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Curves of Genus Seven That Do Not Satisfy the Gieseker-Petri Theorem.

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Sunto. – *Nello spazio dei moduli delle curve di genere g, $\mathcal{M}_g$, indichiamo con $\mathcal{GP}_g$ il luogo delle curve che non soddisfano il teorema di Gieseker-Petri. In questo lavoro noi proviamo che nel caso di genere sette, $\mathcal{GP}_7$ è un divisore di $\mathcal{M}_7$.*

Summary. – *In the moduli space of curves of genus g, $\mathcal{M}_g$, let $\mathcal{GP}_g$ be the locus of curves that do not satisfy the Gieseker-Petri theorem. In the genus seven case we show that $\mathcal{GP}_7$ is a divisor in $\mathcal{M}_7$.*

0. – Introduction

Let $\mathcal{M}_g$ be the moduli space of smooth and irreducible projective curves of genus g. Let $C \in \mathcal{M}_g$ and let K_C be the canonical bundle of C. Let L be a line bundle on C and consider the Petri map $\mu_L : H^0(C,L) \otimes H^0(K_C \otimes L^{-1}) \to H^0(C,K_C)$.

The Gieseker-Petri theorem (see [5], p.285) says that for every line bundle L on a general curve $C \in \mathcal{M}_g$, μ_L is injective. Consider the locus

$$\mathcal{GP}_g := \{C \in \mathcal{M}_g | C \text{ does not satisfy the Gieseker-Petri theorem}\}.$$

By the Gieseker-Petri Theorem, $\mathcal{GP}_g$ is a closed Zariski subset in $\mathcal{M}_g$. Let C be a smooth irreducible projective curve of genus g and $L \to C$ a line bundle of degree d with $r+1 = h^0(C,L)$. The Brill-Noether number is defined by $\rho(g,r,d) := h^0(C,K_C) - h^0(C,L)h^0(C,K_C \otimes L^{-1}) = g-(r+1)(g-d+r)$. So if $\rho(g,r,d) < 0$, the Petri map μ_L is not injective. Let $\mathcal{M}^r_{g,d} := \{C \in \mathcal{M}_g | G^r_d(C) \neq \emptyset\}$. In [7] Steffen showed that if $\rho(g,r,d) < 0$, each component of $\mathcal{M}^r_{g,d}$ has codimension at most $-\rho(g,r,d)$ in $\mathcal{M}_g$. When $\rho = -1$, in [3] Eisenbud and Harris showed that $\mathcal{M}^r_{g,d}$ has a unique irreducible component of codimension one in $\mathcal{M}_g$. M. Teixidor showed (see [8], [9]) that the locus $\mathcal{M}^1_g := \{C \in \mathcal{M}_g | C \text{ has a autoresidual } g^1_{g-1}\}$ is an irreducible divisor in $\mathcal{M}_g$. The above results give us some divisorial components of $\mathcal{GP}_g$.

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We refer the above components as Eisenbud-Harris and Teixidor components respectively. To give all components of $\mathcal{GP}_g$ is a dificult problem, however for specific low genus we can study $\mathcal{GP}_g$ using the projective geometry of curves. For example in [2], the varieties W^r_d for general curves for low genus are described. Using this analysis one can describe all components of $\mathcal{GP}_g$ for $3 \leq g \leq 6$. The genus seven case is a non trivial case for studying $\mathcal{GP}_7$. In this work the main theorem is

THEOREM. $\mathcal{GP}_7$ is a divisor in $\mathcal{M}_7$.

We show the theorem with a degeneration argument. First we study curves of genus seven with a primitive g^r_d, $r = 1, 2$, $d = 1, ..., 6$, for which the Petri map is not injective. In sections 2.1-2.7 we describe two codimension one components of $\mathcal{GP}_7$. These are the Eisenbud-Harris and the Teixidor components. The third codimension one component of $\mathcal{GP}_7$ that we denote by $\mathcal{D}$, is formed by curves of genus seven with a g^1_5 for which the Petri map is not injective. To show that $\mathcal{D} \subset \mathcal{M}_7$ has codimension one we procced as follow: In proposition 2.8 we show that a pentagonal curve C of genus seven does not satisfy the Gieseker-Petri theorem if and only if it has a g^1_5 such that the residual $g^2_7 = |K_C - g^1_5|$ induces a birational morphism on a septic Γ in $\mathbb{P}^2$ with eight double points, seven of them lying on a conic. Now consider $\mathcal{V}^{7,7}$ the Severi variety of reduced and irreducible plane curves of degree seven and geometric genus seven. Consider $\mathcal{V}^{7,7}_8 \subset \mathcal{V}^{7,7}$ the locus consisting of plane curves having eight double points as singularities. In this case, the dimension of $\mathcal{V}^{7,7}_8$ is equal to 27 (see [5], p. 30). The quotient $\mathcal{V} := \mathcal{V}^{7,7}_8/PGL(3, \mathbb{C})$ of $\mathcal{V}^{7,7}_8$ with the automorphisms of $\mathbb{P}^2$ is of dimension 19. Consider the subvariety $\mathcal{D}_0$ of $\mathcal{V}$ defined by $\mathcal{D}_0 := \{\Gamma \in \mathcal{V} |$ seven double points of Γ lying on a conic$\}$. A consequence of the corollary 2.10 shows that $\mathcal{D}_0$ is irreducible and of dimension 17 in $\mathcal{V}$. In section 3 we consider the natural morphism $\phi : \mathcal{V} \to \mathcal{M}_7$, $\Gamma \to \phi(\Gamma) =$ normalization of Γ. Since by excess linear series (see [2], p. 329) a pentagonal curve C of genus seven has $\dim W^1_5(C) = 1$, we have that the image $\mathcal{D} := \phi(\mathcal{D}_0)$ has codimension one in $\mathcal{M}_7$ if for each $C \in \mathcal{D}$, the fiber $\phi^{-1}(C) \simeq W^1_5(C)$ intersects only a finite number of elements of $\mathcal{D}_0$. This means that $\mathcal{D}$ has codimension one in $\mathcal{M}_7$ if for $C \in \mathcal{D}$ we have that for the general element $L \in W^1_5(C)$, the Petri map μ_L is injective. To show this, in 3.2 we degenerate a curve $\Gamma \in \mathcal{D}_0$ to a compact type curve $X_0 = \mathbb{P}^1 \cup Z$, $\{p\} := \mathbb{P}^1 \cap Z$, where Z is a sextic with three not collinear double points. By stable reduction ([5], p. 118) the normalization C_0 of Z is the stable limit of X_0. In Proposition 3.3 we show that for the general linear series $|D| = g^1_5$ on C_0 the Petri map μ_D is injective. This implies that for each $C \in \mathcal{D}$ and for a general element L of $W^1_5(C)$, the Petri map μ_L is injective. So we have that $\mathcal{D}$ is an irreducible component of $\mathcal{GP}_7$ of codimension one in $\mathcal{M}_7$. Thus the components of $\mathcal{GP}_7$ are $\mathcal{M}^1_7$, $\mathcal{M}^1_{7,4}$, $\mathcal{D}$. Thus of this way we will prove the theorem.

It is an interesting open problem to prove that every irreducible component of $\mathcal{GP}_g$ is divisorial in $\mathcal{M}_g$.

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1. – Preliminaries.

1.1. – Let C be a smooth projective irreducible curve of genus g, D a divisor on C and K_C the canonical bundle on C. We say that the linear series $|D|$ is primitive if $|D|$ and $|K_C - D|$ are free of base points.

Let D be a divisor on C. We write $r = r(D) := h^0(C, D) - 1$. Suppose that $|D|$ is not primitive and let $p \in C$ be a base point of $|K_C - D|$, by Riemann-Roch we have that $r(D+p) = \dim\ |D+p| = \deg\ D + 1 - g + h^0(K - (D+p))$, now since $h^0(K_C - D - p) = h^0(K_C - D) = r(D) + g - \deg\ D$, then $r(D+p) = r(D) + 1$, that is, p is not base point of $|D+p|$. In this way we transferred a base point of $|K_C - D|$ to the series $|D|$ obtaining two series $|K_C - (D+p)|$ and $|D+p|$, residual one to the other with respect to the canonical series and of dimension $r - d + g$ and $r+1$ respectively. Iterating this process we can obtain from a pair of non primitive series $(|D|, |K_C - D|)$ a pair of primitive series $(|D'|, |K_C - D'|)$.

LEMMA 1.2. – If there exists $|D|$ such that $\mu_D : H^0(C, D) \otimes H^0(C, K_C - D) \to H^0(C, K_C)$ is not injective then there exists a primitive series $|D'|$ such that $\mu_{D'}$ is not injective.

PROOF. – Let $|D|$ be a non primitive series such that $\mu_{0,D}$ is not injective. We can write $|D| = p_1 + \cdots + p_n + |D_1|$ and $|K_C - D| = q_1 + \cdots + q_m + |D_2|$, where $|D_1|$ and $|D_2|$ are free of base points. Let L_i, $i = 1, 2$ be the line bundles defined by D_i. Consider the map $\mu_1 : H^0(C, L_1) \otimes H^0(C, L_2) \to H^0(C, L_1 \otimes L_2)$. There exists an isomorphism ψ : Kernel $\mu_D \to$ Kernel μ_1 given by $\psi(\sum s_i \otimes t_i) = \sum \frac{s_i}{f_1} \otimes \frac{t_i}{f_2}$, where $s_i \in H^0(C, D)$, $t_i \in H^0(C, K_C - D)$, $(f_1) = p_1 + \cdots + p_n$, $(f_2) = q_1 + \cdots + q_m$. Applying the above process of transfering base points we obtain $D'_1 = D_1 + q_1 + \cdots + q_m$ and $D'_2 = D_2 + p_1 + \cdots + p_n$ such that $D'_2 = K - D'_1$ and $|D'_1|$, $|D'_2|$ are primitive. Since $D_1 \subset D'_1$, $D_2 \subset D'_2$ then $H^0(C, L_i) \subset H^0(C, L'_i)$, $i = 1, 2$, where L'_i is the line bundle defined by D'_i. We have the following commutative diagram

$$\begin{array}{ccc} H^0(C, L_1) \otimes H^0(C, L_2) & \xrightarrow{\mu_1} & H^0(C, L_1 \otimes L_2) \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi}\downarrow \\ H^0(C, L'_1) \otimes H^0(C, K_C \otimes (L'_1)^{-1}) & \xrightarrow{\mu_{L'_1}} & H^0(C, K_C) \end{array}$$

Since Kernel $\mu_1 \neq 0$, then Kernel $\mu_{L'_1} \neq 0$, that is, Kernel $\mu_{D'_1} \neq 0$. $\square$

Thus we consider primitive linear series for which the multiplication map is not injective. Since $W^r_d(C) \simeq W^{g-d+r-1}_{2g-2-d}(C)$, $|D| \to |K_C - D|$, we only consider special primitive linear series $|D|$ such that $r = h^0(C, D) - 1 > 0$, and $g - d + r = h^0(C, K_C - D) > 0$, with $d = \deg D < g$. By Clifford theorem ([2]), a linear series $|D|$ of degree $d \leq 2g - 1$ satisfies $2r \leq d$ with equality if and only if $D = 0$, D is the canonical divisor or C is hyperelliptic and D is a multiple of the hyperelliptic involution. Then for our analysis we only need to consider special linear series $|D|$ such that $r > 0$ and $2r \leq d < g$.

Also we use *The Base point free pencil trick* ([2], p.126): if $|D|$ is a pencil free of base points, we have that Ker $\mu_D \simeq H^0(C, K_C(-2D))$.

2. – The locus $\mathcal{GP}_7$.

Let $C \in \mathcal{M}_7$. We study primitive linear series on C of dimension r and degree d such that $2r \leq d \leq g - 1 = 6$, for which the Petri map is not injective.

2.1. $r = 1$, $d = 2, 3, 4$. In this case we have $\rho(7, 1, d) < 0$, and by gonality, $\mathcal{M}^1_{7,2} \subseteq \mathcal{M}^1_{7,3} \subseteq \mathcal{M}^1_{7,4}$. For $d = 4$, $\rho(7, 1, 4) = -1$, so by [3], $\mathcal{M}^1_{7,4}$ is an irreducible divisor in $\mathcal{M}_7$.

2.2. $r = 1, d = 5$. We postpone this case.

2.3. $r = 1, d = 6$. By the base point free pencil trick, the multiplication map for a g^1_6 on C is not injective if and only if g^1_6 is autoresidual. By [8, 9], the locus $\mathcal{M}^1_7$ is an irreducible divisor in $\mathcal{M}_7$.

2.4. $r = 2$, $d = 4$. By the genus formula, a genus seven curve C with a g^2_4 is hyperelliptic, then $C \in \mathcal{M}^1_{7,2}$.

2.5. $r = 2$, $d = 5$. By the genus formula a curve of genus seven has no primitive g^2_5.

2.6. $r = 2$, $d = 6$. Let $C \in \mathcal{M}_7$ be a non hyperelliptic curve with a g^2_6 that induces a map $\psi : C \to \mathbb{P}^2, X := \psi(C)$. If the degree of X is two or three, either X is trigonal or bielliptic, so $C \in \mathcal{M}^1_{7,4}$. If the degree of X is six, either X has a triple point or it has three double points, in any case X is either trigonal or tetragonal. So a curve C with a g^2_6 belongs to $\mathcal{M}^1_{7,4}$.

2.7. $r = 3, d = 6$. By Castelnuovo's bound ([2], p. 116) a curve of genus seven with a g^3_6 is hyperelliptic.

Now we study the case 2.2. Let C be a pentagonal curve. The residual of a primitive g^1_5 on C is a base point free g^2_7. This g^2_7 defines a birational map of C onto a plane curve in $\mathbb{P}^2$. Since a septic curve of genus seven with a triple point has a g^1_4 cut out by lines through the triple point, we only consider septic curves with double points. Let Γ be such a curve and $f : C \to \Gamma$ the normalization of Γ. We denote by $\triangle_\Gamma$ the scheme of singular points of Γ and $\triangle := f^*(\triangle_\Gamma)$, note that $\triangle$ is a divisor of degree sixteen. By the genus formula the lenght of $(\triangle_\Gamma) = 8$, i.e. $\triangle_\Gamma$ consists of eight points which can be infinitely near. However by our assumption that Γ has only double points the scheme $\triangle_\Gamma$ is in any case curvilinear.

PROPOSITION 2.8. –

a) Let Γ be a plane curve of degree seven and genus seven with only double points and let $f : C \to \Gamma$ be its normalization. Suppose that there is a conic Q such that the scheme theoretic intersection of Q with $\triangle_\Gamma$ has lenght equal to seven, i.e. $f^*(Q)$ contains a divisor of degree fourteen contained in $\triangle$, then C does not satisfy the Gieseker-Petri theorem.

b) Conversely if C is a pentagonal curve of genus seven such that there is a $|D| = g^1_5$ on C for which μ_D is not injective, then there is in $\mathbb{P}^2$ a birational model Γ of C of degree seven with only double points such that the g^1_5 is cut out by lines passing through a double point p and there is a conic Q such that Q contains $\triangle_\Gamma - \{p\}$.

PROOF. – First I will prove the part (a). Also I will only consider the most complicated case in which the support of $\triangle_\Gamma = \{x\}$. The other cases are easier and can be left to the reader.

If the support of $\triangle_\Gamma = \{x\}$, then Γ has eight infinitely near double points. Let $\eta := f^*(x)$, so that η is a divisor of degree two and $\triangle = 8\eta$. Our hypothesis means that the pullback f^*Q on C contains 7η. Consider the $|D| = g^1_5$ cut out on C by the lines through x. Let ℓ_1, ℓ_2 be general such lines, cutting out on C two effective divisors $D_1, D_2 \in |D|$. The pullback of $Q + \ell_1 + \ell_2$ contains $9\eta + D_1 + D_2 \sim 9\eta + 2D$. By adjunction formula ([2],p. 53), one has $K_C \sim \mathcal{O}_C(4)(-\triangle)$, and therefore $K_C - 2D$ is effective. Since ker $\mu_D \simeq H^0(C, K_C - 2D)$, we have the assertion.

The proof of part (b) is as follow: Let $|D| = g^1_5$ be a primitive linear series on C for which μ_D is not injective. Consider $g^2_7 = K_C - D$. This g^2_7 determines a birational morphism $C \to \Gamma \subset \mathbb{P}^2$ and Γ has only double points. Since C fails the Gieseker-Petri theorem for the g^1_5, we have that ker $\mu_D \simeq H^0(C, K_C - 2D)$, but $K_C - 2D \sim g^2_7 - g^1_5$ is effective, so necessarily the g^1_5 is cut out by a pencil of lines through a singular point p of Γ. The existence of the conic Q is now clear. □

REMARK 1. – Suppose we have a curve Γ like in proposition 2.8. Then we claim that the conic Q cannot be singular at any point of $\triangle_\Gamma$. Suppose in fact that $p \in \triangle_\Gamma$ is singular for Q. Suppose that $Q = L_1 + L_2$, where L_1, L_2 are lines through p. Suppose that L_1 contains i double points infinitely near to p.

Therefore Q has to contain $7-i$ more double points of Γ which can be distinc or infinitely near. If j of such points are on L_1 one has $2i+2j\leq 7$. If $L_1=L_2$ we must have $i+j=7$ which gives a contradiction. Suppose then that $L_1\neq L_2$ and suppose that k double points lie on L_2 off p. Then $2+2k\leq 7$ and moreover $i+j+k=7$ which again gives a contradiction. Let $\mathcal{V}^{7,7}$ be the family of reduced, irreducible plane curves of degree seven with geometric genus seven. This is an irreducible variety (see [5] p.30). Let $\mathcal{V}_8^{7,7}$ be the Zariski open subset of $\mathcal{V}^{7,7}$ defined by all irreducible curves in $\mathcal{V}^{7,7}$ with only eight double points. The dimension of $\mathcal{V}_8^{7,7}$ is 27 (see [5] p.30). We will denote by $\mathscr{X}$ the Zariski open subset of the Hilbert Scheme of locally complete intersection zero-dimensional subschemes in $\mathbb{P}^2$ of length 8, formed by curvilinear subschemes. Now consider the subvariety $\mathscr{T}\subseteq\mathcal{V}_8^{7,7}\times\mathscr{U}\times\mathscr{X}$, where $\mathscr{U}\subset|\mathcal{O}_{\mathbb{P}^2}(2)|$ is the open set of smooth conics in $\mathbb{P}^2$. The variety $\mathscr{T}$ consists of all triples $(\Gamma,Q,\triangle)$ such that $\triangle=\triangle_\Gamma$ and $Q\cap\triangle_\Gamma$ contains seven points. Note that by proposition 2.8. the image of the projection map $\mathrm{Pr}_1:\mathscr{T}\to\mathcal{V}_8^{7,7}$ is the subvariety $\Sigma:=\mathrm{Pr}_1(\mathscr{T})$ of $\mathcal{V}_8^{7,7}$ consisting of curves for which the Gieseker-Petri fails for a g^1_5.

PROPOSITION 2.9. – $\mathscr{T}$ is irreducible of dimension 25.

PROOF. – Consider the projection $\pi_3:\mathscr{T}\to\mathscr{X}$. Let $S\in\pi_3(\mathscr{T})$ be any point. Namely S is a curvilinear scheme of lenght 8, seven points of which lie on a irreducible conic. Therefore, if $(\Gamma,Q,S)\in\pi_3^{-1}(S)$, then Q is uniquely determined by S. Moreover Γ belongs to the linear system of plane curves of degree seven which are singular at S. Let $\mathscr{L}_S$ be this linear system.

FIRST CLAIM: The dimension of $\mathscr{L}_S=11$ and the general element of $\mathscr{L}_S$ is irreducible of geometric genus seven.

PROOF OF THE FIRST CLAIM: Let $\mathscr{L}$ be the proper transform of $\mathscr{L}_S$ on the surface X which is $\mathbb{P}^2$ blowed up at S. Let $\tilde{Q}$ the proper transform of Q on X. Consider the exact sequence:

$$0\to\mathscr{L}(-\tilde{Q})\to\mathscr{L}\to\mathscr{L}|_{\tilde{Q}}\to 0$$

we remark that $\mathscr{L}|_{\tilde{Q}}=\mathcal{O}_{\tilde{Q}}$. Moreover $\mathscr{L}(-\tilde{Q})$ is the proper transform on X of the linear system of quintics of $\mathbb{P}^2$ with a double point off Q and seven points on Q. It is easily seen that $h^1(\mathscr{L}(-\tilde{Q}))=0$ and $h^0(\mathscr{L}(-\tilde{Q}))=11$. Hence $h^0(\mathscr{L})=12$. The proof of the irreducibility of the general element of $\mathscr{L}_S$ is easily obtainded by Bertini theorem. We omit the details. In order to finish the proof it is sufficient to show that:

SECOND CLAIM: $\pi_3(\mathscr{T})$ is irreducible of dimension 14.

PROOF OF THE SECOND CLAIM. Let $\pi_{23}:\mathscr{T}\to\mathscr{U}\times\mathscr{X}$ be the projection on the second and third factor and let $\mathscr{Y} := \pi_{23}(\mathscr{T})$ be. The above discussion implies that $\mathscr{Y}$ is the variety formed by pairs $(Q, \triangle)$ such that $Q \cap \triangle$ consists of seven points. Let $\pi_1 : \mathscr{Y} \to \mathscr{U}$ be the projection to the first factor which is dominant. The fiber is of course irreducible of dimension 9. This shows that $\dim \mathscr{Y} = 14$. On the other hand $\pi_3 : \mathscr{Y} \to \mathscr{X}$ is finite, so the assertion follows. □

COROLLARY 2.10. – The subvariety $\Sigma := \mathrm{Pr}_1(\mathscr{T}) \subset \mathcal{V}_8^{7,7}$ is irreducible of dimension 25.

PROOF. – The map $\mathrm{Pr}_1 : \mathscr{T} \to \mathcal{V}_8^{7,7}$ is generically finite. By proposition 2.9. we have that Σ is irreducible of dimension 25.

3. – Proof of the theorem.

In this section we will prove that $\mathcal{GP}_7$ is a divisor in $\mathcal{M}_7$.

Consider the natural morphism $\phi : \Sigma \to \mathcal{M}_7$ where the general fiber of this map has dimension at least 8, because $PGL(3, \mathbb{C})$ acts on Σ and any orbit lies in a fiber of ϕ. Let $\mathcal{V} := \mathcal{V}_8^{7,7}/PGL(3, \mathbb{C})$. Note that $\mathcal{D}_0 := \Sigma/PGL(3, \mathbb{C}) \subset \mathcal{V}$ is of dimension 17. Now we will prove that the general fiber of $\phi : \mathcal{D}_0 \to \mathcal{M}_7$ is zero-dimensional, that is, $\mathcal{D} := \phi(\mathcal{D}_0)$ has codimension one in $\mathcal{M}_7$. This will prove that $\mathcal{D} := \phi(\mathcal{D}_0)$ is an irreducible component of $\mathcal{GP}_7$ of codimension one in $\mathcal{M}_7$. We will prove the theorem with a degeneration argument following the next steps:

3.1. Consider the conic $Q(x, y, w) = y^2 - txw$. When $t \to 0$ we obtain that Q tends to the double line $y^2 = 0$. In $\mathbb{CP}^2$ consider the points $[t : t : 1]$, $[4t : 2t : 1], [9t : 3t : 1] \in Q$. Restricting to $\mathbb{C}^2$ we have the points $p_1(t) = (t, t)$, $p_2(t) = (4t, 2t)$, $p_3(t) = (9t, 3t)$ on the conic $y^2 - tx$. Let $I_1(x) = \langle x - t, y - t \rangle$, $I_2(t) = \langle x - 4t, y - 2t \rangle$, $I_3(t) = \langle x - 9t, y - 3t \rangle$ the ideals that define $p_1(t)$, $p_2(t)$, $p_3(t)$ respectively. The schemes $\mathrm{Spec}\, \mathbb{C}[x, y]/I_k^2(t)$ define $p_k(t)$ as double points for $k = 1, 2, 3$. Set $J(t) := \bigcap_{k=1}^3 (I_k^2(t))$. For $t \neq 0$, the scheme $S_t := \mathrm{Spec}\, \mathbb{C}[x, y]/J(t)$ is the union of these three double points. Using ([4]) we have that a Groebner basis for $J(t)$ is given by the polinomials $x^3 - 12x^2y + 47xy^2 - 60y^3 + 11x^2t - 84xyt + 157y^2t + 36xt^2 - 132yt^2 + 36t^3$, $y^4 - 2xy^2t + x^2t^2$, $xy^3 - x^2yt - 6xy^2t + 11y^3t + 6x^2t^2 - 11xyt^2 - 6y^2t^2 + 6xt^3$, $x^2y^2 - 12x^2yt + 22xy^2t + 36x^2t^2 - 144xyt^2 + 121y^2t^2 + 72xt^3 - 132yt^3 + 36t^4$. So we have that $J(0) = \langle f_1, f_2, f_3, f_4 \rangle$, where $f_1 = x^3 - 12x^2y + 47xy^2 - 60y^3$, $f_2 = y^4$, $f_3 = xy^3$, $f_4 = x^2y^2$. Note that $J(0)$ defines the flat limit for $t \to 0$ of the scheme S_t. Remark that $f_1 = (x - 4y)(x - 3y)(x - 5y)$. It is then clear that $J(0)$ consists of all polinomials $f(x, y)$ such that $f = 0$ defines a curve with an ordinary triple point at the origin with tangent lines $x = 3y, x = 4y, x = 5y$. In conclusion, the limit of the three double points, at $p_1(t), p_2(t), p_3(t)$ is an ordinary triple point with fixed tangent lines. In a similar way when we take the points

$[\frac{1}{t}:1:1], [\frac{2}{t}:4:1], [\frac{3}{t}:9:1]$ and $t \to 0$, the limit of these three points as double points will be another ordinary triple point. Finally we can let another double point $p(t)$ on Q tend for $t = 0$ to the point [1:0:1]. for instance take $p(t) = [1 : \sqrt{t} : 1]$

3.2. If we apply the above specialization to a conic Q on which we have seven double points of an irreducible curve Γ_t of degree seven and genus seven, we have that we can specialize this curve Γ_t to a curve Γ_0 of degree seven with two triple points and one double point on a line ℓ, so that the line ℓ splits off Γ_0, that is, $\Gamma_0 = \ell \cup Z$, where Z is a sextic curve with three double points. Notice that we can make the above limit in such a way that the three double points of Z are not collinear. Let $\psi_t : C_t \to \Gamma_t$ be the normalization of $\Gamma_t, t \neq 0$. $\{C_t\}$ form the fibers of a family $\pi : \mathcal{X}^* \to D(0,1) - \{0\}$, where $D(0,1) := \{t \in \mathbb{C} : |t| < 1\}$. By stable reduction ([5], p. 118), we can make a base change and complete the family $\pi : \mathcal{X}^* \to D(0,1) - \{0\}$ to a family $\pi : \mathcal{X} \to D(0,1)$ of stable curves. In this case $\mathcal{X}$ is smooth, and the stable limit of the C_t is the central fiber of the family $\pi : \mathcal{X} \to D(0,1)$ which is the normalization C_0 of Z. The dimension of $W^1_5(C_0)$ is one: We apply Martens's theorem ([2, p. 191]) and the proof of the Mumford theorem ([2, p. 193]) to the case $d = 5, g = 7$ to deduce that dim $W^1_5(C_0) = 1$.

Remark 2. – We remark that C_0 has only three g^1_4, i.e. the ones cut out by the lines through the double points of Z. We recall that the double points of Z are not collinear. It is clear that C_0 is not trigonal. Let g^1_4 be on Z and $D = q_1 + q_2 + q_3 + q_4 \in g^1_4$ a general divisor. Note that D imposes only three conditions to cubics through the double points p_1, p_2, p_3 of Z. Consider the conic Q passing through p_1, p_2, p_3, q_1, q_2, we claim that $q_3, q_4 \in Q$, otherwise, by monodromy q_3, q_4 both do not lie on Q. Let ℓ_0 be a general line through q_3 so that $q_4 \notin \ell_0$. Then $Q + \ell_0$ contains $p_1, p_2, p_3, q_1, q_2, q_3$ but not q_4 a contradiction. Now I claim that Q splits in the line ℓ_{12} through p_1, p_2 and a line ℓ containing p_3, q_1, q_2, q_3, q_4. In fact, if one uses the Cremona transformation based at p_1, p_2, p_3, then Z is mapped to another sextic curve with three double points and the g^1_4 is now contained in the g^2_6 cut out by the lines, hence it is cut out by the lines through a double point. This implies that also on Z the same happens.

Now note that one component W_1 of $W^1_5(C_0)$ is formed by the family of g^1_5 cut out by lines through a general point of the sextic Z. A second component W_2 is formed by the g^1_5's cut out by conics through the three double points p_1, p_2, p_3 and a general point of Z. We can go from W_1 to W_2 via the quadratic Cremona transformation based at the double points p_1, p_2, p_3 of Z. A third componet W_{p_1} is formed by the g^1_5's given by $g^1_4 + q$, $q \in Z$ general, where the g^1_4 is cut out by lines through the double point p_1. In analogous way we have the components W_{p_2}, W_{p_3}. Now take a g^1_5 not belonging either to W_1 or Wp_i, $i = 1, 2, 3$. Let $D = q_1 + \cdots + q_5 \in g^1_5$ be a general divisor. We have that no three points of D are no collinear, then D lies on a irreducible conic Q_D. Suposse that Q_D does not contain p_1, p_2, p_3. Since the linear system $\mathscr{A} := |K_{C_0} - D|$ is cut out by of cubics through $p_1, p_2, p_3, q_1, , q_5$ and has dimension two, we can split off Q_D for a cubic of $\mathscr{A}$, and the residual $\tilde{D}$ would be a line containing p_1, p_2, p_3 which is not possible. In a similar way we see that it cannot be the case that Q_D does not contain some of the points p_1, p_2, p_3, in other words $p_1, p_2, p_3 \in Q_D$ and therefore $g^1_5 \in W_2$.

PROPOSITION 3.3. – Let $f : C_0 \to Z$ be the normalization of Z. μ_D is not injective only for a finite number of pencils g^1_5 on C_0.

PROOF. – We have that for a $|D| = g^1_5$, ker $\mu_D \simeq H^0(C_0, K_{C_0} - 2D)$, where $K_{C_0} - 2D$ is the pullback under f of the linear system of cubics through p_1, p_2, p_3 and D_1, D_2 with $D_1, D_2 \in |D|$. Note that $H^0(C, K_{C_0} - 2D) = 0$ if D belongs either W_1 or W_2. Let $D \in W_{p_i}$ for some $i = 1, 2, 3$. By simplicity assume that $D \in W_{p_1}$. Thus we have that $|D| = g^1_4 + q$, $q \in Z$ general and the g^1_4 is cut out by the lines through the double point point p_1. Every divisor $D \in |D|$ has four points lying on a line through p_1. A section of $H^0(C, K_{C_0} - 2D)$ will be a cubic G that has four intersection points with two lines ℓ_1, ℓ_2 where the four points of $D_1, D_2 \in |D|$ respectively lie. Thus G splits in $G = \ell_1 \cdot \ell_2 \cdot \ell_3$, where ℓ_3 is the line through p_2, p_3. If $G \neq 0$, q must lie on ℓ_3, that is, q must be one of the other two points z_1, z_2 on Z where ℓ_3 intersects Z. So for $D \in W_{p_1}$, μ_D is not injective only for $|D| = g^1_4 + z_j$ $j = 1, 2$. □

The following corollary is now clear.

COROLLARY 3.4. – Let $C \in \mathcal{D}$. Let L be a general point in $W^1_5(C)$, then the Petri map $\mu_L : H^0(C, L) \otimes H^0(C, K \otimes L^{-1}) \to H^0(C, K)$ is injective.

So we have shown that $\mathcal{D}$ is an irreducible component of $\mathcal{GP}_7$ of codimension one in $\mathcal{M}_7$. Thus $\mathcal{M}^1_7$, $\mathcal{M}^1_{7,4}$, $\mathcal{D}$ are the components of $\mathcal{GP}_7$, then our theorem is proved.

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