x, \xi)$ *is an elementary symbol on* X *if it may be represented as follows*

$$(31)\qquad a(x, \xi) = \sum_{h \in \mathbb{Z}^n_+,\, \lambda \in \mathbb{E}} d_{h,\lambda}(x)\, \psi_{h,\lambda}(\xi),$$

where $\{d_{h,\lambda}\} = \{d_{h,\lambda}(x)\}_{h \in \mathbb{Z}^n_+, \lambda \in \mathbb{E}}$ *is a sequence of functions in* $L^\infty \cap X$, *such that for some* $M > 0$

$$|d_{h,\lambda}(x)| \le M, \quad x \in \mathbb{R}^n, \qquad \|d_{h,\lambda}\| \le M, \quad h \in \mathbb{Z}^n_+, \ \lambda \in \mathbb{E}.$$

$\{\psi_{h,\lambda}\} = \{\psi_{h,\lambda}\}_{h \in \mathbb{Z}^n_+, \lambda \in \mathbb{E}}$ *is a sequence of smooth functions satisfying the following conditions:*

1) *supp* $\psi_{h,\lambda} \subset P^{(H)}_{h,\lambda}$, *for any* $h \in \mathbb{Z}^n_+$, $\lambda \in \mathbb{E}$ *and some* $H > 1$;

2) *for any* $\alpha \in \mathbb{Z}^n_+$ *there exists a positive constant* C_α *such that*

$$|\partial^\alpha \psi_{h,\lambda}(\xi)| \le C_\alpha 2^{-h\cdot\alpha}, \qquad \textit{for any } \xi \in \mathbb{R}^n,\, h \in \mathbb{Z}^n_+,\, \lambda \in \mathbb{E}.$$

REMARK 9. – Definition 6.2 is well-posed, since for a fixed $\xi \in \mathbb{R}^n$ all but a finite number of terms in (31) are zero.

Moreover thanks to the assumptions 1) and 2) it is easy to see that an elementary symbol belongs to the symbol class XM^0_Λ.

PROPOSITION 6.1. – *Let $a(x, \xi)$ be a symbol in XM^0_Λ. Then there exists a sequence of elementary symbols $a_m(x, \xi)$, $m \in \mathbb{Z}^n$, such that*

$$a(x, \xi) = \sum_{m \in \mathbb{Z}^n} \frac{1}{(1 + |m|^2)^{2n}} a_m(x, \xi), \qquad x, \xi \in \mathbb{R}^n,$$

with absolute convergence in $L^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$.

LEMMA 6.1. – *For $\{\varphi_{h,\lambda}\} \in \Phi^{(H)}$, $H > 1$, $a(x, \xi) \in XM^0_\Lambda$ let us set*

$$a_{h,\lambda}(x, \xi) := \varphi_{h,\lambda}(\xi)\, a(x, \xi), \qquad h \in \mathbb{Z}^n_+,\ \lambda \in \mathbb{E}\,.$$

Then the following statements hold, for any $\alpha, h \in \mathbb{Z}^n_+$, $\lambda \in \mathbb{E}$ and some $C_\alpha > 0$:

$$a_{h,\lambda}(x, \xi) = 0, \qquad \text{for } x \in \mathbb{R}^n \text{ and } \xi \notin P^{(H)}_{h,\lambda};$$

$$|\partial^\alpha_\xi a_{h,\lambda}(x, \xi)| \leq C_\alpha 2^{-\alpha \cdot h}, \qquad x, \xi \in \mathbb{R}^n; \tag{32}$$

$$\|\partial^\alpha_\xi a_{h,\lambda}(.\,, \xi)\| \leq C_\alpha 2^{-\alpha \cdot h}, \qquad \xi \in \mathbb{R}^n.$$

PROOF. – Since supp $\varphi_{h,\lambda} \subset P^{(H)}_{h,\lambda}$, the first statement is obviously true. By Leibnitz formula we get for any $\alpha \in \mathbb{Z}^n_+$

$$|\partial^\alpha_\xi a_{h,\lambda}(x, \xi)| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial^\beta_\xi a(x, \xi)|\, |\partial^{\alpha-\beta}_\xi \varphi_{h,\lambda}(\xi)|, \qquad x, \xi \in \mathbb{R}^n.$$

From (30) it follows that for any $x \in \mathbb{R}^n$ and $\xi \in \operatorname{supp} \varphi_{h,\lambda}$

$$|\partial^\beta_\xi a(x, \xi)| \leq C_\beta \left(\prod_{j=1}^n (1 + \xi_j^2)^{\frac{\beta_j}{2}} \right)^{-1} \leq H^{|\beta|} 2^{-h \cdot \beta}.$$

On the other hand $|\partial^{\alpha-\beta}_\xi \varphi_{h,\lambda}(\xi)| \leq C_{\alpha-\beta} 2^{-h\cdot(\alpha-\beta)}$, $\xi \in \mathbb{R}^n$, then (32) is proved.

The last assertion follows as a repetition of the above argument, where the absolute value $|\,.\,|$ is replaced by the norm $\|\,.\,\|$ in X. ■

PROOF (of Proposition 6.1). – For $\{\varphi_{h,\lambda}\} \in \Phi^{(H)}$ and $a_{h,\lambda}(x, \xi)$ as in Lemma 6.1 we have:

$$a(x, \xi) = \sum_{h,\lambda} a_{h,\lambda}(x, \xi), \qquad x, \xi \in \mathbb{R}^n. \tag{33}$$

For every $h \in \mathbb{Z}^n_+$ and $\lambda \in \mathbb{E}$ let us set

$$b_{h,\lambda}(x, \xi) := a_{h,\lambda}(x, 2^{h_1 + \theta_{h_1}} \xi_1 + \lambda_1 c_{h_1}, \dots, 2^{h_n + \theta_{h_n}} \xi_n + \lambda_n c_{h_n}), \tag{34}$$

where θ_h and c_h are defined by (11).

For any $\alpha, h \in \mathbb{Z}_+^n$, $\lambda \in \mathbb{E}$ and some constants $C_\alpha > 0$, $K = K(H) > 1$, we can easily deduce from Lemma 6.1 the following properties of $b_{h,\lambda}(x,\xi)$:

$$b_{h,\lambda}(x,\xi) = 0, \quad \text{when } x \in \mathbb{R}^n \text{ and } \xi \notin Q_1 := \left[-\frac{K}{2}, \frac{K}{2} \right]^n;$$

$$|\partial_\xi^\alpha b_{h,\lambda}(x,\xi)| \leq C_\alpha, \qquad x, \xi \in \mathbb{R}^n; \tag{35}$$

$$\|\partial_\xi^\alpha b_{h,\lambda}(.,\xi)\| \leq C_\alpha, \qquad \xi \in \mathbb{R}^n. \tag{36}$$

Arguing now as in Coifman-Meyer [6], we can choose the constant $K > 1$ so that $Q_1 \subset [-\pi, \pi]^n$ and set for any $h \in \mathbb{Z}_+^n$ and $\lambda \in \mathbb{E}$

$$B_{h,\lambda}(x,\xi) := \sum_{m \in \mathbb{Z}^n} b_{h,\lambda}(x, \xi - 2m\pi), \qquad x, \xi \in \mathbb{R}^n. \tag{37}$$

The above function is well-defined since for every $(x,\xi) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^n$, all terms but one in the right-hand side of (37) are equal to zero.

$B_{h,\lambda}(x,\xi)$ is nothing else but the 2π-periodic function in the ξ variable obtained by extension of $b_{h,\lambda}(x,\xi)$ on each n-cube of $\mathbb{R}_\xi^n$ with sides of length 2π.

Moreover if $\phi(\xi) \in C_0^\infty(\mathbb{R}^n)$ vanishes outside $[-\pi,\pi]^n$ and is equal to 1 on Q_1, we have

$$b_{h,\lambda}(x,\xi) = \phi(\xi)\, B_{h,\lambda}(x,\xi), \qquad x, \xi \in \mathbb{R}^n. \tag{38}$$

For any fixed $x \in \mathbb{R}^n$, we can write $B_{h,\lambda}(x,\xi)$ in terms of its Fourier expansion:

$$\begin{aligned} B_{h,\lambda}(x,\xi) &= \sum_{m \in \mathbb{Z}^n} e^{im\cdot\xi} \int_{[-\pi,\pi]^n} B_{h,\lambda}(x,\eta)\, e^{-im\cdot\eta}\, đ\eta \\ &= \sum_{m \in \mathbb{Z}^n} e^{im\cdot\xi} \int_{[-\pi,\pi]^n} b_{h,\lambda}(x,\eta)\, e^{-im\cdot\eta}\, đ\eta, \end{aligned}$$

with convergence in $L^2([-\pi,\pi]^n)$ with respect to ξ, where $đ\eta = (2\pi)^{-n} d\eta$.

Integrating by parts we can write $B_{h,\lambda}(x,\xi)$ as

$$\begin{aligned} &\sum_{m \in \mathbb{Z}^n} e^{im\cdot\xi} \frac{1}{(1+|m|^2)^{2n}} (1+|m|^2)^{2n} \int_{[-\pi,\pi]^n} b_{h,\lambda}(x,\eta)\, e^{-im\cdot\eta}\, đ\eta = \\ &= \sum_{m \in \mathbb{Z}^n} e^{im\cdot\xi} \frac{1}{(1+|m|^2)^{2n}} \int_{[-\pi,\pi]^n} (I - \Delta_\eta)^{2n} b_{h,\lambda}(x,\eta)\, e^{-im\cdot\eta}\, đ\eta, \end{aligned}$$

where $\Delta_\eta := \sum_{j=1}^n \partial_{\eta_j}^2$.

For any $m \in \mathbb{Z}^n$, $h \in \mathbb{Z}_+^n$ and $\lambda \in \mathbb{E}$ let us define

$$d_{h,\lambda}^m(x) := \int_{[-\pi,\pi]^n} (I - \Delta_\eta)^{2n} b_{h,\lambda}(x,\eta)\, e^{-im\cdot\eta}\, đ\eta, \qquad x \in \mathbb{R}^n. \tag{39}$$

From (35) we obtain for any $h \in \mathbb{Z}_+^n$, $m \in \mathbb{Z}^n$, $\lambda \in \mathbb{E}$ and $C_n > 0$ depending only on the dimension n:

$$|d_{h,\lambda}^m(x)| \leqslant \int_{[-\pi,\pi]^n} |(I - \Delta_\eta)^{2n} b_{h,\lambda}(x,\eta)|\, đ\eta \leqslant C_n, \qquad x \in \mathbb{R}^n;$$

so we have that $\{d_{h,\lambda}^m(x)\}_{h,\lambda}$ is a bounded sequence in $L^\infty(\mathbb{R}^n)$.

If we look at the right-hand side in (39) as an integral of a measurable function taking its values in a Banach space X, arguing on (36) as before we have $d_{h,\lambda}^m \in X$ and $\|d_{h,\lambda}^m\| \leqslant C_n$ for all $h \in \mathbb{Z}_+^n$, $m \in \mathbb{Z}^n$ and $\lambda \in \mathbb{E}$.

At this point, we may represent $B_{h,\lambda}(x,\xi)$ as follows

$$B_{h,\lambda}(x,\xi) = \sum_{m \in \mathbb{Z}^n} \frac{1}{(1+|m|^2)^{2n}} e^{im\cdot\xi} d_{h,\lambda}^m(x).$$

Let us remark that the above expansion is convergent in $L^2([-\pi,\pi]^n)$ with respect to ξ for any $x \in \mathbb{R}^n$ and actually converges to $B_{h,\lambda}(x,\xi)$ uniformly in $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$.

In fact for every $m \in \mathbb{Z}^n$:

$$\left| \frac{1}{(1+|m|^2)^{2n}} e^{im\cdot\xi} d_{h,\lambda}^m(x) \right| \leqslant \frac{1}{(1+|m|^2)^{2n}} C_n, \qquad x, \xi \in \mathbb{R}^n.$$

Setting for any $m \in \mathbb{Z}^n$ $\phi_m(\xi) := e^{im\cdot\xi}\phi(\xi)$, from (38) we obtain:

$$b_{h,\lambda}(x,\xi) = \sum_{m \in \mathbb{Z}^n} \frac{1}{(1+|m|^2)^{2n}} \phi_m(\xi)\, d_{h,\lambda}^m(x), \qquad x, \xi \in \mathbb{R}^n, \tag{40}$$

with uniform convergence on $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$.

With the change of variables $\zeta_j = 2^{h_j + \theta_{h_j}} \xi_j + \lambda_j c_{h_j}$, $j = 1, \dots, n$, where θ_{h_j} and c_{h_j} are defined by (11), we get from (40) the following representation for $a_{h,\lambda}(x,\xi)$:

$$a_{h,\lambda}(x,\zeta) = \sum_{m \in \mathbb{Z}^n} \frac{1}{(1+|m|^2)^{2n}} d_{h,\lambda}^m(x)\, \psi_{m,h,\lambda}(\zeta), \qquad x, \zeta \in \mathbb{R}^n, \tag{41}$$

with $\psi_{m,h,\lambda}(\zeta) = \phi_m(2^{-h_1-\theta_{h_1}}(\zeta_1 - \lambda_1 c_{h_1}), \dots, 2^{-h_1-\theta_{h_1}}(\zeta_1 - \lambda_1 c_{h_1}))$. The convergence in (41) is uniform on $\mathbb{R}_x^n \times \mathbb{R}_\zeta^n$.

It is easy to check that for every $m \in \mathbb{Z}^n$ the function sequence $\{\psi_{m,h,\lambda}(\zeta)\}$ satisfies conditions 1) and 2) of Definition 6.2. More precisely the following statements hold, for every $h \in \mathbb{Z}^n_+$, $\lambda \in \mathbb{E}$ and some $C_{\alpha, m} > 0$:

$$\operatorname{supp} \psi_{m,h,\lambda} \subset P^{(H)}_{h,\lambda}, \qquad m \in \mathbb{Z}^n;$$
$$|\partial^\alpha_\xi \psi_{m,h,\lambda}(\xi)| \le C_{\alpha,m} 2^{-h\cdot\alpha}, \qquad \xi \in \mathbb{R}^n;$$

Deeply arguing on the functions $\psi_{m,h,\lambda}(\xi)$ we get for some $M_0 > 0$:

$$\sum_{h\in\mathbb{Z}^n_+,\, \lambda\in\mathbb{E}} |\psi_{m,h,\lambda}(\xi)| \le M_0, \quad \text{for every } \xi \in \mathbb{R}^n \text{ and } m \in \mathbb{Z}^n. \tag{42}$$

Replacing now in (33) the expression of $a_{h,\lambda}(x, \xi)$ given in (41) we obtain for $x, \xi \in \mathbb{R}^n$

$$a(x,\xi) = \sum_{h\in\mathbb{Z}^n_+,\, \lambda\in\mathbb{E}} \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|^2)^{2n}} d^m_{h,\lambda}(x)\, \psi_{m,h,\lambda}(\xi). \tag{43}$$

The expansion is absolutely convergent in $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$, since using (42)

$$\sum_{h\in\mathbb{Z}^n_+,\, \lambda\in\mathbb{E}} \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|^2)^{2n}} |d^m_{h,\lambda}(x)|\, |\psi_{m,h,\lambda}(\xi)| \le$$

$$C_n M_0 \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|^2)^{2n}} \le C C_n M_0.$$

So we may change the order of the two sums in (43) and conclude

$$a(x,\xi) = \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|^2)^{2n}} a_m(x,\xi), \qquad x, \xi \in \mathbb{R}^n, \tag{44}$$

where the functions $a_m(x,\xi) := \sum_{h,\lambda} d^m_{h,\lambda}(x)\, \psi_{m,h,\lambda}(\xi)$, $m \in \mathbb{Z}^n$ are elementary symbols.

Since $|a_m(x,\xi)| \le M_0 C_n$, for all $m \in \mathbb{Z}^n$, the series in (44) is absolutely convergent to $a(x,\xi)$ in $L^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$. ■

7. – Action on Sobolev spaces.

For any symbol $a(x,\xi)$ in the class XM^m_Λ, $m \in \mathbb{R}$, we can define as usual the pseudodifferential operator:

$$a(x,D)\, u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,\xi)\, \widehat{u}(\xi)\, d\xi, \qquad u \in \mathcal{S}(\mathbb{R}^n). \tag{45}$$

REMARK 10. – In view of Remark 8, the integral on the right-hand side of (45) is well-defined in classical sense and moreover $a(x,D) : \mathcal{S}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ continuously.

This section will be devoted to prove the following mapping property of pseudodifferential operators whose symbol belongs to the classes $H_\Lambda^{r,p}M_\Lambda^m$, $\Lambda(\xi)$ weight function, $r \in \mathbb{R}$, $1 < p < \infty$.

THEOREM 7.1. – *Let $\Lambda(\xi)$ be a weight function and assume moreover that there exists a number $0 < \delta < 1$ such that for some $C > 0$*

$$\Lambda(\xi+\eta) \leqslant C(\Lambda(\xi) + \Lambda(\eta) + \Lambda(\xi)^\delta \Lambda(\eta)^\delta), \qquad \xi, \eta \in \mathbb{R}^n. \tag{46}$$

Let $a(x, \xi)$ be a symbol in $H_\Lambda^{r,p}M_\Lambda^m$ with $r > \dfrac{n}{(1-\delta)\mu_0 p}$, $1 < p < \infty$ and $m \in \mathbb{R}$. Then:

$$a(x, D): H_\Lambda^{s+m,p} \to H_\Lambda^{s,p}, \tag{47}$$

continuously for every $0 \leqslant s \leqslant r$.

Before starting the proof of Theorem 7.1, let us make some useful remarks.

1) The proof of Theorem 7.1 may be restricted to the case $m = 0$. It suffices to observe that, for any $m \in \mathbb{R}$, $\Lambda(D)^m$: $H_\Lambda^{s+m,p} \to H_\Lambda^{s,p}$ continuously for all $s \in \mathbb{R}$, $1 < p < \infty$ (cf. [11]) and $a(x, D)\Lambda(D)^{-m}$ has symbol $a(x, \xi)\Lambda(\xi)^{-m} \in H_\Lambda^{r,p}M_\Lambda^0$.

So $a(x, D) = (a(x, D)\Lambda(D)^{-m})(\Lambda(D)^m)$ maps continuously $H_\Lambda^{s+m,p}$ into $H_\Lambda^{s,p}$, when $0 \leqslant s \leqslant r$.

2) In view of Proposition 6.1 and the dominated convergence theorem it comes that a pseudodifferential operator $a(x, D)$ with symbol $a(x, \xi) \in H_\Lambda^{r,p}M_\Lambda^0$ can be written for any $u(x) \in \mathcal{S}(\mathbb{R}^n)$:

$$a(x, D)u(x) = \sum_{m \in \mathbb{Z}^n} \frac{1}{(1+|m|^2)^{2n}} a_m(x, D)\, u(x), \tag{48}$$

where $a_m(x, \xi)$, $m \in \mathbb{Z}^n$ are elementary symbols and the convergence is given in $\mathcal{S}'(\mathbb{R}_x^n)$.

At first we prove Theorem 7.1 for $a(x, \xi) = \sum_{h,\lambda} d_{h,\lambda}(x)\, \psi_{h,\lambda}(\xi)$, elementary symbol.

3) For every $u(x) \in \mathcal{S}(\mathbb{R}^n)$, we can show that for any elementary symbol $a(x, \xi)$ we have:

$$a(x, D)\, u(x) = \sum_{h,\lambda} d_{h,\lambda}(x)\, u_{h,\lambda}(x), \tag{49}$$

with absolute convergence in $L^\infty(\mathbb{R}^n)$.

In fact since $\sum_{h,\lambda} |d_{h,\lambda}(x)|\,|\psi_{h,\lambda}(\xi)| \leqslant MCN_1(H)$ for any $x, \xi \in \mathbb{R}^n$ (here $N_1(H)$ is the number of terms in the sum which do not vanish), it follows that for $u(x) \in \mathcal{S}(\mathbb{R}^n)$

$$a(x, D)\, u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \sum_{h,\lambda} d_{h,\lambda}(x)\, \psi_{h,\lambda}(\xi)\, \widehat{u}(\xi)\, d\xi =$$

$$(2\pi)^{-n} \sum_{h,\lambda} d_{h,\lambda}(x) \int e^{ix\cdot\xi} \psi_{h,\lambda}(\xi)\, \widehat{u}(\xi)\, d\xi = \sum_{h,\lambda} d_{h,\lambda}(x)\, u_{h,\lambda}(x).$$

Moreover for any $h \in \mathbb{Z}_+^n$, $\lambda \in \mathbb{E}$ and $T > 0$ we also have

$$\|u_{h,\lambda}\|_\infty \leqslant C \int |\psi_{h,\lambda}(\xi)\, \widehat{u}(\xi)|\, d\xi \leqslant C \int_{P_{h,\lambda}^{(H)}} (1 + |\xi|^2)^T\, |\psi_{h,\lambda}(\xi)|\, \frac{|\widehat{u}(\xi)|}{(1 + |\xi|^2)^T}\, d\xi \leqslant$$

$$\frac{C}{\left(1 + \frac{1}{H^2} \sum_{j=1}^{n} \chi_{h_j} 2^{2h_j}\right)^T} \int (1 + |\xi|^2)^T\, |\widehat{u}(\xi)|\, d\xi \leqslant C_T\, a_{h_1} \ldots a_{h_n},$$

with a suitable $C > 0$, where $C_T := \int (1 + |\xi|^2)^T\, |\widehat{u}(\xi)|\, d\xi$ and

$$\chi_h := \begin{cases} 0, & h = 0 \\ 1, & h > 0, \end{cases} \qquad a_h := \begin{cases} 1, & h = 0, \\ \dfrac{1}{\frac{1}{H^2} 2^{\frac{2hM}{n}}}, & h > 0. \end{cases} \tag{50}$$

Thus $\sum_{h,\lambda} \|d_{h,\lambda}\|_\infty \|u_{h,\lambda}\|_\infty \leqslant M 2^n C_T \sum_{h_1=0}^{\infty} a_{h_1} \ldots \sum_{h_n=0}^{\infty} a_{h_n} < \infty$, which shows that (49) is absolutely convergent in $L^\infty(\mathbb{R}^n)$.

4) By a similar argument, for $\{\varphi_{h,\lambda}(\xi)\} \in \Phi^{(H)}$ and $v \in \mathcal{S}'(\mathbb{R}^n)$ it holds:

$$v = \sum_{h,\lambda} \varphi_{h,\lambda}(D)\, v = \sum_{h,\lambda} v_{h,\lambda}, \qquad \text{with convergence in } \mathcal{S}'(\mathbb{R}^n). \tag{51}$$

5) For $\{\phi_{k,\varepsilon}(\xi)\}_{k,\varepsilon} \in \Phi^{(K)}$, $K > 1$, we may apply the decomposition (51) to each term $d_{h,\lambda}(x) \in H_\Lambda^{r,p}$ in (49). Then, by setting $d_{h,\lambda}^{k,\varepsilon}(x) := \phi_{k,\varepsilon}(D)\, d_{h,\lambda}(x)$, we get

$$a(x, D)\, u(x) = \sum_{h,\lambda} \sum_{k,\varepsilon} d_{h,\lambda}^{k,\varepsilon}(x)\, u_{h,\lambda}(x). \tag{52}$$

For $r > \frac{n}{\mu_0 p}$ the expansion (52) is absolutely convergent in $L^\infty(\mathbb{R}_x^n)$. Since $\{d_{h,\lambda}\}$ is bounded in $H_\Lambda^{r,p}$, we can in fact find, in view of Proposition 5.4, a number $M > 0$ such that for any $h, k \in \mathbb{Z}_+^n$ and $\lambda \in \mathbb{E}$: $\|d_{h,\lambda}^{k,\varepsilon}\|_\infty \leqslant M\Lambda(c^{(K),k,\varepsilon})^{-(r - \frac{n}{\mu_0 p})}$.

Thus for a_{h_j} defined by (50), $j = 1, \ldots, n$, and provided $r - \frac{n}{\mu_0 p} > 0$ we can show that $\sum_{h,\lambda}\sum_{k,\varepsilon}\|d_{h,\lambda}^{k,\varepsilon}\|_\infty\|u_{h,\lambda}\|_\infty$ is bounded by:

$$M2^n\sum_{k,\varepsilon}\Lambda(c_{k,\varepsilon}^{(K)})^{-(r-\frac{n}{\mu_0 p})}\sum_{h_1=0}^{\infty}a_{h_1}\dots\sum_{h_n=0}^{\infty}a_{h_n}<\infty.$$

6) The absolute convergence of the series in the right hand-side of (52) makes possible to change its terms according to a useful order.

For this purpose, let us introduce some preliminary notations.

Let $N_0\in\mathbb{N}$ be given and $h\in\mathbb{Z}_+$ arbitrary; we set

$$E_{1,h}^{(N_0)}:=\begin{cases}\emptyset, & h\leqslant N_0;\\ \mathbb{Z}_+\cap[0,h-N_0[, & h>N_0;\end{cases}\tag{53}$$

$$E_{2,h}^{(N_0)}:=\mathbb{Z}_+\cap[h-N_0,h+N_0[;\tag{54}$$

$$E_{3,h}^{(N_0)}:=\mathbb{Z}_+\cap[h+N_0,\infty[.\tag{55}$$

Moreover, we denote by B^A the set of all functions $\sigma:A\to B$ with $A:=\{1,2,\dots,n\}$ and $B:=\{1,2,3\}$.

For any $h=(h_1,h_2,\dots,h_n)\in\mathbb{Z}_+^n$ and $\sigma\in B^A$ we set:

$$E_{\sigma,h}^{(N_0)}:=E_{\sigma(1),h_1}^{(N_0)}\times E_{\sigma(2),h_2}^{(N_0)}\times\dots\times E_{\sigma(n),h_n}^{(N_0)}.\tag{56}$$

According to notations (53)-(56), we can write (52) as follows

$$a(x,D)\,u(x)=\sum_{\sigma\in B^A}\sum_{h,\lambda}\sum_{k\in E_{\sigma,h}^{(N_0)},\,\varepsilon\in\mathbb{E}}d_{h,\lambda}^{k,\varepsilon}(x)\,u_{h,\lambda}(x).\tag{57}$$

7) For every $h,k\in\mathbb{Z}_+^n$ and $\lambda,\varepsilon\in\mathbb{E}$: $\operatorname{supp}(\widehat{d_{h,\lambda}^{k,\varepsilon}u_{h,\lambda}})\subset P_{k,\varepsilon}^{(K)}+P_{h,\lambda}^{(H)}$, for some $K,H>1$.

Given $r,s\in\mathbb{Z}_+$ and $\sigma,\delta\in\{-1,1\}$, according to the notations introduced in (9) we can say that the n-cubes $P_{h,\lambda}^{(H)}$ and $P_{k,\varepsilon}^{(K)}$ are obtained as superposition of n intervals of the type $L_{r,\sigma}^{(H)}$ and $L_{s,\delta}^{(K)}$. Therefore we may restrict to argue on the sum $L_{r,\sigma}^{(H)}+L_{s,\delta}^{(K)}$.

Let us then prove the following technical lemma.

LEMMA 7.1. – *Let us consider $r,s\in\mathbb{Z}_+$, $\sigma,\delta\in\{-1,1\}$, H, K greater than* 1. *For any N_0 positive integer, $N_0>\log_2(2HK)$, we can always find two positive constants T, M such that $T>H+K$, $\frac{1}{T}<\min\left\{\frac{1}{K}-\frac{2H}{2^{N_0}},\frac{1}{H}-\frac{2K}{2^{N_0}}\right\}$ and $M>2^{N_0+1}K+2H$, which fulfill the following statements:*

(*a*) *if $s\in E_{1,r}^{(N_0)}$ and $r>N_0$ then*

$$L_{r,\sigma}^{(H)}+L_{s,\delta}^{(K)}\subset\left\{\theta\in\mathbb{R}:\frac{2^r\eta_r}{T}\leqslant\sigma\theta\leqslant T2^{r+1}\right\}=:L_{r,\sigma}^{(T)};\tag{58}$$

(b) if $s \in E_{2,r}^{(N_0)}$ *then*

(59) $$L_{r,\sigma}^{(H)} + L_{s,\delta}^{(K)} \subset \{\theta \in \mathbb{R} : |\theta| \leq M2^r\} =: [-M2^r, M2^r];$$

(c) if $s \in E_{3,r}^{(N_0)}$ *then*

(60) $$L_{r,\sigma}^{(H)} + L_{s,\delta}^{(K)} \subset \left\{\theta \in \mathbb{R} : \frac{2^s \eta_s}{T} \leq \delta\theta \leq T2^{s+1}\right\} =: L_{s,\delta}^{(T)}.$$

Here $\eta_h = -1$ *if* $h = 0$ *and* $\eta_h = 1$ *if* $h > 0$.

PROOF. – At a first glance, we have to distinguish four cases:

$(i) : (\sigma, \delta) = (1, 1)$; $(ii) : (\sigma, \delta) = (-1, 1)$;
$(iii) : (\sigma, \delta) = (-1, -1)$; $(iv) : (\sigma, \delta) = (1, -1)$.

It is easy see that $\theta \in L_{r,\sigma}^{(H)} + L_{s,\delta}^{(K)}$ for $(\sigma, \delta) = (1, 1)$ [or $(-1, 1)$] if and only if $-\theta \in L_{r,\sigma}^{(H)} + L_{s,\delta}^{(K)}$ for $(\sigma, \delta) = (-1, -1)$ [or $(1, -1)$]. So we are actually reduced to argue on the cases (i) and (ii).

For the sub-case $(a\text{-}i)$ suppose firstly that $s = 0$. It easily follows:

$$L_{r,1}^{(H)} + L_{0,1}^{(K)} \subset \left\{\theta \in \mathbb{R} : -\frac{1}{K} + \frac{1}{H}2^r \leq \theta \leq 2K + H2^{r+1}\right\}.$$

From $r > N_0$ it comes that

$$-\frac{1}{K} + \frac{1}{H}2^r = \left(-\frac{1}{K2^r} + \frac{1}{H}\right)2^r > \left(-\frac{1}{K2^{N_0}} + \frac{1}{H}\right)2^r;$$

on the other hand, $2K + H2^{r+1} < (K + H)2^{r+1}$.

The inclusion (58) then follows by choosing a constant T with the required properties; let us notice in particular that $N_0 > \log_2(2HK)$ yields $-\frac{2K}{2^{N_0}} + \frac{1}{H} > 0$ and then we can always find T such that $\frac{1}{T} < -\frac{2K}{2^{N_0}} + \frac{1}{H} < -\frac{1}{K2^{N_0}} + \frac{1}{H}$.

We get $L_{r,1}^{(H)} + L_{s,1}^{(K)} \subset \left\{\theta \in \mathbb{R} : \frac{1}{K}2^s + \frac{1}{H}2^r \leq \theta \leq K2^{s+1} + H2^{r+1}\right\}$ when $s > 0$; then (58) easily follows with the same T before considered, since $s < r - N_0$.

Let us assume now $s > 0$ in the sub-case $(a\text{-}ii)$, then we have

$$L_{r,-1}^{(H)} + L_{s,1}^{(K)} \subset \left\{\theta \in \mathbb{R} : \frac{1}{K}2^s - H2^{r+1} \leq \theta \leq K2^{s+1} - \frac{1}{H}2^r\right\}.$$

Inclusion (58) follows by observing that for $s < r - N_0$: $K2^{s+1} - \frac{1}{H}2^r < \left(\frac{2K}{2^{N_0}} - \frac{1}{H}\right)2^r < -\frac{1}{T}2^r$ and $\frac{1}{K}2^s - H2^{r+1} > -(H + K)2^{r+1} > -T2^{r+1}$, with the same T of case (a-i). By means of similar computations the statement follows also for $s=0$.

In the case (b-i) we can see that for $s=0$:

$$L_{r,1}^{(H)}+L_{0,1}^{(K)}\subset\left[-\left(\frac{1}{H}+\frac{1}{K}\right)2^r,(H+K)2^{r+1}\right],$$

while for $s>0$ from $r-N_0\leqslant s<r+N_0$ we get:

$$L_{r,1}^{(H)}+L_{s,1}^{(K)}\subset\left[-\left(\frac{1}{H}-\frac{1}{2^{N_0}K}\right)2^r,\ (2^{N_0}K+H)2^{r+1}\right].$$

In the case (b-ii), for $s=0$ we obtain

$$L_{r,-1}^{(H)}+L_{0,1}^{(K)}\subset\left[-\left(H+\frac{1}{K}\right)2^{r+1},\ \left(2K+\frac{1}{H}\right)2^r\right];$$

for $s>0$ it follows

$$L_{r,-1}^{(H)}+L_{s,1}^{(K)}\subset\left[-\left(H-\frac{1}{2^{N_0+1}K}\right)2^{r+1},\ \left(2^{N_0+1}K+\frac{1}{H}\right)2^r\right];$$

let us notice that $\frac{1}{H}-\frac{1}{2^{N_0}K}$ and $H-\frac{1}{2^{N_0+1}K}$ are positive as $N_0>\log_2(2HK)$.

Thus inclusion (59) holds in both cases (b-i) and (b-ii), for $M>2^{N_0+1}K+2H$.

Finally, inclusion (60) easily follows by observing that $s\geqslant r+N_0$ if and only if $r\leqslant s-N_0$ and then arguing as in (a) with r and s, σ and δ, H and K interchanged.

Let us remark, at the end, that we may always find $M=T$ satisfying the inclusions (58)-(60). ■

Coming back to (57), it follows from Lemma 7.1 that the support of $\widehat{d_{h,\lambda}^{k,\varepsilon}u_{h,\lambda}}$ is contained in the product of n real intervals of type (58)-(60).

This suggests to split B^A in the following way:

$$\begin{aligned}
C_1 &:= \left\{\sigma\in B^A\colon \sigma(A)=\{1\}\right\};\\
C_2 &:= \left\{\sigma\in B^A\colon \sigma(A)=\{2\}\right\};\\
C_3 &:= \left\{\sigma\in B^A\colon \sigma(A)=\{3\}\right\};\\
C_4 &:= \left\{\sigma\in B^A\colon \sigma(A)=\{1,2\}\right\};\\
C_5 &:= \left\{\sigma\in B^A\colon \sigma(A)=\{1,3\}\right\};\\
C_6 &:= \left\{\sigma\in B^A\colon \sigma(A)=\{2,3\}\right\};\\
C_7 &:= \left\{\sigma\in B^A\colon \sigma(A)=\{1,2,3\}\right\}.
\end{aligned}$$

The sets C_1, C_2 and C_3 reduce to a single constant function σ, while C_4-C_7 contain several functions, for any dimension $n \geqslant 2$.

For any elementary symbol in $H_\lambda^{r,p} M_\lambda^0$ we can write:

$$a(x, D)\, u(x) = \sum_{j=1}^{7} T_j u(x),$$

where for $j = 1, 2, \dots, 7$:

$$T_j u(x) := \sum_{\sigma \in C_j} \sum_{h, \lambda} \sum_{k \in E_{\sigma, h}^{(N_0)},\, \varepsilon \in \mathbb{E}} d_{h,\lambda}^{k,\varepsilon}(x)\, u_{h,\lambda}(x), \qquad u \in \mathcal{S}(\mathbb{R}^n).$$

In the following Propositions 7.1-7.3 we assume $a(x, \xi)$ to be an elementary symbol in $H_\Lambda^{r,p} M_\Lambda^0$, with $1 < p < \infty$, $\Lambda(\xi)$ weight function and $r > \dfrac{n}{\mu_0 p}$

PROPOSITION 7.1.

$$T_1 \colon H_\Lambda^{s,p} \to H_\Lambda^{s,p}, \qquad \text{continuously for every } s \in \mathbb{R}.$$

In order to show the previous result, we will use a consequence of the Nikol'skij type representation for Besov and Triebel spaces (see Triebel [28] Theorem 2.1/1); namely

LEMMA 7.2. – *Let us consider* $u = \sum_{h,\lambda} u_{h,\lambda}$, *with convergence in* $\mathcal{S}'(\mathbb{R}^n)$ *and assume that supp* $\widehat{u}_{h,\lambda} \subset P_{h,\lambda}^{(H)}$, *for every* $h \in \mathbb{Z}_+^n$, $\lambda \in \mathbb{E}$ *and some constant* $H > 1$.

Then for any $s \in \mathbb{R}$ *and* $1 < p < \infty$ *there exists a constant* $C = C_{s,p} > 0$ *such that:*

$$\|u\|_{H_\Lambda^{s,p}} \leqslant C \left\| \left(\sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{2s}\, |u_{h,\lambda}|^2 \right)^{\frac{1}{2}} \right\|_p . \tag{61}$$

PROOF (of Proposition 7.1.). – Assuming $N_0 > \log_2(2HK)$, from Lemma 7.1 it follows that for some constant $T > 1$ we have supp $\widehat{d_{h,\lambda}^{k,\varepsilon} u_{h,\lambda}} \subset P_{h,\lambda}^{(T)}$, for any $h, k \in \mathbb{Z}_+^n$, with $k_j < h_j - N_0$ $(j = 1, \dots, n)$, and any $\lambda, \varepsilon \in \mathbb{E}$.

In view of Lemma 7.2 for every $s \in \mathbb{R}$ and $1 < p < \infty$ we get:

$$\|T_1 u\|_{H_\Lambda^{s,p}} \leqslant C \left\| \left(\sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(T)})^{2s}\, |u_{h,\lambda}(x)|^2 \left| \sum_{\substack{k \in E_{1,h}^{(N_0)} \\ \varepsilon \in \mathbb{E}}} d_{h,\lambda}^{k,\varepsilon}(x) \right|^2 \right)^{\frac{1}{2}} \right\|_p ,$$

where $E_{1,h}^{(N_0)} := \prod_{j=1}^{n} E_{1,h_j}^{(N_0)}$.

Since the sequence $\{d_{h,\lambda}\}_{h,\lambda}$ is bounded in $H_\Lambda^{r,p}$ and $r > \dfrac{n}{\mu_0 p}$, from Proposition 5.4 and Remark 7 there exists a positive constant M, depending only on

r, p, μ_0, K, n, such that for any $x \in \mathbb{R}^n$,

$$\left| \sum_{\substack{k \in E_{1,h}^{(N_0)} \\ \varepsilon \in \mathbb{E}}} d_{h,\lambda}^{k,\varepsilon}(x) \right| \leqslant M \left(\sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^{-\left(r - \frac{n}{\mu_0 p}\right)} \right) \sup_{h,\lambda} \|d_{h,\lambda}\|_{H_\Lambda^{r,p}}.$$

On the other hand, it could be shown (see Triebel [27]) that for two arbitrary numbers $H, T > 1$ there exists a constant $C > 0$ independent of h and λ such that:

$$\frac{1}{C} \Lambda(c_{h,\lambda}^{(H)}) \leqslant \Lambda(c_{h,\lambda}^{(T)}) \leqslant C \Lambda(c_{h,\lambda}^{(H)}), \qquad h \in \mathbb{Z}_+^n, \lambda \in \mathbb{E}, \tag{63}$$

By the estimates (62), (63) and Proposition 5.1, we get for any $u \in \mathcal{S}(\mathbb{R}^n)$

$$\|T_1 u\|_{H_\Lambda^{r,p}} \leqslant M' \sup_{h,\lambda} \|d_{h,\lambda}\|_{H_\Lambda^{r,p}} \left\| \left(\sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{2s} |u_{h,\lambda}(x)|^2 \right)^{\frac{1}{2}} \right\|_p \leqslant$$

$$M'' \sup_{h,\lambda} \|d_{h,\lambda}\|_{H_\Lambda^{r,p}} \|u\|_{H_\Lambda^{s,p}},$$

M', M'' depending only on r, s, p, μ_0, n, H and K.

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_\Lambda^{s,p}$ for any real s and $1 < p < \infty$, the prof is concluded. ■

REMARK 11. – Similarly to (63), it may be also proved that $\frac{1}{C} < \frac{\Lambda(c_{h,\lambda}^{(H)})}{\Lambda(c_{k,\varepsilon}^{(H)})} < C$ when $|h - k| \leqslant A$ (see [27]), for some positive constants C and A, independent of $h, k \in \mathbb{Z}_+^n$ and $\lambda, \varepsilon \in \mathbb{E}$.

PROPOSITION 7.2.

$$T_2 \colon H_\Lambda^{s,p} \to H_\Lambda^{s + r - \frac{n}{\mu_0 p} - \theta, p},$$

continuously for every $s > -r + \frac{n}{\mu_0 p}$ *and* $0 < \theta < s + r - \frac{n}{\mu_0 p}$.

In order to prove Proposition 7.2 and moreover the continuity of the terms T_j, $3 \leqslant j \leqslant 7$, we need the following

LEMMA 7.3. – *Let us consider* $u = \sum_{h,\lambda} u_{h,\lambda}$, *with convergence in* $\mathcal{S}'(\mathbb{R}^n)$. *We assume moreover that there exists a constant* $H > 1$ *such that for any* $h \in \mathbb{Z}_+^n$ *and* $\lambda \in \mathbb{E}$:

$$\operatorname{supp} \widehat{u}_{h,\lambda} \subset J_{h_1,\lambda_1}^{(H)} \times \ldots \times J_{h_n,\lambda_n}^{(H)}$$

where $J_{h,\lambda}^{(H)}$ *is either* $L_{h,\lambda}^{(H)}$ *defined in* (9) *or* $[-H2^{h+1}, H2^{h+1}]$.

Then for every $s \geqslant 0$, $\gamma > 0$ *and* $1 < p < \infty$ *there exists a constant* $C =$

$C_{s,\gamma,p} > 0$ *such that:*

$$\|u\|_{H^{s,p}_\Lambda} \leqslant C \left\| \left(\sum_{h,\lambda} \Lambda(c^{(H)}_{h,\lambda})^{2s} 2^{2\gamma\chi(h)\cdot h} |u_{h,\lambda}|^2 \right)^{\frac{1}{2}} \right\|_p, \tag{64}$$

where $\chi(h) := (\chi(h_1), \dots, \chi(h_n))$ *and*

$$\chi(h_j) := \begin{cases} 1, & \textit{if } J^{(H)}_{h_j,\lambda_j} = [-H2^{h_j+1}, H2^{h_j+1}], \\ 0, & \textit{otherwise}. \end{cases}$$

PROOF. – For any $\{\varphi_{k,\varepsilon}\} \in \Phi^{(K)}$, $K > 1$, arguing similarly to the proof of Proposition 4.1, we see that for $N_0 := \log_2(2HK)$ the supports of $\widehat{u}_{h,\lambda}$ and $\varphi_{k,\varepsilon}$ are disjoint when at least one index $1 \leqslant j \leqslant n$ satisfies one of the following assumptions,

$$h_j < k_j - N_0 \ \text{ or } \ h_j > k_j + N_0, \qquad \text{if } J^{(H)}_{h_j,\lambda_j} = L^{(H)}_{h_j,\lambda_j}$$

or

$$h_j < k_j - N_0, \qquad \text{if } J^{(H)}_{h_j,\lambda_j} = [-H2^{h_j+1}, H2^{h_j+1}],$$

whatever are $\lambda, \varepsilon \in \mathbb{E}$.

Hereafter we suppose that $J^{(H)}_{h_j,\lambda_j} = [-H2^{h_j+1}, H2^{h_j+1}]$ for $j = 1, 2, \dots, n_1$ and $J^{(H)}_{h_j,\lambda_j} = L^{(H)}_{h_j,\lambda_j}$ for the remaining indices $j = n_1 + 1, \dots, n$ ($1 \leqslant n_1 \leqslant n$), without any loss of generality.

From the above arguments it follows that:

$$\varphi_{k,\varepsilon}(D)\, u = \sum_{\substack{h \in E_k^{(N_0),\, n_1} \\ \lambda \in \mathbb{E}}} \varphi_{k,\varepsilon}(D) u_{h,\lambda},$$

where, for sake of simplicity, for any $k \in \mathbb{Z}^n_+$, we set:

$$E_k^{(N_0),\, n_1} := \left\{ h \in \mathbb{Z}^n_+ : \begin{array}{ll} h_j \geqslant k_j - N_0, & j = 1, \dots, n_1 \\ k_j - N_0 \leqslant h_j \leqslant k_j + N_0, & j = n_1 + 1, \dots, n \end{array} \right\}.$$

By the characterization of $H^{s,p}_\Lambda$ given by Proposition 5.1 there exists a positive $C = C_{s,p}$ such that

$$\|u\|_{H^{s,p}_\Lambda} \leqslant C \left\| \left(\sum_{k,\varepsilon} \Lambda(c^{(K)}_{k,\varepsilon})^{2s} \left| \sum_{\substack{h \in E_k^{(N_0),\, n_1} \\ \lambda \in \mathbb{E}}} \varphi_{k,\varepsilon}(D)\, u_{h,\lambda} \right|^2 \right)^{\frac{1}{2}} \right\|_p = \tag{65}$$

$$C \left\| \left(\sum_{k,\varepsilon} \Lambda(c^{(K)}_{k,\varepsilon})^{2s} \left| \sum_{\substack{t \in E^{(N_0),\, n_1} \\ \lambda \in \mathbb{E}}} \varphi_{k,\varepsilon}(D)\, u_{k+t,\lambda} \right|^2 \right)^{\frac{1}{2}} \right\|_p,$$

where $t = h - k$ and

$$E^{(N_0),\,n_1} := \left\{ t \in \mathbb{Z}^n : \begin{array}{ll} t_j \geqslant -N_0, & j = 1, \dots, n_1 \\ -N_0 \leqslant t_j \leqslant N_0, & j = n_1 + 1, \dots, n \end{array} \right\},$$

agreeing that $u_{k+t,\lambda} \equiv 0$, when $k_j + t_j < 0$ for some $1 \leqslant j \leqslant n$.

By the triangular inequality in $L^p(\ell^2)$ we obtain

$$\left\| \left(\sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^{2s} \left| \sum_{\substack{t \in E^{(N_0),\,n_1} \\ \lambda \in \mathbb{E}}} \varphi_{k,\varepsilon}(D)\, u_{k+t,\lambda} \right|^2 \right)^{\frac{1}{2}} \right\|_p \leqslant \tag{66}$$

$$\sum_{t \in E^{(N_0),\,n_1}} \left\| \left\{ \varphi_{k,\varepsilon}(D) \left(\sum_{\lambda} \Lambda(c_{k,\varepsilon}^{(K)})^s u_{k+t,\lambda} \right) \right\}_{k,\varepsilon} \right\|_{L^p(\ell^2)}.$$

On the other hand the system $\{\varphi_{k,\varepsilon}\}$ satisfies the hypothesis of Theorem 5.1, assuming that $m_{j,l} = 0$ when $j \neq l$.

Then in view of (29) we may find a constant $C' = C'_{p,n} > 0$ such that:

$$\left\| \left\{ \varphi_{k,\varepsilon}(D) \left(\sum_{\lambda} \Lambda(c_{k,\varepsilon}^{(K)})^s u_{k+t,\lambda} \right) \right\}_{k,\varepsilon} \right\|_{L^p(\ell^2)} \leqslant \tag{67}$$

$$C' \left\| \left\{ \sum_{\lambda} \Lambda(c_{k,\varepsilon}^{(K)})^s u_{k+t,\lambda} \right\}_{k,\varepsilon} \right\|_{L^p(\ell^2)} = C' \left\| \left(\sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^{2s} \left| \sum_{\lambda} u_{k+t,\lambda} \right|^2 \right)^{\frac{1}{2}} \right\|_p \leqslant$$

$$C' c_n \left\| \left(\sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^{2s} \sum_{\lambda} |u_{k+t,\lambda}|^2 \right)^{\frac{1}{2}} \right\|_p,$$

for any $t \in E^{(N_0),\,n_1}$ and a suitable $c_n > 0$.

Thanks to the inequalities (65), (66) and (67) we obtain then

$$\|u\|_{H^{s,p}_\Lambda} \leqslant CC' c_n \sum_{t \in E^{(N_0),\,n_1}} \left\| \left(\sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^{2s} \sum_{\lambda} |u_{k+t,\lambda}|^2 \right)^{\frac{1}{2}} \right\|_p. \tag{68}$$

Since we have considered $s \geqslant 0$, using now the estimates (63), the assumption 3 in Definition 2.1 and Remark 11, we may find a constant $C'' = C''_{s,n,H,K} > 0$ such that: $\Lambda(c_{k,\varepsilon}^{(K)})^{2s} \leqslant C'' \Lambda(c_{k+t,\lambda}^{(H)})^{2s}$, for all $k \in \mathbb{Z}^n_+$, $t \in E^{(N_0),\,n_1}$, $\lambda, \varepsilon \in \mathbb{E}$.

Thus from (68) it follows:

$$\|u\|_{H^{s,p}_\Lambda} \leqslant CC' c'_n \sum_{t \in E^{(N_0),\,n_1}} \left\| \left(\sum_{k,\lambda} \Lambda(c_{k+t,\lambda}^{(H)})^{2s} |u_{k+t,\lambda}|^2 \right)^{\frac{1}{2}} \right\|_p. \tag{69}$$

Let us multiply any term of the sum in t, in the right-hand side of (69), by $2^{\gamma t_j}$

and $2^{-\gamma t_j}$ as $j=1,\ldots,n_1$; for $t'=(t_1,\ldots,t_{n_1})$ and $k'=(k_1,\ldots,k_{n_1})$ we have

$$\sum_{t\in E^{(N_0),n_1}}\left\|\left(\sum_{k,\lambda}\Lambda(c^{(H)}_{k+t,\lambda})^{2s}\,|u_{k+t,\lambda}|^2\right)^{\frac12}\right\|_p\leqslant \tag{70}$$

$$\leqslant\sum_{t\in E^{(N_0),n_1}}2^{-\gamma|t'|}\left\|\left(\sum_{k,\lambda}\Lambda(c^{(H)}_{k+t,\lambda})^{2s}2^{2\gamma|t'+k'|}\,|u_{k+t,\lambda}|^2\right)^{\frac12}\right\|_p.$$

By observing that $\sum\limits_{\substack{-N_0\leqslant t_j\leqslant N_0\\ j=n_1+1,\ldots,n}}\sum\limits_{\substack{t_j\geqslant N_0\\ j=1,\ldots,n_1}}2^{-\gamma|t'|}\leqslant C'''_{\gamma,N_0}$ as $\gamma>0$, we get the statement. ■

REMARK 12. – Under the same assumptions of Lemma 7.3, by means of the inequalities (23), we immediately deduce from estimate (64) the following:

$$\|u\|_{H^{s,p}_\Lambda}\leqslant C\left\|\left(\sum_{h,\lambda}\Lambda(c^{(H)}_{h,\lambda})^{2\left(s+\frac{\gamma n_1}{\mu_0}\right)}\,|u_{h,\lambda}|^2\right)^{\frac12}\right\|_p,$$

where n_1 has the same meaning as in the proof of Lemma 7.3.

PROOF (of Proposition 7.2). – For $E^{(N_0)}_{2,h}:=\prod\limits_{j=1}^{n}E^{(N_0)}_{2,h_j}$ let us set $U_{h,\lambda}(x):=\sum\limits_{\substack{k\in E^{(N_0)}_{2,h}\\ \varepsilon\in\mathbb{E}}}d^{k,\varepsilon}_{h,\lambda}(x)\,u_{h,\lambda}(x)$. It follows from Lemma 7.1 that $T_2u(x)=\sum\limits_{h,\lambda}U_{h,\lambda}(x)$ fulfills the assumption of Lemma 7.3.

Since $s+r-\dfrac{n}{\mu_0 p}-\theta>0$, we may estimate the $H_\Lambda^{s+r-\frac{n}{\mu_0 p}-\theta,p}$-norm of $T_2u(x)$ by means of (64) with $\gamma=\dfrac{\mu_0\theta}{n}$; using moreover Remark 12, we obtain for $C=C(r,s,p,\mu_0,n)>0$

$$\|T_2u\|_{H_\Lambda^{s+r-\frac{n}{\mu_0p}-\theta,p}}\leqslant C\left\|\left(\sum_{h,\lambda}\Lambda(c^{(T)}_{h,\lambda})^{2\left(s+r-\frac{n}{\mu_0p}\right)}\,|U_{h,\lambda}|^2\right)^{\frac12}\right\|_p. \tag{71}$$

But from Proposition 5.4, estimates (26) and the boundedness of the sequence $\{d_{h,\lambda}\}$ in $H^{r,p}_\Lambda$, we may estimate $U_{h,\lambda}(x)$ as follows:

$$|U_{h,\lambda}(x)|\leqslant|u_{h,\lambda}(x)|\sum_{\substack{k\in E^{(N_0)}_{2,h}\\ \varepsilon\in\mathbb{E}}}|d^{k,\varepsilon}_{h,\lambda}(x)|\leqslant \tag{72}$$

$$M\,\sup\nolimits_{h,\lambda}\|d_{h,\lambda}\|_{H^{r,p}_\Lambda}\sum_{\substack{k\in E^{(N_0)}_{2,h}\\ \varepsilon\in\mathbb{E}}}\Lambda(c^{(K)}_{k,\varepsilon})^{-\left(r-\frac{n}{\mu_0p}\right)}\,|u_{h,\lambda}(x)|\leqslant$$

$$M'\sup_{h,\lambda}\|d_{h,\lambda}\|_{H^{r,p}_\Lambda}\Lambda(c^{(H)}_{h,\lambda})^{-\left(r-\frac{n}{\mu_0p}\right)}\,|u_{h,\lambda}(x)|,$$

where we also used estimates (63) and Remark 11 about the weight function

$\Lambda(\xi)$ to get the last inequality and the constants M and M' depend only on H, K, r, s, p, μ_0, θ, n.

Now the statement readily follows from (71) and (72). ∎

REMARK 13. – For $r - \frac{n}{\mu_0 p} - \theta > 0$, $H_\Lambda^{s+r-\frac{n}{\mu_0 p}-\theta, p} \subset H_\Lambda^{s,p}$ with continuous embedding. Thus for $0 < \theta < \min\left\{r - \frac{n}{\mu_0 p}, s + r - \frac{n}{\mu_0 p}\right\}$, Proposition 7.2 yields that T_2 continuously maps $H_\Lambda^{s,p}$ into itself.

PROPOSITION 7.3. –

$$T_3 \colon H_\Lambda^{s-r+\theta+\frac{n}{\mu_0 p}, p} \to H_\Lambda^{s,p}, \tag{73}$$

continuously for every $s \leqslant r$ *and* $\theta > 0$.

PROOF. – Thanks to the absolute convergence of the expansion (52), we may write:

$$T_3 u(x) = \sum_{k,\varepsilon} \sum_{\substack{h \in E_{1,k}^{(N_0-1)} \\ \lambda \in \mathbb{E}}} d_{h,\lambda}^{k,\varepsilon}(x)\, u_{h,\lambda}(x), \qquad u \in \mathcal{S}(\mathbb{R}^n),$$

For any k and ε the support of $V_{k,\varepsilon}(x) := \sum_{\substack{h \in E_{1,k}^{(N_0-1)} \\ \lambda \in \mathbb{E}}} d_{h,\lambda}^{k,\varepsilon}(x)\, u_{h,\lambda}(x)$ is included in the n-cube $P_{k,\varepsilon}^{(K)}$ (see Lemma 7.1), then by use of Proposition 5.1 for any $s \leqslant r$ and $1 < p < \infty$ there exists a constant $C = C_{s,p} > 0$ such that:

$$\|T_3 u\|_{H_\Lambda^{s,p}} \leqslant C \left\| \left(\sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^{2s} |V_{k,\varepsilon}|^2 \right)^{\frac{1}{2}} \right\|_p \leqslant C \sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^s \|V_{k,\varepsilon}\|_p, \tag{74}$$

where the second inequality is given by the continuous inclusion $B_{p,1}^{s,\Lambda} \subset F_{p,2}^{s,\Lambda}$.

Let us assume now $s < r$ and take a number $1 < p_1 < \infty$ such that $\frac{1}{p_1} = \frac{1}{p} + \eta$ with $0 < \eta < 1$. Using Proposition 5.3 (with p instead of p_2 and $q = 1$) we obtain

$$\sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^s \|V_{k,\varepsilon}\|_p \leqslant C' \sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^{s+\frac{n}{\mu_0}\eta} \|V_{k,\varepsilon}\|_{p_1}.$$

On the other hand, from the triangular and Hölder's inequalities we have:

$$\|V_{k,\varepsilon}\|_{p_1} \leqslant \sum_{\substack{h \in E_{1,k}^{(N_0-1)} \\ \lambda \in \mathbb{E}}} \|d_{h,\lambda}^{k,\varepsilon}\|_p \|u_{h,\lambda}\|_{\frac{1}{\eta}}, \qquad k \in \mathbb{Z}_+^n,\ \varepsilon \in \mathbb{E}. \tag{75}$$

Since $H_\Lambda^{r,p} \subset B_{p,\infty}^{r,\Lambda}$ with continuous embedding and the sequence $\{d_{h,\lambda}\}_{h,\lambda}$ is bounded in $H^{r,p}_\Lambda$, it follows from (74) and (75) that for any $\eta > 0$, setting $\Lambda_k := \Lambda(c^{(K)}_{k,\varepsilon})$, we have

$$(76)\qquad \|T_3 u\|_{H^{s,p}_\Lambda} \leqslant C \sup_{h,\lambda} \|d_{h,\lambda}\|_{B^{r,\Lambda}_{p,\infty}} \sum_{k,\varepsilon} \left\{ \Lambda_k^{s-r+\frac{n}{\mu_0}\eta} \sum_{\substack{h \in E^{(N_0-1)}_{1,k} \\ \lambda \in \mathbb{E}}} \|u_{h,\lambda}\|_{\frac{1}{\eta}} \right\} \leqslant$$

$$C \sup_{h,\lambda} \|d_{h,\lambda}\|_{B^{r,\Lambda}_{p,\infty}} \sum_{k,\varepsilon} \Lambda_k^{s-r+\frac{n}{\mu_0}\eta+\eta'} \Lambda_k^{-\eta'} \sum_{\substack{h \in E^{(N_0-1)}_{1,k} \\ \lambda \in \mathbb{E}}} \|u_{h,\lambda}\|_{\frac{1}{\eta}}.$$

Since $k_j \geqslant h_j + N_0$ for all $1 \leqslant j \leqslant n$, from (5), jointly with (63), we deduce that:

$$\Lambda(c^{(K)}_{k,\varepsilon}) \geqslant T\Lambda(c^{(H)}_{h,\lambda}),$$

with a constant $T = T_{H,K} > 0$ independent of h, k, λ and ε. Then for any $k \in \mathbb{Z}^n_+$, $\varepsilon, \lambda \in \mathbb{E}$:

$$(77)\qquad \Lambda(c^{(K)}_{k,\varepsilon})^{s-r+\frac{n}{\mu_0}\eta+\eta'} \leqslant T' \Lambda(c^{(H)}_{h,\lambda})^{s-r+\frac{n}{\mu_0}\eta+\eta'}, \qquad h \in E^{(N_0)}_{\sigma_1,k},$$

with a suitable $T' > 0$ depending on $H, K, r, s, \mu_0, \eta, \eta', n$, provided we choose η and η' small enough such that $s - r + \frac{n}{\mu_0}\eta + \eta' < 0$.

From (76), (77) jointly with Proposition 5.2, statement 3), we obtain then

$$(78)\qquad \|T_3 u\|_{H^{s,p}_\Lambda} \leqslant T' S \sup_{h,\lambda} \|d_{h,\lambda}\|_{B^{r,\Lambda}_{p,\infty}} \sum_{h,\lambda} \Lambda(c^{(H)}_{h,\lambda})^{s-r+\frac{n}{\mu_0}\eta+\eta'} \|u_{h,\lambda}\|_{\frac{1}{\eta}} \leqslant$$

$$T' S' \sup_{h,\lambda} \|d_{h,\lambda}\|_{B^{r,\Lambda}_{p,\infty}} \left(\sum_{h,\lambda} \Lambda(c^{(H)}_{h,\lambda})^{p\left(s-r+\frac{n}{\mu_0}\eta+\eta'+\eta''\right)} \|u_{h,\lambda}\|^p_{\frac{1}{\eta}} \right)^{\frac{1}{p}} =$$

$$CT' S' \sup_{h,\lambda} \|d_{h,\lambda}\|_{B^{r,\Lambda}_{p,\infty}} \|u\|_{B^{s-r+\frac{n}{\mu_0}\eta+\eta'+\eta'',\Lambda}_{\frac{1}{\eta},p}},$$

where $\eta'' > 0$ is arbitrary, $S = S_{H,\eta'} := \sum_{k,\varepsilon} \Lambda(c^{(K)}_{k,\varepsilon})^{-\eta'} < \infty$ and the constant $S' > 0$ only depends on H, η', η'' and p.

Let us consider now an arbitrary positive θ; in all the above arguments we can pick η, η' and η'' so that $\eta' + \eta'' < \theta$ and $\frac{1}{\eta} > p$.

By Corollary 5.1, with $s - r + \theta + \frac{n}{\mu_0 p}$ instead of s, $\frac{1}{\eta}$ instead of p_2 and $\theta - (\eta' + \eta'')$ instead of δ_2, we have

$$H^{s-r+\theta+\frac{n}{\mu_0 p},p}_\Lambda \subset B^{s-r+\frac{n}{\mu_0}\eta+\eta'+\eta'',\Lambda}_{\frac{1}{\eta},p}$$

with continuous embedding. This proves (73) for $s < r$.

For the case $s = r$ let us come back to the first inequality in (74); using also the Cauchy-Schwarz inequality and setting for some $t > 0$ $\Gamma_t(u) := \sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^t \|u_{h,\lambda}\|_\infty$ we can estimate $\|T_3 u\|_{H_\Lambda^{r,p}}$ using

$$\Gamma_t(u) \left\| \left(\sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^{2r} \sum_{\substack{h \in E_{1,k}^{(N_0-1)} \\ \lambda \in \mathbb{E}}} \Lambda(c_{h,\lambda}^{(H)})^{-2t} \,|d_{h,\lambda}^{k,\varepsilon}(x)|^2 \right)^{\frac{1}{2}} \right\|_p . \tag{79}$$

By changing the order of the sums in k, ε and h, λ we have now

$$\left\| \left(\sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^{2r} \sum_{\substack{h \in E_{1,k}^{(N_0-1)} \\ \lambda \in \mathbb{E}}} \Lambda(c_{h,\lambda}^{(H)})^{-2t} \,|d_{h,\lambda}^{k,\varepsilon}(x)|^2 \right)^{\frac{1}{2}} \right\|_p \leqslant$$

$$\sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{-t} \left\| \left(\sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^{2r} \,|d_{h,\lambda}^{k,\varepsilon}|^2 \right)^{\frac{1}{2}} \right\|_p \leqslant CS \sup_{h,\lambda} \|d_{h,\lambda}\|_{H_\Lambda^{r,p}},$$

where $S = S_t := \sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{-t} < \infty$.

Let $0 < \theta' < \theta'' < \theta$ be arbitrary, set $t = \theta'$ and then use estimate (19) with $q_1 = p$, $q_2 = 1$, $s = \theta'$, $\varepsilon = \theta'' - \theta'$, $b_{h,\lambda} = \|u_{h,\lambda}\|_\infty$ and estimate (27) with $p_2 = \infty$, $s = \dfrac{n}{\mu_0 p} + \theta$ and $\delta_2 = \theta - \theta''$; we obtain then

$$\Gamma_{\theta'}(u) \leqslant C_1 \left(\sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{p\theta''} \|u_{h,\lambda}\|_\infty^p \right)^{\frac{1}{p}} \leqslant C_2 \|u\|_{H_\Lambda^{\frac{n}{\mu_0 p}+\theta,\,p}}, \tag{80}$$

with suitable positive constants C_1, C_2.

Estimates (79) for $t = \theta'$ and (80) complete the proof. ■

REMARK 14. – Let us notice that, under the assumption $r > \dfrac{n}{\mu_0 p}$, we may always find $\theta > 0$ such that $H_\Lambda^{s,p} \subset H_\Lambda^{s-r+\theta+\frac{n}{\mu_0 p},\,p}$ continuously, then it follows from Proposition 7.3:

$$T_3 \colon H_\Lambda^{s,p} \to H_\Lambda^{s,p}$$

continuously for every $s \leqslant r$.

Let us remark that any the operators T_j, $j = 4, \dots, 7$, may be expressed as a finite sum of operators with the following form

$$Ru(x) = \sum_{h,\lambda} \sum_{\substack{k \in E_h^{(N_0),\,n_1,\,n_2,\,\pi} \\ \lambda \in \mathbb{E}}} d_{h,\lambda}^{k,\varepsilon}(x)\, u_{h,\lambda}(x), \qquad u \in \mathcal{S}(\mathbb{R}^n). \tag{81}$$

Here n_1, n_2 are integers such that $0 \leqslant n_1 \leqslant n_2 \leqslant n$ and two at least of these inequalities must be strict; π is any permutation of the set $\{1, 2, \dots, n\}$, $E_{1,h_{\pi(j)}}^{(N_0)}$,

$E^{(N_0)}_{2, h_{\pi(j)}}$, $E^{(N_0)}_{3, h_{\pi(j)}}$ are defined by (53)-(55) and

$$E_h^{(N_0),\, n_1,\, n_2,\, \pi} := \prod_{j=1}^{n_1} E^{(N_0)}_{1, h_{\pi(j)}} \times \prod_{j=n_1+1}^{n_2} E^{(N_0)}_{2, h_{\pi(j)}} \times \prod_{j=n_2+1}^{n} E^{(N_0)}_{3, h_{\pi(j)}}.$$

At the moment it only needs to study the $H^{s,p}_{\Lambda}$-continuity of an operator taking the form (81). We need the following

PROPOSITION 7.4. – *Let us assume that the weight function $\Lambda(\xi)$ satisfies* (46); *let R be defined by* (81), $1 < p < \infty$ *and* $r > \dfrac{n}{(1-\delta)\mu_0 p}$. *Then*

$$R : H^{s,p}_{\Lambda} \to H^{s,p}_{\Lambda},$$

continuously for every $0 \leq s \leq r$.

In order to prove the previous statement we need a result of Caldéron [5] about complex interpolation. Following then the notations of Triebel [23] we write $[.,.]_{\Theta}$, $0 < \Theta < 1$, for the complex interpolation functor.

PROPOSITION 7.5. – *Let (B^0, B^1) and (C^0, C^1) be two interpolation couples. Let L be a linear mapping from $B^0 + B^1$ to $C^0 + C^1$ such that $x \in B^i$ implies $L(x) \in C^i$ and*

$$\|L(x)\|_{C^i} \leq M_i \|x\|_{B^i}, \qquad i = 0, 1.$$

Then $x \in B_{\Theta} := [B^0, B^1]_{\Theta}$ implies $L(x) \in C_{\Theta} := [C^0, C^1]_{\Theta}$ and

$$\|L(x)\|_{C_{\Theta}} \leq M_0^{1-\Theta} M_1^{\Theta} \|x\|_{B_{\Theta}}.$$

REMARK 15. – From Triebel [27], we get also the following complex interpolation formula:

$$[L^p, H^{r,p}_{\Lambda}]_{\Theta} = H^{\Theta r, p}_{\Lambda}, \qquad 0 < \Theta < 1, \quad r > 0.$$

So if R is L^p and $H^{r,p}_{\Lambda}$ bounded, then its $H^{s,p}_{\Lambda}$-continuity follows from Proposition 7.5, for any $0 \leq s \leq r$.

For an exhaustive introduction to complex interpolation methods we address to Caldéron [5] and Triebel [23].

PROOF (of Proposition 7.4). – In order to simplify all the next technical calculus, we assume, without loss of generality, that the permutation π in (81) is the identity of $\{1, 2, \ldots, n\}$ and restrict ourselves to the case $n_1 = 1$, $n_2 = 2$ and $n = 3$.

Then the operator R takes the form

$$Ru(x) := \sum_{h, \lambda} \sum_{\substack{k_j \in E^{(N_0)}_{j, h_j},\ j = 1, 2, 3 \\ \lambda \in \mathbb{E}}} d^{k, \varepsilon}_{h, \lambda}(x)\, u_{h, \lambda}(x), \qquad u \in \mathcal{S}(\mathbb{R}^3). \tag{82}$$

Because of the absolute convergence of the expansion in (82), we may write R

as follows

$$Ru(x) := \sum_{\substack{h_1, h_2, k_3 \\ \lambda_1, \lambda_2, \varepsilon_3}} \sum_{\substack{k_j \in E^{(N_0)}_{j, h_j}, j=1,2, \\ h_3 \in E^{(N_0-1)}_{1, k_3}, \\ \varepsilon_1, \varepsilon_2, \lambda_3}} d^{k,\varepsilon}_{h,\lambda}(x)\, u_{h,\lambda}(x).$$

From Lemma 7.1, we know there exists a number $T>1$ such that $\operatorname{supp} d^{k,\varepsilon}_{h,\lambda} u_{h,\lambda} \subset L^{(T)}_{h_1,\lambda_1} \times [-T2^{h_2}, T2^{h_2}] \times L^{(T)}_{k_3,\varepsilon_3}$, for any (k_1, k_2, h_3), such that $k_1 < h_1 - N_0$, $h_2 - N_0 \leqslant k_2 < h_2 + N_0$ and $h_3 \leqslant k_3 - N_0$, and all $\varepsilon_1, \varepsilon_2, \lambda_3$.

For shortness, later on we set $t := (h_1, h_2, k_3)$, $\sigma := (\lambda_1, \lambda_2, \varepsilon_3)$ and $E^{(N_0)}_t := E^{(N_0)}_{1,h_1} \times E^{(N_0)}_{2,h_2} \times E^{(N_0-1)}_{1,k_3}$; moreover we will write $e^{(K)}_{1,r,\varepsilon} := (c_{1,K}\varepsilon 2^r, 0, 0)$, $e^{(K)}_{2,r,\varepsilon} := (0, c_{2,K}\varepsilon 2^r, 0)$, $e^{(K)}_{3,r,\varepsilon} := (0, 0, c_{3,K}\varepsilon 2^r)$, for any integer r, $K>1$, $\varepsilon \in \{-1, 1\}$ and $c_{j,K} := K \pm \dfrac{1}{2K}$, $j = 1, 2, 3$.

Using now Lemma 7.3 we have that for every $s \geqslant 0$, $1 < p < \infty$ and $\gamma > 0$ there exists $C = C_{s,p,\gamma} > 0$ such that

$$\|Ru\|_{H^{s,p}_\Lambda} \leqslant C \left\| \left(\sum_{t,\sigma} \Lambda(c^{(T)}_{t,\sigma})^{2s} 2^{2\gamma h_2} |U_{t,\sigma}|^2 \right)^{\frac{1}{2}} \right\|_p, \tag{83}$$

where $U_{t,\sigma}(x) := \sum_{\substack{(k_1,k_2,h_3) \in E^{(N_0)}_t \\ \varepsilon_1, \varepsilon_2, \lambda_3}} d^{k,\varepsilon}_{h,\lambda}(x)\, u_{h,\lambda}(x)$ and

$$c^{(T)}_{t,\sigma} = \left(c_{T,1}\lambda_1 2^{h_1}, c_{T,2}\lambda_2 2^{h_2}, c_{T,3}\varepsilon_3 2^{k_3}\right), \quad c_{T,j} := T \pm \frac{1}{2T}, \ j = 1, 2, 3.$$

In order to prove the $H^{r,p}_\Lambda$-continuity of R, let us notice that $c^{(T)}_{t,\sigma} = c^{(T)}_{h,\sigma} + \tau_3 e^{(T)}_{3,k_3,\varepsilon_3}$, with $\tau_3 := 1 - 2^{h_3 - k_3}$, that is $0 < \tau_3 < 1$ as $h_3 \leqslant k_3 - N_0$; by using (46) and (5) we have:

$$\Lambda(c^{(T)}_{t,\sigma}) \leqslant C\left(\Lambda(c^{(T)}_{h,\sigma}) + \Lambda(e^{(T)}_{3,k_3,\varepsilon_3}) + \Lambda(c^{(T)}_{h,\sigma})^\delta \Lambda(e^{(T)}_{3,k_3,\varepsilon_3})^\delta\right),$$

for any t, σ, $k_3 \geqslant h_3 + N_0$ and some $C > 0$ independent of t and σ.

For $\tau := (k_1, k_2, h_3) \in E^{(N_0)}_t$, $e = (\varepsilon_1, \varepsilon_2, \lambda_3)$ and $s = r$ it follows from (83):

$$\begin{aligned}
\|Ru\|_{H^{r,p}_\Lambda} \leqslant\ & C \left\| \left(\sum_{t,\sigma} 2^{2\gamma h_2} \Big(\sum_{\tau, e} \Lambda(c^{(T)}_{h,\sigma})^r |d^{k,\varepsilon}_{h,\lambda}(x)|\,|u_{h,\lambda}(x)| \Big)^2 \right)^{\frac{1}{2}} \right\|_p + \\
& C \left\| \left(\sum_{t,\sigma} 2^{2\gamma h_2} \Lambda(e^{(T)}_{3,k_3,\varepsilon_3})^{2r} \Big(\sum_{\tau, e} |d^{k,\varepsilon}_{h,\lambda}(x)|\,|u_{h,\lambda}(x)| \Big)^2 \right)^{\frac{1}{2}} \right\|_p + \\
& C \left\| \left(\sum_{t,\sigma} 2^{2\gamma h_2} \Lambda(e^{(T)}_{3,k_3,\varepsilon_3})^{2r\delta} \Big(\sum_{\tau, e} \Lambda(c^{(T)}_{h,\sigma})^{r\delta} |d^{k,\varepsilon}_{h,\lambda}(x)|\,|u_{h,\lambda}(x)| \Big)^2 \right)^{\frac{1}{2}} \right\|_p \leqslant \\
& C(I_1 + I_2 + I_3).
\end{aligned}$$

In order to estimate I_1, we use Proposition 5.4, jointly with Remark 7 and the boundedness of $\{d_{h,\lambda}\}$ in $H_\Lambda^{r,p}$ to get:

$$\sum_{\tau, e} \Lambda(c_{h,\sigma}^{(T)})^r \, |d_{h,\lambda}^{k,\varepsilon}(x)| \, |u_{h,\lambda}(x)| \leqslant$$

$$M \sup_{h,\lambda} \|d_{h,\lambda}\|_{H_\Lambda^{r,p}} \sum_{h_3, \lambda_3} \Lambda(c_{h,\sigma}^{(T)})^r \, |u_{h,\lambda}| \sum_{\substack{k_j, \varepsilon_j \\ j=1,2}} \Lambda(c_{k,\varepsilon}^{(K)})^{-\left(r - \frac{n}{\mu_0 p}\right)},$$

where h_3 runs through $E_{1,k_3}^{(N_0-1)}$ and k_j takes its values in $E_{j,h_j}^{(N_0)}$ for $j = 1, 2$.

Let now $\theta > 0$ be such that $r - \dfrac{n}{\mu_0 p} - \theta > 0$; in view of (5) and Remark 11, $\Lambda(c_{k,\varepsilon}^{(K)})^{-\left(r - \frac{n}{\mu_0 p}\right)}$ may be bounded by

$$\Lambda(e_{1,k_1,\varepsilon_1}^{(K)})^{-\left(r - \frac{n}{\mu_0 p} - \theta\right)} \Lambda(e_{3,k_3,\varepsilon_3}^{(K)})^{-\frac{\theta}{3}} \Lambda(e_{2,h_2,\varepsilon_2}^{(K)})^{-\frac{\theta}{3}} \Lambda(e_{3,h_3,\varepsilon_3}^{(K)})^{-\frac{\theta}{3}}, \tag{84}$$

for all $k \in \mathbb{Z}_+^3$, $k_2 - N_0 < h_2 \leqslant k_2 + N_0$, $h_3 \leqslant k_3 - N_0$ and $\varepsilon \in \mathbb{E}$.

From (84) it follows for k_j, $j = 1, 2$ running as above:

$$\sum_{\substack{k_j, \varepsilon_j \\ j=1,2}} \Lambda(c_{k,\varepsilon}^{(K)})^{-\left(r - \frac{n}{\mu_0 p}\right)} \leqslant C_2 \Lambda(e_{3,k_3,\varepsilon_3}^{(K)})^{-\frac{\theta}{3}} \Lambda(e_{2,h_2,\varepsilon_2}^{(K)})^{-\frac{\theta}{3}} \Lambda(e_{3,h_3,\varepsilon_3}^{(K)})^{-\frac{\theta}{3}},$$

whence, if τ, e, h_3 run as before, by the Cauchy-Schwarz inequality we have:

$$\sum_{\tau, e} \Lambda(c_{h,\sigma}^{(T)})^r \, |d_{h,\lambda}^{k,\varepsilon}(x)| \, |u_{h,\lambda}(x)| \leqslant$$

$$M C_2 K_3^\theta \sup_{h,\lambda} \|d_{h,\lambda}\|_{H_\Lambda^{r,p}} \sum_{h_3, \lambda_3} \Lambda(c_{h,\sigma}^{(T)})^r \Lambda(e_{3,h_3,\varepsilon_3}^{(K)})^{-\frac{\theta}{3}} |u_{h,\lambda}| \leqslant$$

$$M C_3 K_3^\theta \sup_{h,\lambda} \|d_{h,\lambda}\|_{H_\Lambda^{r,p}} \left(\sum_{h_3, \lambda_3} \Lambda(c_{h,\sigma}^{(T)})^{2r} |u_{h,\lambda}|^2 \right)^{\frac{1}{2}},$$

for $K_3^\theta := \Lambda(e_{3,k_3,\varepsilon_3}^{(K)})^{-\frac{\theta}{3}} \Lambda(e_{2,h_2,\varepsilon_2}^{(K)})^{-\frac{\theta}{3}}$ and C_2, C_3 depending only on θ.

From the previous estimate and Proposition 5.1, we obtain then:

$$I_1 \leqslant C_4 \sup_{h,\lambda} \|d_{h,\lambda}\|_{H_\Lambda^{r,p}} \left\| \left(\sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{2r} |u_{h,\lambda}|^2 \right)^{\frac{1}{2}} \right\|_p \leqslant C_5 \sup_{h,\lambda} \|d_{h,\lambda}\|_{H_\Lambda^{r,p}} \|u\|_{H_\Lambda^{r,p}},$$

as $\sum_{k_3, \varepsilon_3} \Lambda(e_{3,k_3,\varepsilon_3}^{(K)})^{-\frac{\theta}{3}}$ is finite, $2^{2\gamma h_2} \Lambda(e_{2,h_2,\varepsilon_2}^{(K)})^{-\frac{\theta}{3}}$ is bounded from above, for γ sufficiently small, and $\Lambda(c_{h,\sigma}^{(T)})^{2r} \leqslant C' \Lambda(c_{h,\lambda}^{(H)})^{2r}$ by (5).

Let t, q be two arbitrary positive numbers; by means of multiplication and division for $\Lambda\big(e_{1,k_1,\varepsilon_1}^{(K)}\big)^t$, $\Lambda\big(e_{3,h_3,\varepsilon_3}^{(K)}\big)^q$ and the Cauchy-Schwarz inequality we get, for $C_1 = C_1(t, q)$ and τ, e, h_3, k_1, k_2 as before:

$$\sum_{\tau, e} |d_{h,\lambda}^{k,\varepsilon}(x)|\,|u_{h,\lambda}(x)| \leqslant \tag{85}$$

$$C_1 \Lambda(e_{1,h_1,\varepsilon_1}^{(K)})^t \left(\sum_{h_3,\lambda_3} \Lambda(e_{3,h_3,\varepsilon_3}^{(K)})^{2q} |u_{h,\lambda}|^2 \sum_{\substack{k_j,\varepsilon_j \\ j=1,2}} |d_{h,\lambda}^{k,\varepsilon}(x)|^2 \right)^{\frac{1}{2}}.$$

Here we also used the estimate $\Lambda\big(e_{1,k_1,\varepsilon_1}^{(K)}\big)^t \leqslant c\Lambda\big(e_{1,h_1,\varepsilon_1}^{(K)}\big)^t$, due to $k_1 < h_1 - N_0$ and $2^{k_1} = \tau_1 2^{h_1}$ for $0 \leqslant \tau_1 = 2^{k_1 - h_1} < 1$.

Since $\Lambda(e_{1,h_1,\varepsilon_1}^{(K)})^t \Lambda(e_{3,h_3,\varepsilon_3}^{(K)})^q 2^{\gamma h_2} \leqslant C' \Lambda(c_{h,\lambda}^{(H)})^{t+q+\frac{\gamma}{\mu_0}}$, from (85) the following estimate of I_2 follows:

$$I_2 \leqslant C_2 \left\| \sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{t+q+\frac{\gamma}{\mu_0}} |u_{h,\lambda}| \left(\sum_{k,\varepsilon} \Lambda(c_{k,\varepsilon}^{(K)})^{2r} |d_{h,\lambda}^{k,\varepsilon}|^2 \right)^{\frac{1}{2}} \right\|_p \leqslant \tag{86}$$

$$C_3 \sup_{h,\lambda} \|d_{h,\lambda}\|_{H_\Lambda^{r,p}} \sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{t+q+\frac{\gamma}{\mu_0}} \|u_{h,\lambda}\|_\infty .$$

We may always assume $t + q + \dfrac{\gamma}{\mu_0} = r - \dfrac{n}{\mu_0 p} - \theta$; arguing as in the proof of (80), we obtain now

$$\sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{r - \frac{n}{\mu_0 p} - \theta} \|u_{h,\lambda}\|_\infty \leqslant \tag{87}$$

$$C_1 \left(\sum_{h,\lambda} \Lambda(c_{h,\lambda}^{(H)})^{p\left(r - \frac{n}{\mu_0 p} - \frac{\theta}{2}\right)} \|u_{h,\lambda}\|_\infty^p \right)^{\frac{1}{p}} \leqslant C_2 \|u\|_{H_\Lambda^{r,p}}.$$

The estimates (86) and (87) imply:

$$I_2 \leqslant C_3 \sup_{h,\lambda} \|d_{h,\lambda}\|_{H_\Lambda^{r,p}} \|u\|_{H_\Lambda^{r,p}}. \tag{88}$$

It remains to estimate I_3; as we did to obtain (85), using the Cauchy-Schwarz inequality with $(1-\delta)\,r$ instead of q and $\Lambda(c_{h,\sigma}^{(K)})^{\delta r} u_{h,\lambda}$ instead of $u_{h,\lambda}$, setting $\Lambda_j := \Lambda\big(e_{j,h_j,\varepsilon_j}^{(K)}\big)$, $j = 1, 2, 3$, we have:

$$\sum_{\tau, e} \Lambda(c_{h,\sigma}^{(K)})^{\delta r} |d_{h,\lambda}^{k,\varepsilon}(x)|\,|u_{h,\lambda}(x)| \leqslant$$

$$C_1 \Lambda_1^t \left(\sum_{h_3,\lambda_3} \Lambda(c_{h,\sigma}^{(K)})^{2\delta r} \Lambda_3^{2(1-\delta) r} |u_{h,\lambda}|^2 \sum_{\substack{k_j,\varepsilon_j \\ j=1,2}} |d_{h,\lambda}^{k,\varepsilon}(x)|^2 \right)^{\frac{1}{2}},$$

whence, observing that $\Lambda(e^{(K)}_{3,h_3,\varepsilon_3})^{2(1-\delta)r} \leqslant C' \Lambda(e^{(K)}_{3,k_3,\varepsilon_3})^{2(1-\delta)r}$ when $h_3 \leqslant k_3 - N_0$ and arguing as from (85) to (86), we obtain:

$$I_3 \leqslant C_2 \sup_{h,\lambda} \|d_{h,\lambda}\|_{H^{r,p}_\Lambda} \sum_{h,\lambda} \Lambda(c^{(H)}_{h,\lambda})^{t+\delta r+\frac{\gamma}{\mu_0}} \|u_{h,\lambda}\|_\infty .$$

Lastly, we obtain for I_3 an estimate like (88), by choosing t, γ and $0<\theta<r-\frac{n}{\mu_0 p}$ such that $t+\delta r+\frac{\gamma}{\mu_0}=r-\frac{n}{\mu_0 p}-\theta$ $\left(\text{this is always possible in view of the assumption } r>\frac{n}{(1-\delta)\mu_0 p}\right)$ and then by repeating the arguments which lead us to (87).

This completes the proof of the $H^{r,p}_\Lambda$-continuity of R.

Concerning the L^p-continuity of R, it suffices to repeat step by step the argument used to estimate I_1, starting from (83) with $s=0$ and $\gamma>0$ suitably small.

If we replace the indices h_1, h_2, k_3 by the more general systems of indices $\{h_{\pi(1)}, \dots, h_{\pi(n_1)}\}$, $\{h_{\pi(n_1+1)}, \dots, h_{\pi(n_2)}\}$, $\{k_{\pi(n_2+1)}, \dots, k_{\pi(n)}\}$ respectively, the same proof runs for a general dimension n. ■

PROOF (of Theorem 7.1). – By using the Propositions 7.1-7.4 we immediately get the statement for an elementary symbol $a(x,\xi) \in H^{r,p}_\Lambda M^0_\Lambda$. More precisely for any $0 \leqslant s \leqslant r$ and $1<p<\infty$

$$\|a(x,D)u\|_{H^{s,p}_\Lambda} \leqslant C \sup_{h,\lambda} \|d_{h,\lambda}\|_{H^{r,p}_\Lambda} \|u\|_{H^{s,p}_\Lambda}, \qquad u \in \mathcal{S}(\mathbb{R}^n), \tag{89}$$

where the constant $C>0$ depends only on r, s, p and n.

Let us take now an arbitrary symbol $a(x,\xi)$ in $H^{r,p}_\Lambda M^0_\Lambda$; in view of (48) we obtain for every $0 \leqslant s \leqslant r$, $1<p<\infty$ and $u \in \mathcal{S}(\mathbb{R}^n)$

$$\|a(x,D)\,u\|_{H^{s,p}_\Lambda} \leqslant C \|u\|_{H^{s,p}_\Lambda} \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|^2)^{2n}} \sup_{h,\lambda} \|d^m_{h,\lambda}\|_{H^{r,p}_\Lambda},$$

$C>0$ depending only on r, s, p and the dimension n.

Since the sequences $\{d^m_{h,\lambda}\}_{h,\lambda}$ are bounded in $H^{r,p}_\Lambda$ uniformly in $m \in \mathbb{Z}^n$, $\sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|^2)^{2n}}$ is finite and $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{s,p}_\Lambda$, it follows that $a(x,D)$ is $H^{s,p}_\Lambda$ bounded.

Lastly when the symbol $a(x,\xi) \in H^{r,p}_\Lambda M^m_\Lambda$ has an arbitrary order m, we easily reduce to the case of order zero as already noticed in this section. ■

COROLLARY 7.1. – *Let $\Lambda(\xi)$ be a weight function which fulfills* (46). *Then for* $1 < p < \infty$, $r > \dfrac{n}{(1-\delta)\mu_0 p}$ *and* $0 \leq s \leq r$ *there exists a constant* $C > 0$ *such that:*

$$\|uv\|_{H_\Lambda^{s,p}} \leq C\|u\|_{H_\Lambda^{s,p}} \|v\|_{H_\Lambda^{r,p}}, \tag{90}$$

for all $u \in H_\Lambda^{s,p}$ *and* $v \in H_\Lambda^{r,p}$.

In particular, the space $H_\Lambda^{r,p}$ *is a multiplication algebra.*

PROOF. – For any fixed $v \in H_\Lambda^{r,p}$ the multiplication operator $M_v(u) := uv$ is a special pseudodifferential operator whose symbol $a(x, \xi) = v(x) \in H_\Lambda^{r,p} M_\Lambda^0$ may be written as an elementary symbol: $v(x) = \sum_{h,\lambda} v(x)\, \psi_{h,\lambda}(\xi)$, where $\{\psi_{h,\lambda}(\xi)\}$ is any non-homogeneous partition of unity.

Then (90) easily follows by applying (89) to the operator M_v. ■

COROLLARY 7.2. – *Let* $F : \mathbb{C} \to \mathbb{C}$ *be an entire analytic function such that* $F(0) = 0$.

Then for every $u \in H_\Lambda^{r,p}$, $1 < p < \infty$ *and* $r > \dfrac{n}{(1-\delta)\mu_0 p}$, $F(u) \in H_\Lambda^{r,p}$ *and moreover*

$$\|F(u)\|_{H_\Lambda^{r,p}} \leq C\|u\|_{H_\Lambda^{r,p}}, \qquad \text{with } C = C(r, p, F, \|u\|_{H_\Lambda^{r,p}}).$$

PROOF. – Since $F(0) = 0$, for a suitable sequence of complex numbers $\{\lambda_j\}$ we have $F(\zeta) = \sum_{j=0}^{\infty} \lambda_j \zeta^{j+1}$, $\zeta \in \mathbb{C}$, with absolute convergence.

On the other hand, from (90) we obtain for $u \in H_\Lambda^{r,p}$

$$\|u^{j+1}\|_{H_\Lambda^{r,p}} \leq C^j \|u\|_{H_\Lambda^{r,p}}^{j+1}, \qquad j = 0, 1, \ldots,$$

with a positive C depending only on r, p, μ_0 and the dimension n.

Thus it follows that:

$$\|F(u)\|_{H_\Lambda^{r,p}} \leq \sum_{j=0}^{\infty} |\lambda_j| C^j \|u\|_{H_\Lambda^{r,p}}^{j+1} \leq F_1(\|u\|_{H_\Lambda^{r,p}}) \|u\|_{H_\Lambda^{r,p}}, \tag{91}$$

where $F_1(\zeta) := \sum_{j=0}^{\infty} |\lambda_j| C^j \zeta^j$, $\zeta \in \mathbb{C}$, with absolute convergence. ■

REMARK 16. – For $p = 2$ the continuity of $H_\Lambda^{r,2} S_\Lambda^m$ and the algebra property of $H_\Lambda^{r,2} = H_\Lambda^r$ are known for a more general class of weight functions $\Lambda(\xi)$ independently of Theorem 7.1; see Garello [9], [10] where more precise estimates are also given.

In Marschall [18] the reader can find results of $H^{s,p}$-continuity for pseudodifferential operators with non regular symbol, where $H^{s,p} = H_{\langle\xi\rangle}^{s,p}$.

8. – Examples and applications.

Let us recall that a convex polyhedron $\mathcal{P} \subset \mathbb{R}^n$ can be obtained as the convex hull of a finite subset $\mathcal{V}(\mathcal{P}) \subset \mathbb{R}^n$ of convex-linearly independent points, called *vertices* of $\mathcal{P}$ and univocally determined by $\mathcal{P}$ itself. More precisely if $\mathcal{P}$ has non empty interior, it is completely described by

$$\{\zeta \in \mathbb{R}^n;\ \nu \cdot \zeta \geq 0,\ \forall \nu \in \mathcal{N}_0(\mathcal{P})\} \cap \{\zeta \in \mathbb{R}^n;\ \nu \cdot \zeta \leq 1,\ \forall \nu \in \mathcal{N}_1(\mathcal{P})\},$$

where $\mathcal{N}_0(\mathcal{P}) \subset \{\nu \in \mathbb{R}^n;\ |\nu| = 1\}$, $\mathcal{N}_1(\mathcal{P}) \subset \mathbb{R}^n$ are finite sets univocally determined by $\mathcal{P}$, as usual, $\nu \cdot \xi = \sum_{j=1}^{N} \nu_j \xi_j$.

We say that a convex polyhedron $\mathcal{P} \subset \mathbb{R}^n_+ = [0, \infty)^n$ is a *complete polyhedron* if:

i) $\mathcal{V}(P) \subset \mathbb{N}^n$;

ii) $(0, \dots, 0) \in \mathcal{V}(\mathcal{P})$, and $\mathcal{V}(\mathcal{P}) \neq \{(0, \dots, 0)\}$;

iii) $\mathcal{N}_0(\mathcal{P}) = \{e_1, \dots, e_n\}$ with $e_j = (0, \dots, 1_{j\text{-entry}}, \dots 0) \in \mathbb{R}^n_+$;

iv) every $\nu \in \mathcal{N}_1(\mathcal{P})$ has components $\nu_j > 0$ $(j = 1, \dots, n)$.

The boundary of $\mathcal{P}$ is made of faces which are the convex hull of the vertices of $\mathcal{P}$ lying on the hyperplane H_ν orthogonal to $\nu \in \mathcal{N}_0(\mathcal{P}) \cup \mathcal{N}_1(\mathcal{P})$ of equation:

$$\nu \cdot \zeta = 0 \ \text{ if } \nu \in \mathcal{N}_0(\mathcal{P}), \qquad \nu \cdot \zeta = 1 \ \text{ if } \nu \in \mathcal{N}_1(\mathcal{P}).$$

Particularly we define $\mathcal{F}(\mathcal{P}) := \bigcup_{\nu \in \mathcal{N}_1(\mathcal{P})} H_\nu \cap \mathcal{P}$, the set of the faces which do not lie on the coordinate hyperplanes.

Given a complete polyhedron $\mathcal{P}$, we set

$$\Lambda_{\mathcal{P}}(\xi) := \Big(\sum_{\alpha \in \mathcal{V}(P)} \xi^{2\alpha} \Big)^{\frac{1}{2}}, \qquad \xi \in \mathbb{R}^n. \tag{92}$$

One easily proves that $\Lambda_{\mathcal{P}}(\xi)$ satisfies the following estimates

$$\frac{1}{C}(1 + |\xi|)^{\mu_0} \leq \Lambda_{\mathcal{P}}(\xi) \leq C(1 + |\xi|)^{\mu_1}, \qquad \xi \in \mathbb{R}^n, \tag{93}$$

with a suitable $C > 1$ and

$$\mu_0 := \min_{\alpha \in \mathcal{V}(P) \setminus \{0\}} |\alpha| \qquad \mu_1 := \max_{\alpha \in \mathcal{V}(P)} |\alpha|.$$

LEMMA 8.1. – *Let $\mathcal{P}$ be a complete polyhedron of $\mathbb{R}^n$. Then for any multi-indices $\alpha, \gamma \in \mathbb{Z}_+^n$ there exists $C_{\alpha,\gamma} > 0$ such that*

$$\prod_{j=1}^{n}(1+\xi_j^2)^{\frac{\gamma_j}{2}}\,|\partial^{\alpha+\gamma}\Lambda_{\mathcal{P}}(\xi)| \leqslant C_{\alpha,\gamma}\Lambda_{\mathcal{P}}(\xi)^{1-\frac{1}{\mu}|\alpha|}, \qquad \xi \in \mathbb{R}^n,$$

where $\mu := \max\left\{\dfrac{1}{\nu_j} : j = 1, \dots, n \text{ and } \nu \in \mathcal{N}_1(\mathcal{P})\right\}$.

PROOF. – First of all let us observe that for any $\gamma \in \mathbb{Z}_+^n$ we have:

$$\prod_{j=1}^{n}(1+\xi_j^2)^{\frac{\gamma_j}{2}} \leqslant \prod_{j=1}^{n}(1+|\xi_j|)^{\gamma_j} = \sum_{\sigma \leqslant \gamma}\binom{\gamma}{\sigma}|\xi^\sigma|. \tag{94}$$

Moreover we can show that for any $\alpha, \beta \in \mathbb{Z}_+^n$ there exists a constant $C_{\alpha,\beta} > 0$ such that:

$$|\xi^\beta \partial^{\alpha+\beta}\Lambda_{\mathcal{P}}(\xi)| \leqslant C_{\alpha,\beta}\Lambda_{\mathcal{P}}(\xi)^{1-\frac{1}{\mu}|\alpha|}, \qquad \xi \in \mathbb{R}^n. \tag{95}$$

In fact for $|\alpha+\beta| = 0$ the estimate (95) is trivially verified, with $C_{0,0} = 1$. For a fixed $k \in \mathbb{Z}_+$, let us assume that (95) holds for any $\alpha, \beta \in \mathbb{Z}_+^n$ with $|\alpha+\beta| \leqslant k$ and consider $\alpha, \beta \in \mathbb{Z}_+^n$ such that $|\alpha+\beta| = k+1$.

From (92) we obtain:

$$\partial^{\alpha+\beta}(\Lambda_{\mathcal{P}}(\xi)^2) = \sum_{\substack{\chi \in \mathcal{V}(\mathcal{P}) \\ 2\chi \geqslant \alpha+\beta}} (\alpha+\beta)!\binom{2\chi}{\alpha+\beta}\xi^{2\chi-\alpha-\beta}.$$

So, by Leibnitz formula, we get

$$\partial^{\alpha+\beta}\Lambda_{\mathcal{P}}(\xi) = \frac{1}{2\Lambda_{\mathcal{P}}(\xi)}\left\{\sum_{\substack{\chi \in \mathcal{V}(\mathcal{P}) \\ 2\chi \geqslant \alpha+\beta}} (\alpha+\beta)!\binom{2\chi}{\alpha+\beta}\xi^{2\chi-\alpha-\beta} - \sum_{\substack{\delta \leqslant \beta, \eta \leqslant \alpha \\ (\eta,\delta) \neq (0,0) \\ (\eta,\delta) \neq (\alpha,\beta)}} \binom{\alpha}{\eta}\binom{\beta}{\delta}\partial^{\eta+\delta}\Lambda_{\mathcal{P}}(\xi)\,\partial^{\alpha-\eta+\beta-\delta}\Lambda_{\mathcal{P}}(\xi)\right\},$$

whence

$$|\xi^\beta \partial^{\alpha+\beta}\Lambda_{\mathcal{P}}(\xi)| \leqslant \frac{1}{2\Lambda_{\mathcal{P}}(\xi)}\left\{\sum_{\substack{\chi \in \mathcal{V}(\mathcal{P}) \\ 2\chi \geqslant \alpha+\beta}} (\alpha+\beta)!\binom{2\chi}{\alpha+\beta}|\xi^{2\chi-\alpha}| + \sum_{\substack{\delta \leqslant \beta, \eta \leqslant \alpha \\ (\eta,\delta) \neq (0,0) \\ (\eta,\delta) \neq (\alpha,\beta)}} \binom{\alpha}{\eta}\binom{\beta}{\delta}|\xi^\delta \partial^{\eta+\delta}\Lambda_{\mathcal{P}}(\xi)|\,|\xi^{\beta-\delta}\partial^{\alpha-\eta+\beta-\delta}\Lambda_{\mathcal{P}}(\xi)|\right\}. \tag{96}$$

From the inductive assumption, we have

$$|\xi^{\delta}\partial^{\eta+\delta}\Lambda_{\mathcal{P}}(\xi)| \leq C_{\eta,\delta}\Lambda_{\mathcal{P}}(\xi)^{1-\frac{|\eta|}{\mu}}, \qquad \xi\in\mathbb{R}^n \tag{97}$$

and

$$|\xi^{\beta-\delta}\partial^{\alpha-\eta+\beta-\delta}\Lambda_{\mathcal{P}}(\xi)| \leq C_{\alpha,\beta,\eta,\delta}\Lambda_{\mathcal{P}}(\xi)^{1-\frac{|\alpha-\eta|}{\mu}}, \qquad \xi\in\mathbb{R}^n. \tag{98}$$

Let us observe now that

$$|\xi^{2\chi-\alpha}| \leq \Lambda_{\mathcal{P}}(\xi)^{2-\frac{|\alpha|}{\mu}}, \qquad \xi\in\mathbb{R}^n. \tag{99}$$

In fact, if $2\chi=\alpha$ $(\beta=0)$, $\xi^{2\chi-\alpha}\equiv 1$ and $|\alpha|=2|\chi|\leq 2\mu_1\leq 2\mu$, so that $\Lambda_{\mathcal{P}}(\xi)^{2-\frac{|\alpha|}{\mu}}\geq 1$ and the inequality (99) is trivially verified.

When $2\chi>\alpha$, for $\chi\in\mathcal{V}(\mathcal{P})\subset\mathcal{P}$, we have $\chi\cdot\nu\leq 1$ and, in view of definition of μ, $\alpha\cdot\nu\geq\frac{1}{\mu}|\alpha|$, when $\nu\in\mathcal{N}_1(\mathcal{P})$.

Since $2\mu-|\alpha|>0$, the previous inequalities yield $\frac{\mu}{2\mu-|\alpha|}(2\chi-\alpha)\cdot\nu\leq 1$, for $\nu\in\mathcal{N}_1(\mathcal{P})$, and then $\frac{\mu}{2\mu-|\alpha|}(2\chi-\alpha)\in\mathcal{P}$. So $|\xi^{2\chi-\alpha}|^{\frac{\mu}{2\mu-|\alpha|}}\leq\Lambda_{\mathcal{P}}(\xi)$, whence the estimate (99) follows.

So estimates (97), (98) and (99), jointly with (96), give (95) for $|\alpha+\beta|=k+1$.

Now the statement follows from (94) and (95); in fact we have:

$$\prod_{j=1}^{n}(1+\xi_j^2)^{\frac{\gamma_j}{2}}|\partial^{\alpha+\gamma}\Lambda_{\mathcal{P}}(\xi)| \leq \sum_{\sigma\leq\gamma}\binom{\gamma}{\sigma}|\xi^{\sigma}\partial^{\alpha+\gamma}\Lambda_{\mathcal{P}}(\xi)|$$

and for every $\sigma\leq\gamma$

$$|\xi^{\sigma}\partial^{\alpha+\varepsilon+\sigma}\Lambda_{\mathcal{P}}(\xi)| \leq C_{\sigma,\varepsilon,\alpha}\Lambda_{\mathcal{P}}(\xi)^{1-\frac{1}{\mu}|\alpha+\varepsilon|} \leq C'_{\sigma,\varepsilon,\alpha}\Lambda_{\mathcal{P}}(\xi)^{1-\frac{1}{\mu}|\alpha|},$$

where $\varepsilon=\gamma-\sigma$, since $\Lambda_{\mathcal{P}}(\xi)\geq c>0$ as a consequence of the left inequality in (93). ■

REMARK 17. – It is easy to see that $\Lambda_{\mathcal{P}}(\xi)$ satisfies (5), then thanks to (93) and Lemma 8.1 we conclude that, for any complete polyhedron $\mathcal{P}$ of $\mathbb{R}^n$, $\Lambda_{\mathcal{P}}(\xi)$ provides a weight function according to Definition 2.1.

It could be also proved that $\Lambda_{\mathcal{P}}(\xi)$ satisfies the estimate (46) with

$$\delta=\max_{\beta\in\mathcal{P}\setminus\mathcal{F}(P)} k(\mathcal{P},\beta)<1, \tag{100}$$

where $k(\mathcal{P},\beta)=\max_{\nu\in N_1(\mathcal{P})}\nu\cdot\beta$ for every $\beta\in\mathbb{Z}^n_+$ (see Garello [9], Proposition 3.2).

For any complete polyhedra $\mathcal{P}$ of $\mathbb{R}^n$ let us fix the attention on a semilinear partial differential equation of the type

$$p(x, \partial) u = F(x, \partial^\alpha u, f)_{\alpha \in \mathcal{P} \setminus \mathcal{F}(P)}, \tag{101}$$

where $p(x, \partial) := \sum_{\alpha \in \mathcal{P}} c_\alpha(x) \partial_x^\alpha$, $c_\alpha(x) \in C^\infty(V_{x_0})$ and $V_{x_0} \subset \mathbb{R}^n$ is an open neighborhood of $x_0 \in \mathbb{R}^n$.

About the nonlinear part, we assume that, for $M := 1 + \sum_{\alpha \in \mathbb{P} \setminus \mathcal{F}(\mathcal{P})} 1$, the function F maps $V_{x_0} \times \mathbb{C}^M$ into $\mathbb{C}$, it is locally smooth with respect to the real variable x and entire analytic in the complex variable $\zeta \in \mathbb{C}^M$; namely:

$$F(x, \zeta) = \sum_{\beta \in \mathbb{Z}_+^M} c_\beta(x) \zeta^\beta, \qquad c_\beta \in C^\infty(V_{x_0}), \quad \zeta \in \mathbb{C}^M,$$

where $\sup_{x \in K} |\partial_x^\alpha c_\beta(x)| \leq c_{\alpha, \beta} \lambda_\beta$ for any compact $K \subset V_{x_0}$, $\alpha \in \mathbb{Z}_+^n$ $\beta \in \mathbb{Z}_+^M$, and $F_1(\zeta) := \sum_{\beta \in \mathbb{Z}_+^M} \lambda_\beta \zeta^\beta$ is entire analytic.

For $\Lambda(\xi)$ weight function, $1 < p < \infty$ and $s \in \mathbb{R}$, we write $H^{s,p}_{\Lambda, \text{loc}}(\Omega)$, $\Omega \subset \mathbb{R}^n$ open set, for the localization of the Sobolev space $H^{s,p}_\Lambda$ given by $u \in \mathcal{D}'(\Omega)$ such that $\varphi u \in H^{s,p}_\Lambda$ for every $\varphi \in C_0^\infty(\Omega)$.

Applying then Corollary 7.2 we obtain that $F(x, g) \in H^{s,p}_{\Lambda_\mathcal{P}, \text{loc}}(V_{x_0})$ when all the components of the vector $g = (g_1, \ldots, g_M)$ belong to $H^{s,p}_{\Lambda_\mathcal{P}, \text{loc}}(V_{x_0})$ and $s > \dfrac{n}{(1-\delta)\mu_0 p}$ for δ given in (100).

PROPOSITION 8.1. – *For $\mathcal{P}$ complete polyhedron of $\mathbb{R}^n$ and $1 < p < \infty$ let us consider the equation* (101) *with $f(x) \in H^{t,p}_{\Lambda_\mathcal{P}, \text{loc}}(V_{x_0})$, where $t > \dfrac{n}{(1-\delta)\mu_0 p} + \delta$ and δ is defined by* (100). *Let us assume moreover that the linear part $p(x, \partial)$ is multi-quasi-elliptic, that is for some positive constants c, C:*

$$|p_1(x, \xi)| \geq c\Lambda_\mathcal{P}(\xi), \qquad \text{for } x \in V_{x_0}, \quad |\xi| > C, \tag{102}$$

where $p_1(x, \xi) = \sum_{\alpha \in \mathcal{F}(P)} c_\alpha(x)(-i\xi)^\alpha$ is the $\mathcal{P}$-principal symbol of $p(x, \partial)$.

Then any solution of (101) *taken in $H^{s,p}_{\Lambda_\mathcal{P}, \text{loc}}(V_{x_0})$, $\dfrac{n}{(1-\delta)\mu_0 p} + \delta < s \leq t$, belongs to the local space $H^{t+1,p}_{\Lambda_\mathcal{P}, \text{loc}}(V_{x_0})$.*

PROOF. – Let $u \in H^{s,p}_{\Lambda_\mathcal{P}, \text{loc}}(V_{x_0})$ be a solution of (101). It follows that $\partial^\alpha u \in H^{s-\delta,p}_{\Lambda_\mathcal{P}, \text{loc}}(V_{x_0})$, for any $\alpha \in \mathcal{P} \setminus \mathcal{F}(P)$, see [11].

Using (91), we obtain $F(x, \partial^\alpha u, f(x))_{\alpha \in \mathcal{P} \setminus \mathcal{F}(P)} \in H^{s-\delta,p}_{\Lambda_\mathcal{P}, \text{loc}}(V_{x_0})$, since $s > \dfrac{n}{(1-\delta)\mu_0 p} + \delta$. $p(x, \partial) u = F(x, \partial^\alpha u, f(x))_{\alpha \in \mathcal{P} \setminus \mathcal{F}(P)} \in H^{s-\delta,p}_{\Lambda_\mathcal{P}, \text{loc}}(V_{x_0})$, then under the assumption (102) we have $u \in H^{s+1-\delta,p}_{\Lambda_\mathcal{P}, \text{loc}}(V_{x_0})$ (see again [11]).

We can iterate the above argument N-times provided that $s + N(1-\delta) - \delta \leq t$.

Let N_0 be the first integer such that $s+N_0(1-\delta)-\delta>t$; we obtain

$$\partial^\alpha u \in H^{s+N_0(1-\delta)-\delta,\,p}_{\Lambda_{\mathcal{P}},\,\mathrm{loc}}(V_{x_0}) \subset H^{t,\,p}_{\Lambda_{\mathcal{P}},\,\mathrm{loc}}(V_{x_0})$$

which assures $p(x,\partial)\,u \in H^{t,\,p}_{\Lambda_{\mathcal{P}},\,\mathrm{loc}}(V_{x_0})$ and we can conclude u belongs to $H^{t+1,\,p}_{\Lambda_{\mathcal{P}},\,\mathrm{loc}}(V_{x_0})$. ■

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