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Rosenthal and Semi-Tauberian Linear Relations in Normed Spaces (*).

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Sunto. – *Si introduce la classe delle relazioni lineari di Rosenthal in spazi normati e si studia in termini dei suoi coniugati primi e secondi. Si analizza il rapporto fra una relazione lineare di Rosenthal e il suo coniugato. Nell'articolo si studiano inoltre le relazioni lineari semi-Tauberiane che seguono il modello adottato nello studio delle relazioni lineari Tauberiane. Si dimostra che le relazioni lineari semi-Tauberiane condividono alcune delle proprietá delle relazioni lineari Tauberiane e che stanno in relazione alle relazioni lineari di Rosenthal nello stesso modo in cui le relazioni lineari Tauberiane si trovano in relazione con le relazioni lineari debolmente compatte. Si descrivono esempi e si discutono casi particolari,* F_+ *e le relazioni lineari strettamente singolari.*

Summary. – *The class of Rosenthal linear relations in normed spaces is introduced and studied in terms of their first and second conjugates. We investigate the relationship between a Rosenthal linear relation and its conjugate. In this paper, we also study the semi-Tauberian linear relations following the pattern followed for the study of the Tauberian linear relations. We prove that the semi-Tauberian linear relations share some of the properties of Tauberian linear relations and they are related to Rosenthal linear relations in the same way as Tauberian linear relations are related to weakly compact linear relations. We describe examples and investigate special cases: in particular,* F_+ *and strictly singular linear relations.*

1. – Introduction.

Let $T : X \to Y$ be a bounded operator where X and Y are Banach spaces. T is called Tauberian (resp. semi-Tauberian) if every bounded sequence (x_n) in X such that (Tx_n) is weakly convergent (resp. weak Cauchy) has a weakly convergent (resp. weak Cauchy) subsequence.

Bounded Tauberian operators in Banach spaces were originally introduced

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by Kalton and Wilansky [18] and they have found application in many different situations: summability [13], factorisation of operators [11] and [22], equivalence between Radon-Nikodym property and Krein-Milman property [24], generalised Fredholm operators [25] and [27], etc.

The class of bounded semi-Tauberian operators in Banach spaces is studied in [14] (under the name of SRo_+-operators) in the context of semigroups of generalised Fredholm operators and also in [6] and [17] where they are applied to study isomorphic properties of $L_1(\mu, X)$.

Tauberian linear relations in arbitrary normed spaces were introduced and studied by Cross [10] generalising the bounded case.

The purpose of this paper is to study the classes of Rosenthal and semi-Tauberian operators in the more general setting of linear relations between normed spaces.

In Section 2, the class of Rosenthal linear relations is introduced and studied in terms of their first and second conjugates. We analyse the connection between a Rosenthal linear relation and its conjugate. We also give a condition under which Rosenthal linear relations are strictly singular.

In Section 3 we introduce and study the class of semi-Tauberian linear relations following the pattern followed for the study of Tauberian linear relations. We prove that they share some of the properties of Tauberian linear relations. We show that semi-Tauberian linear relations are related to Rosenthal linear relations in the same way as Tauberian linear relations are related to weakly compact linear relations.

Examples of Rosenthal and semi-Tauberian linear relations are exhibited.

NOTATIONS. – We follow the notation and terminology of the book [10]: X and Y are normed spaces, B_X the closed unit ball of X, X' and X'' the first and second dual spaces of X respectively. If $M \subseteq X$ and $N \subseteq X'$ are subspaces, then $M^{\perp} = \{x' \in X' : x'(x) = 0 \text{ for } \textit{all } x \in M\}$, $N^{\top} = \{x \in X : x'(x) = 0 \textit{ for all } x' \in N\}$, J_M^X (or simply J_M) is the natural injection of M into X, Q_M is the quotient map of X onto X/M and J is the canonical injection of a given normed space into its second dual. We write J_X for the injection of X into its completion $\tilde{X}$ and $K(X) := \{x'' \in X'' : \text{there exists a sequence } (x_n) \text{ in } X \text{ such that } x'' = \sigma(X'', X') - \lim Jx_n\}$.

A linear relation or multivalued linear operator $T : X \to Y$ is a mapping from a subspace $D(T) \subseteq X$, called the domain of T, into $P(Y) \setminus \{\phi\}$ (the collection of nonempty subsets of Y) such that $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all nonzero scalars α, $\beta \in K$ and $x_1, x_2 \in D(T)$. The class of such relations T is denoted by $LR(X, Y)$. If T maps the points of its domain to singletons, then T is said to be a single valued linear relation or operator. Continuous everywhere defined linear operators referred to as bounded operators.

The graph $G(T)$ of $T \in LR(X, Y)$ is $G(T) := \{(x, y) \in X \times Y : x \in D(T),$

$y \in Tx\}$. Let M be a subspace of $D(T)$. Then the restriction $T\,|_M$ is defined by $G(T\,|_M\,) := \{(m, y) : m \in M,\ y \in Tm\}$. For any subspace M of X we write $T\,|_M = T\,|_{M \cap D(T)}$. The inverse of T is the linear relation T^{-1} defined by $G(T^{-1}) := \{(y, x) \in Y \times X : (x, y) \in G(T)\}$. If T^{-1} is single valued, then T is called injective, that is, T is injective if and only if its null space $N(T) := T^{-1}(0) = \{0\}$, and T is said to be surjective if its range $R(T) := T(D(T)) = Y$.

The closure and completion of a linear relation T, denoted $\overline{T}$ and $\widetilde{T}$, respectively, are defined in terms of their corresponding graphs: $G(\overline{T}) := \overline{G(T)} \subseteq X \times Y$, $G(\widetilde{T}) := \widetilde{G(T)} \subseteq \widetilde{X} \times \widetilde{Y}$.

A linear relation $T \in LR(X, Y)$ is said to be closed if $G(T)$ is a closed subspace, closable if $\overline{T}$ is an extension of T, continuous if for any neighbourhood $\Omega \subseteq R(T)$, the inverse image $T^{-1}(\Omega)$ is a neighbourhood in $D(T)$, open if its inverse T^{-1} is continuous, partially continuous (resp. F_+) if there exists a finite codimensional subspace M of X such that $T\,|_M$ is continuous (resp. $T\,|_M$ is injective and open) and strictly singular if there is no infinite dimensional subspace M of $D(T)$ for which $T\,|_M$ is injective and open.

If $T \in LR(X, Y)$, we shall denote $Q^Y_{\overline{T(0)}}$ by Q_T. Clearly $Q_T T$ is single valued and it can be shown that T is continuous if and only if $\|T\| := \|Q_T T\| < \infty$ [10, II. 3.2]. Given $T \in LR(X, Y)$, let D_T denote the vector space $D(T)$ normed by $\|x\|_T := \|x\| + \|Tx\|$ for $x \in D(T)$, and $G_T \in LR(D_T, X)$ the identity injection of D_T into X, that is, $D(G_T) = D_T$, $G_T(x) = x$, for $x \in D_T$.

The minimum modulus of T is the quantity $\gamma(T) := \sup\{\lambda \geq 0 : \|Tx\| \geq \lambda d(x, N(T))\ \ for\ x \in D(T)\}$. T is open if and only if $\gamma(T) > 0$ [10, II. 3.2].

As remarked by Wilcox [26], single valued maps were favoured as the natural morphisms in the rigorous development of topology at the start of the 20th century. Nevertheless, limits of sequences of sets were considered by Painlevé in 1909 (see, e.g., [5]) and later by Kuratowski [19] in 1958. Furthermore, extension problems in topology led to the study of selections or single valued parts of upper and lower semi-continuous set valued maps [21]. Multivalued maps, of course, occur quite naturally, but the earnest development of mathematical methods for set valued or multivalued problems came in the 1960's.

Linear relations were introduced into Functional Analysis by J. von Neumann [23] motivated by the need of considering conjugates of non-densely defined linear differential operators.

Problems in optimisation and control also lead to the study of set valued maps and differential inclusions (see, e.g., Aubin and Cellina [4], Clarke [7], among others). Studies of convex processes, tangent cones, etc, form part of the theory of convex analysis developed to deal with nonsmooth problems in viability and control theory, for example. Some of the basic topological properties from this area coincide with the core of the topological results for multivalued linear operators.

Other works on multivalued linear operators include the treatise on partial differential relations by Gromov [16] and the application of multivalued methods to the solution of differential equations by Favini and Yagi [12].

A recent work on linear relations semi-Fredholm type and other classes related to them is the book «Multivalued linear Operators» by Cross [10]. This is the first book that have been published on these classes of linear relations. It contains an impressive amount of information, including many unpublished results and open problems.

2. – Rosenthal Linear Relations.

Recall that a bounded operator acting between Banach spaces is called weakly compact (resp. Rosenthal) if it transforms bounded sequences into sequences having weakly convergent (resp. weak Cauchy) subsequences. (See, e.g., [6] and [14]). A Banach space X is said to be reflexive if $X'' = JX$, almost reflexive if every bounded sequence in X has a weak Cauchy subsequence and weakly sequentially complete if every weak Cauchy sequence is weakly convergent. Clearly c_o is almost reflexive and l_1 is weakly sequentially complete.

The notion of weakly compact operator is generalised to linear relations in normed spaces as follows:

DEFINITION 1. – [10]. *Let $T \in LR(X, Y)$. Then T is called weakly compact if $TB_{D(T)}$ is relatively $\sigma(Y, D(T'))$-compact.*

It is clear that T is weakly compact if and only if the operator $Q_T T$ is weakly compact.

We have the following properties:

THEOREM 2. – [10, *VIII*. 2.7, *VIII*. 2.8 *and VIII*. 2.10]. *Let $T \in LR(X, Y)$. Then:*

i) *If T is weakly compact, then T' is weakly compact.*

ii) *If Y is a Banach space and T' is weakly compact, then T is weakly compact.*

iii) *T' is $\sigma(D(T'), JY) - \sigma(X', D(T)'')$-continuous if and only if T' is continuous and $R(T'') \subseteq JY + T''(0)$.*

iv) *If T' is continuous and $R(T'') \subseteq JY + T''(0)$, then T is weakly compact.*

Our aim in this Section is to obtain analogue properties to those of Theorem 2 for Rosenthal linear relations.

DEFINITION 3. – *We say that $T \in LR(X, Y)$ is Rosenthal if $Q_T T$ maps bounded sequences in $D(T)$ into sequences having $\sigma(Q_T Y, D(T'))$-Cauchy subsequences.*

The corresponding class of linear relations will be abbreviated $Ro(X,Y)$ or simply Ro.

Every weakly compact linear relation is trivially Rosenthal but, in general, the converse is false. Indeed, the identity on c_o is Rosenthal but not weakly compact.

PROPOSITION 4. – *Let $T \in LR(X,Y)$ be Rosenthal and let $S \in LR(Y,Z)$ be continuous with $R(T) + \overline{T(0)} \subseteq D(S)$. Then ST is Rosenthal.*

PROOF. – First suppose that T is single valued. Then $ST(0) = S(0)$, so $Q_{ST}ST = (Q_S S)T$ and we may assume that S is single valued. Note that $D(ST) = D(T)$ because $R(T) + \overline{T(0)} = R(T) \subseteq D(S)$. Let (x_n) be a bounded sequence in $D(ST)$. Then, there exists a subsequence (x_{n_j}) of (x_n) such that (Tx_{n_j}) is $\sigma(Y, D(T'))$-Cauchy. Since $D(S') = Z'$ and $R(T) \subseteq D(S)$, it follows by [10, III. 1.6] that $(ST)' = T'S'$. Thus, for each $z' \in D((ST)')$ we have $z'S \in D(T')$ and consequently there exists $\lim z'S(Tx_{n_j})$. Therefore ST is Rosenthal.

For the general case, consider the linear relation $SQ_T^{-1}Q_T T$. Since $Q_T^{-1}(0) = \overline{T(0)} \subseteq D(S)$ and S is continuous, by [10, II. 3.13] SQ_T^{-1} is continuous. Hence $SQ_T^{-1}Q_T T$ is Rosenthal. Now, as $\overline{ST(0)} = \overline{SQ_T^{-1}Q_T T(0)}$, by the continuity of S and $\overline{T(0)} \subseteq D(S)$, using [10, V. 2.9] we deduce that $Q_{ST}ST = Q_{ST}SQ_T^{-1}Q_T T$ is Rosenthal and consequently so is ST. ∎

Next we prove a result which reduces the study of a Rosenthal linear relation to the case of a continuous single valued linear relation. For this purpose, we need the following Proposition.

PROPOSITION 5. – [10, *IV.* 3.17]. *Let $T \in LR(X,Y)$. There exists a normed space Z_T and a bounded operator H_T mapping Y onto Z_T with the following properties:*

i) $\|H_T\| \leq 1$.

ii) *$H_T T$ is continuous and single valued with $\|H_T T\| \leq 1$.*

iii) $Z'_T = D_{T'}$.

iv) *$H'_T = G_{T'}$ (and hence $(H_T T)' = T'G_{T'}$).*

v) *$N(H_T) = \overline{T}(0)$; in particular H_T is injective if and only if T is closable and single valued.*

PROPOSITION 6. – *Given a linear relation $T \in LR(X,Y)$, we have that T is Rosenthal if and only if $H_T T$ is Rosenthal.*

PROOF. – The proof is along the lines of the proof of [10, VIII. 2.5], with only minor changes. ∎

In order to investigate the relationship between a Rosenthal linear relation and its conjugate we recall the following result.

PROPOSITION 7. – [15, 2.8]. *Let $T : X \to Y$ be a bounded operator, where X and Y are Banach spaces. Then T is Rosenthal if T' is Rosenthal.*

THEOREM 8. – *Let $T \in LR(X, Y)$ such that Y is complete. If T' is continuous and Rosenthal, then T is Rosenthal.*

PROOF. – Let us assume that T' is a continuous Rosenthal linear relation, and prove that $T'G_{T'}$ is Rosenthal. Indeed, we note that $Q_{T'G_{T'}}T'G_{T'} = (Q_{T'}T')G_{T'}$ (as $G_{T'}$ is single valued) and a bounded sequence in $D_{T'}$ is mapped by $G_{T'}$ to a bounded sequence in $D(T')$. Moreover, if T' is continuous, then $D(T'') = D(T)''$ by [10, III. 8.10] and so $D((T'G_{T'})') = D(T'') = D(T)''$. Thus, as T' is Rosenthal, we get that $T'G_{T'}$ is also Rosenthal.

We are going to prove that T is Rosenthal. Since $T' = \widetilde{T}'$ it follows from Proposition 7 that $H_{\tilde{T}}\widetilde{T}$ is Rosenthal and applying Proposition 6 we deduce that $\widetilde{T}$ is Rosenthal. But, since $\widetilde{T}$ coincides with the complete closure of $J_Y^{\tilde{Y}}T$, from the definition of Rosenthal linear relation it follows that T is Rosenthal whenever $\widetilde{T}$ is Rosenthal and Y is complete. ■

The converse of Theorem 8 is false even for bounded operators in Banach spaces. For example, let T be the identity on c_o. Then T is Rosenthal but T' is not Rosenthal.

Cross shows in [10] that there are no closed unbounded weakly compact linear relations in Banach spaces. We don't know if the same property is true for Rosenthal linear relations.

THEOREM 9. – *Let $T \in LR(X, Y)$ such that T' is continuous and $R(T'') \subseteq K(Y) + T''(0)$. Then T is Rosenthal.*

PROOF. – Let (x_n) be a bounded sequence in $D(T)$. Then by the Banach-Alaoglu Theorem, the sequence (Jx_n) has a subsequence (Jx_{n_j}) such that $Jx_{n_j} \to x''$ with respect to $\sigma(D(T)'', D(T)')$, for some $x'' \in B_{D(T)''}$. Since T' is continuous, $D(T'') = D(T)''$ and by the weak*-weak* continuity of $Q_{T''}T''$ [10, VIII. 1.8], we have $Q_{T''}T''x'' = \sigma(Q_{T''}Y'', D(T')) - \lim Q_{T''}T''Jx_{n_j}$. By hypothesis, there exists $y'' \in K(Y)$ such that $Q_{T''}T''x'' = Q_{T''}y''$. Hence for $y' \in D(T')$, $Q_{T''}T''Jx_{n_j}(y') = Q_{T''}JTx_{n_j}(y') \to Q_{T''}y''(y')$ or equivalently $y'(Q_TTx_{n_j}) \to Q_{T''}y''(y')$. This shows that $(Q_TTx_{n_j})$ is $\sigma(Q_TY, D(T'))$ Cauchy. ■

The converse of Theorem 9 is false even for bounded operators in Banach spaces. Indeed, let T be the identity on $c_o(\Gamma)$ with Γ uncountable. Then it is clear that T is Rosenthal and $R(T'') \not\subseteq K(c_o(\Gamma))$.

In the classical case of bounded operators in Banach spaces the previous Theorem was obtained by Bombal and Hernando [6, 2.3].

Let us exhibit some examples of Rosenthal linear relations.

EXAMPLE 10. – *Let $T : X \to Y$ be a bounded operator, where X and Y are Banach spaces. If X is almost reflexive and separable, then T is Rosenthal.*

In effect, a result of Odell and Rosenthal (see, e.g., [20, 2.e.7]) states that a separable Banach space Z contains no isomorphic copy of l_1 if and only if $Z'' = K(Z)$ and also it is known that a Banach space Z is almost reflexive if and only if l_1 does not embed in Z (see, e.g., [20, 2.e.5]). Now, the assertion follows immediately.

EXAMPLE 11. – *Let $T \in LR(X, Y)$ be everywhere defined with $D(T') = \{0\}$. Then T is Rosenthal.*

Since $G(T'') = X'' \times Y''$, we have $T''(0) = Y''$ and thus T is Rosenthal by Theorem 9.

Propositions 13 and 14 below give conditions under which Rosenthal operators are F_+ or strictly singular. In order to obtain these Propositions we recall some definitions.

A subclass $\mathcal{A}$ of all Banach spaces has the three-space property if it satisfies the following conditions: If M is a closed subspace of a Banach space Z such that $Z/M \in \mathcal{A}$ and $M \in \mathcal{A}$, then $Z \in \mathcal{A}$.

DEFINITION 12. – [3]. *An operator $T : D(T) \subseteq X \to Y$ (X, Y normed spaces) is called thin if there is no infinite dimensional subspace N of Y such that $TB_{D(T)}$ almost absorbs B_N, namely, for every $\varepsilon > 0$ there exists $\lambda > 0$ such that $B_N \subseteq \lambda TB_{D(T)} + \varepsilon B_Y$.*

PROPOSITION 13. – *Let Y be a Banach space and $T : D(T) \subseteq X \to Y$ a partially continuous operator. Consider the following properties:*

i) *T is Rosenthal and Y is hereditarily-l_1.*
ii) *T is thin.*
iii) *T is strictly singular.*

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii).

PROOF. – (i)$\Rightarrow$(ii) Since T is partially continuous and Y is complete, there exists a finite dimensional subspace E of Y such that $Q_E T$ is continuous [10, V. 9.3]. Assume (i) holds and T is not thin. Then $Q_E T$ is not thin by [3, 3.1]. Moreover, since the class of all Banach spaces containing no l_1 isomorphically has the three-space property (see, e.g., [1]), Y/E is hereditarily -l_1 and by [22, II. C. 11], we obtain that $Q_E T$ is not Rosenthal which contradicts the assumption «T Rosenthal» by virtue of Proposition 4.

(ii)$\Rightarrow$(iii) See [3, 3.3]. ■

This Proposition generalises the homologous result for bounded operators in Banach spaces [22, II. C. 1 and II. C. 11].

PROPOSITION 14. – *Let $T \in LR(X, Y)$ be a Rosenthal linear relation. Then $D(T)$ is almost reflexive if T is F_+.*

PROOF. – Since $T \in F_+$ if and only if $Q_T T \in F_+$ [10, V. 1.1], we may suppose (substituting $Q_T T$ for T if necessary) that T is single valued.

Let $T \in F_+(X, Y)$. Then, there is a closed finite codimensional subspace M of X for which $T\,|_M$ is injective and open [8, 2.2]. We have $(T\,|_M)^{-1}(T\,|_M) = I\,|_M$. Moreover $D(T') = D((T\,|_M)')$ by [2, 3.8] and thus it follows immediately from Definition 3 that $T\,|_M$ is a Rosenthal linear relation and so by Proposition 4 we have that $I\,|_M$ is Rosenthal. Therefore M is almost reflexive, equivalently $\widetilde{M}$ is an almost reflexive Banach space. This fact combined with $\dim \widetilde{D(T)}/\widetilde{M} < \infty$ and the three-space property of almost reflexive Banach spaces permit us to conclude that $\widetilde{D(T)}$ is almost reflexive and so is $D(T)$. ■

In [2, 3.5], it is proved the analogue of Proposition 14 for unbounded weakly compact operators; namely, if T is a weakly compact F_+-operator, then $\widetilde{D(T)}$ is reflexive. Since $T \in LR(X, Y)$ is weakly compact (resp. F_+) if and only if the operator $Q_T T$ is weakly compact (resp. F_+) we can extend this property to linear relations.

3. – Semi-Tauberian Linear Relations.

The notion of bounded Tauberian operator acting between Banach spaces was originally extended to linear relations in normed spaces as follows: $T \in LR(X, Y)$ is Tauberian if $(T'')^{-1}JY \subseteq J(\widetilde{D(T)})$. This concept is due to Cross [10, VIII. 5.1], who proves that Tauberian linear relations can be defined as those linear relations which respect the relatively $\sigma(Y, D(T'))$-compact subsets of Y, in the sense that for every bounded subset A of $D(T)$ if TA is relatively $\sigma(Y, D(T'))$-compact, then A is relatively $\sigma(D(T), X')$-compact; hence they appear as opposite to weakly compact linear relations. This characterisation combined with the notions of bounded semi-Tauberian operators in Banach spaces, suggests to consider the semi-Tauberian linear relations which are defined as follows:

DEFINITION 15. – *We say that $T \in LR(X, Y)$ is semi-Tauberian if every bounded sequence (x_n) in $D(T)$ for which $(Q_T T x_n)$ is $\sigma(Q_T Y, D(T'))$-Cauchy has a $\sigma(D(T), X')$-Cauchy subsequence.*

The corresponding class of linear relations will be denoted $ST(X,Y)$ or simply ST.

We note the following stability property of semi-Tauberian linear relations.

THEOREM 16. – *Let $T \in ST(X,Y)$ and let $S \in LR(X,Y)$ be a continuous Rosenthal linear relation with $D(T) \subseteq D(S)$ and $S(0) \subseteq \overline{T(0)}$. Then $T+S$ is semi-Tauberian.*

PROOF. – Since $S(0) \subseteq \overline{T(0)}$ we have $\overline{(T+S)(0)} = \overline{T(0)}$, and hence $Q_{T+S} = Q_T$ and $Q_T = Q_A Q_S$ where $A := \overline{T(0)}/\overline{S(0)}$ by [10, IV. 5.2]. Now, using Proposition 4 we obtain that $Q_T S$ is a continuous Rosenthal operator and consequently $D(T') \subseteq \overline{T(0)}^{\perp} = (Y/\overline{T(0)})' = D((Q_T S)')$. Since S is continuous and $S(0) \subseteq \overline{T(0)}$, it follows by [10, III. 4.6] that $D(T') \subseteq T(0)^{\perp} \subseteq S(0)^{\perp} = D(S')$ and applying [10, III. 1.5] we have that $(T+S)' = T' + S'$. Combining these facts, we deduce that $D(T') = D((Q_T(T+S))')$.

Let (x_n) be a bounded sequence in $D(T+S)$ such that $(Q_T(T+S)x_n)$ is $\sigma(Q_T Y, D((T+S)')$-Cauchy. Then, (x_n) has a subsequence (z_n) which is $\sigma(Q_T Y, D(Q_T S)')$-Cauchy. Hence $Q_T T z_n$ is $\sigma(Q_T Y, D(T'))$-Cauchy and since T is semi-Tauberian, it follows that there exists a subsequence (z_{n_j}) of (z_n) such that (z_{n_j}) is $\sigma(D(T), X')$-Cauchy. Therefore $T+S$ is semi-Tauberian, as desired. ∎

For bounded operators in Banach spaces this property was obtained by González and Onieva [14].

The homologous result for Tauberian linear relations was proved by [10, VIII. 5.4] with a different scheme of proof.

THEOREM 17. – *Let $T \in ST(X,Y)$ be a bounded operator and $S \in ST(Y,Z)$. Then ST is semi-Tauberian.*

PROOF. – Clearly $Q_{ST} = Q_S$, $D(T') = Y'$ and it is easy to verify that $D(S') \subseteq D((ST)')$; indeed, if $y' \in D(S')$, then $y'S$ is single valued and continuous on $D(ST)$, so $y' \in D((ST)')$.

Let (x_n) be a bounded sequence in $D(ST)$ such that the sequence $(Q_{ST}STx_n)$ is $\sigma(Q_{ST}Z, D((ST)')$-Cauchy. Then, since (Tx_n) is bounded in $D(S)$ and S is semi-Tauberian, there exists a subsequence (z_n) of (x_n) such that (Tz_n) is $\sigma(D(T), Y')$-Cauchy. By hypothesis T is semi-Tauberian and hence, (z_n) has a subsequence (z_{n_j}) such that (z_{n_j}) is $\sigma(D(ST), X')$-Cauchy. Therefore ST is semi-Tauberian. ∎

For Tauberian linear relations, the homologous of this Theorem was obtained by Cross [10, VIII. 5.13].

THEOREM 18. – *Let $S \in LR(Y,Z)$ be a bounded operator and $T \in LR(X,Y)$. If ST is semi-Tauberian, then so is T.*

PROOF. – Let (x_n) be a bounded sequence in $D(T)$ such that $(Q_T T x_n)$ is $\sigma(Q_T Y, D(T'))$-Cauchy. We have $D(T) = D(ST)$ (as $D(S) = Y$) and $(ST)' = T'S'$ by [10, III.1.6]. Hence for $z' \in D((ST)')$ we have that $S'z' \in D(T')$, so $S'z'(Q_T T x_n) = z'(Q_{ST} ST x_n)$ is convergent. But ST is semi-Tauberian, so we conclude that (x_n) has a $\sigma(D(ST), X')$-Cauchy subsequence. Therefore T is semi-Tauberian. ■

The homologous result for Tauberian linear relations was proved in [10, VIII. 5.14].

Let $T \in LR(X, Y)$. Then, the regular contraction of T, $Q_{D(T')^\top} T$ is single valued and closable with $D(T') = D((Q_{D(T')^\top} T)')$, [10, III. 4.11 and III. 4.12].

COROLLARY 19. –*The operator T is semi-Tauberian if and only if its regular contraction $Q_{D(T')^\top} T$ is semi-Tauberian.*

PROOF. – . Let $Q_{D(T')^\top} T$ be semi-Tauberian, then so is T by Theorem 18. For the converse, assume that T is semi-Tauberian. Let (x_n) be a bounded sequence in $D(T) = D((Q_{D(T')^\top} T)$ such that $(Q_{D(T')^\top} T x_n)$ is $\sigma(Y/D(T')^\top, D(T'))$-Cauchy. Then for $y' \in D(T')$ we have $y'(Tx_n) = y'(Q_{D(T')^\top} T x_n)$ (since $y' \in D(T')^{\top\perp}$) and thus $(Q_T T x_n)$ is $\sigma(Q_T Y, D(T'))$-Cauchy. Consequently (x_n) has a $\sigma(D(T), X')$- Cauchy subsequence and the assertion follows. ■

For Tauberian linear relations the analogue property is true by [10, VIII. 5.16].

Notice that some properties enjoyed by Tauberian linear relations are not satisfied by semi-Tauberian linear relations, as we can see from Theorem 20 and Example 22.

THEOREM 20. – *Let $T \in LR(X, Y)$ such that $N(T'') \subseteq K(\widetilde{D(T)})$. Then T is semi-Tauberian.*

PROOF. – Since $N(T'') = N((Q_T T)'')$ by [10, VIII. 5.2], we may suppose that T is single valued.

Assume first that T is continuous. Then as $D(\widetilde{T}) = \widetilde{D(T)}$ and $T'' = \widetilde{T}''$ we have that $\widetilde{T}$ is a continuous single valued between Banach spaces with $N(\widetilde{T}'') \subseteq K(D(\widetilde{T}))$ and therefore $\widetilde{T}$ is semi-Tauberian by [6, 2.1]. Now, observing that $y'\widetilde{T}x = y'Tx$ for $y' \in D(T')$ and $x \in D(T)$, it follows that T is semi-Tauberian.

For the general case, consider the continuous single valued $H_T T$. Then $N(T'') = N((H_T T)'')$. Indeed, $N((H_T T)'') = R((H_T T)')^\perp = R(T' G_{T'})^\perp = R(T')^\perp = N(T'')$ (since $N(T') = R(T)^\perp$ by [10, III. 1.4] and $H_T' = G_{T'}$ which maps $D_{T'}$ onto $D(T')$ by Proposition 5). Hence $H_T T$ is semi-Tauberian, so Theorem 18 states that T is semi-Tauberian. ■

This result generalises the corresponding result for bounded operators in Banach spaces of [6, 2.1].

The converse of Theorem 20 is false. An example of bounded semi-Tauberian operator $T : X \to Y$ (X, Y Banach spaces) such that $N(T'') \not\subseteq K(X)$ has been constructed by Bombal and Hernando [6, 2.2].

We note that the homologous of Theorem 20 for Tauberian linear relations is not true. In fact, follows immediately from the definition of Tauberian linear relation that $N(T'') \subseteq J(\widetilde{D(T)})$ if T is a Tauberian linear relation. But the converse is not holds even for bounded operators in Banach spaces. For example, let $T : l_\infty \to l_2$ be the bounded operator defined by $T(a_n) := (a_n/n)$, $(a_n) \in l_\infty$. Then $N((T\,|_{c_o})'') = \{0\}$ and $T\,|_{c_o}$ is not Tauberian by [22].

Upon noting that every F_+-relation is Tauberian [10, VIII. 8.4], an application of Theorem 20 yields immediately the next result.

Corollary 21. --*Let $T \in LR(X, Y)$. Consider the following properties:*

i) *T is F_+.*

ii) *T is Tauberian.*

iii) *T is semi-Tauberian.*

Then (i) $\Rightarrow$(ii)$\Rightarrow$(iii).

In [10, VIII. 6.4], it is showed that $TB_{D(T)}$ is closed if T is Tauberian. However, we have:

Example 22. – *There exist a Banach space X and a bounded semi-Tauberian operator from X into X such that TB_X is not closed.*

Define $T : c_o \to c_o$ by $T(a_n) := ((a_{n+1} - a_n)/n)$, $(a_n) \in c_o$. It is easy to see that T is a bounded injective operator and T'' is not injective. Consequently T is not Tauberian [22, I.D.7]. Let M be a closed subspace of c_o such that $T\,|_M B_M$ is not closed. (See [22, I. D. 4] for the existence of such M). Clearly $T\,|_M$ is semi-Tauberian.

The following results illustrate that the semi-Tauberian linear relations share some of the properties of Tauberian linear relations.

Theorem 23. – *Let $T \in LR(X, Y)$. We have:*

i) *If T is semi-Tauberian, then $\widetilde{N(T)}$ is almost reflexive.*

ii) *If T is open and $N(T)$ is closed, then T is semi-Tauberian if and only if $\widetilde{N(T)}$ is almost reflexive.*

Proof. – (i) Since (Tx_n) is the null sequence for every (x_n) in $N(T)$, the assertion is clear from the definitions.

(ii) Let $N(T)$ be closed and $\gamma(T) > 0$. We can consider the canonical factorisation $T = \widehat{T}Q_{N(T)}$ of T, where the injective component $\widehat{T}$ of T is the linear relation $\widehat{T} \in LR(X/N(T), Y)$ given by $G(\widehat{T}) := \{(Q_{N(T)}x, y) : (x, y) \in G(T)\}$. According to Theorem 17, T is semi-Tauberian if $\widehat{T}$ and $Q_{N(T)}$ are both semi-Tauberian.

Since T is open if and only if so is $\widehat{T}$ by [10, V.13.5], we obtain from [10, V. 5.1] that $\widehat{T} \in F_+$. Hence Corollary 21 yields $\widehat{T}$ is semi-Tauberian.

Now assume that $\widetilde{N(T)}$ is a closed almost reflexive. Let $\overline{N(T)}$ denote the closure of $N(T)$ in $\widetilde{X}$. Then $\overline{N(T)}$ is almost reflexive. Indeed, if $\overline{N(T)}$ is not almost reflexive, then by [20, 2.e.5], there exists a closed subspace M of $\overline{N(T)}$ isomorphic to l_1. Consequently $M \cap X$ is a subspace of $N(T)$ such that $\widetilde{M \cap X}$ has a closed subspace isomorphic to l_1. This last fact contradicts the assumption «$N(T)$ almost reflexive» since, in [9, 3] Cross proves that a normed space X is almost reflexive if and only if $\widetilde{X}$ is almost reflexive, equivalently X contains no subspace whose completion is isomorphic to l_1. Therefore $\overline{N(T)}$ is an almost reflexive Banach space. In this situation, we have that $Q_{\overline{N(T)}}$ is a bounded operator on Banach spaces such that $N(Q_{\overline{N(T)}})$ does not have a closed subspace isomorphic to l_1 and thus by [14, 1], $Q_{\overline{N(T)}}$ is semi-Tauberian and thus by Theorem 17, $Q_{\overline{N(T)}} J_X$ is semi-Tauberian. From the canonical equality $\widetilde{X}/\overline{N(T)} = \widetilde{X/N(T)}$, the operator $J_{X/N(T)} Q_{N(T)}$ is naturally identified with $Q_{\overline{N(T)}} J_X$. Therefore $J_{X/N(T)} Q_{N(T)}$ is semi-Tauberian and by Theorem 18 $Q_{N(T)}$ is semi-Tauberian. ■

For bounded operators in Banach spaces, this result was obtained by González and Onieva [14, 1].

Cross proves in [10, VIII. 7.1 and VIII. 7.6] the homologous result for Tauberian linear relations, namely that if $T \in LR(X,Y)$, then : (a) If T is Tauberian , then $\widetilde{N(T)}$ is reflexive. (b) If T is open and $N(T)$ is closed, then T is Tauberian if and only if $\widetilde{N(T)}$ is reflexive. We remark that the scheme of the proof of this last result is false for semi-Tauberian linear relations. In fact, the proof of Cross is based in the property : If T is an open linear relation, then T is Tauberian if and only if $N(T'') \subseteq J(\widetilde{D(T)})$, (see [10, VIII. 7.1]). But, it follows from [6, 2.2] that if $T \in LR(X,Y)$ is open, then the properties, T semi-Tauberian, and, $N(T'') \subseteq K(\widetilde{D(T)})$, are not equivalent.

It follows from Open Mapping Theorem for linear relations [26, 3.3.6], that if X and Y are Banach spaces and $T \in LR(X,Y)$ is closed, then T is open if and only if $R(T)$ is closed. Combining this observation with Theorem 23 yields immediately the following Corollary.

COROLLARY 24. –*Let X and Y be Banach spaces, and let $T \in LR(X,Y)$ be closed with closed range. Then T is semi-Tauberian if and only if $N(T)$ is almost reflexive.*

This Corollary has been proved in [14, 1] in the context of bounded operators in Banach spaces.

THEOREM 25. – *Let $T \in LR(X,Y)$. Then T is F_+ if and only if T is semi-Tauberian and $T\mid_M$ is F_+ for all subspaces M of $D(T)$ with almost reflexive completion.*

PROOF. – Let $T \in F_+$. Restrictions of linear relations in F_+ are linear relations of F_+ [10, V. 2.4]; moreover, by virtue of Corollary 21, T is semi-Tauberian.

Conversely, assume that T is semi-Tauberian and $T\,|_M$ is F_+ for all subspaces M of $D(T)$ such that $\widetilde{M}$ is almost reflexive. Since $T \in F_+$ if and only if $Q_T T \in F_+$ by [10, V. 1.1], we may suppose without loss of generality that T is single valued. According to [10, V. 7.11], the assertion follows if we verify that $N(T+K)$ is finite dimensional for every precompact operator $K \in LR(X, Y)$.

Let $K \in LR(X, Y)$ be a precompact single valued. Thus trivially K is Rosenthal. Let us consider two cases for K :

Case 1: $D(T) \subseteq D(K)$. Then by Theorem 16, $T + K$ is semi-Tauberian if so is T. Then we infer from Theorem 23 that $\widetilde{N(T+K)}$ is almost reflexive and by hypothesis $T\,|_{N(T+K)} \in F_+$.

Case 2: $D(T) \supseteq D(K)$. Then, as $D(T') \subseteq D((T\,|_{D(K)}\,)')$ it follows that $T\,|_{D(K)}$is semi-Tauberian and proceeding exactly as in Case 1 we obtain that $T\,|_{N(T+K)} \in F_+$, equivalently, each bounded sequence in $N(T+K)$ whose image under T is Cauchy has a Cauchy subsequence [9, 1].

Now, take a bounded sequence (x_n) in $N(T+K)$; since $(Tx_n) = (-Kx_n)$ and K is precompact, there exists a subsequence (x_{n_j}) of (x_n) such that $(Tx_{n_j}) = (-Kx_{n_j})$ is Cauchy and thus (x_n) has a Cauchy subsequence. In consequence, $N(T+K)$ is finite dimensional, as required. ■

This Theorem is the multivalued version of the corresponding result of González and Onieva [14, 2] for bounded operators in Banach spaces.

We note that the statement of Theorem 25 can be used to prove the homologous result for Tauberian linear relations of [10, VIII. 8.4], namely that if $T \in LR(X, Y)$, then T is F_+ if and only if T is Tauberian and $T\,|_M$ is F_+ for all subspaces M of $D(T)$ with reflexive completion.

The *VIAR*-property for Banach spaces is defined in [14] where it is shown that if X is a Banach space, then X has the *VIAR*-property if and only if for every Banach space Y the class of bounded F_+-operators from X into Y coincides with the class of bounded semi-Tauberian operators from X into Y.

The generalisation of this property for linear relations in normed spaces will now be obtained.

DEFINITION 26. – *A normed space X has the VIAR-property if no infinite dimensional subspace of X contains an infinite dimensional subspace with almost reflexive completion.*

THEOREM 27. – *Let $T \in LR(X, Y)$ and Y be a Banach space. Then $D(T)$ has the VIAR -property if and only if the following properties on T are equivalent:*

i) *T is F_+.*

ii) *T is semi-Tauberian.*

PROOF. – Suppose that $D(T)$ has the *VIAR*-property. Then every subspace of $D(T)$ with almost reflexive completion must be finite dimensional. Moreover, on the one hand linear relations with finite dimensional range are continuous [10, V. 5.19] and on the other hand a result of Cross [10, V. 1.7] states that if $S \in LR(X, Y)$ is closed, where X and Y are Banach spaces, then $S \in F_+$ if and only if S has closed range and finite dimensional null space. Thus the restrictions of semi-Tauberian linear relations to subspaces of $D(T)$ with almost reflexive completion are F_+ and so the implication (ii)⇒(i) is true by Theorem 25.

The converse implication follows from Corollary 21.

Now, assume that (i) and (ii) are equivalent. Let M be an infinite dimensional subspace of $D(T)$ such that $\widetilde{M}$ is almost reflexive. Then, since $J_{X/M}Q_M = \widetilde{Q_{\overline{M}}J_X}$ (see the proof of Theorem 23) and $\overline{M}$ is a closed almost reflexive subspace of $D(T)$ we obtain that $Q_{\overline{M}}$ is a bounded operator in Banach spaces with closed range and null space $\overline{M}$ almost reflexive and so by Theorem 23 $Q_{\overline{M}}$ is semi-Tauberian; but $Q_{\overline{M}}$ is not F_+ since $dim\overline{M}=\infty$. ■

We remark that the method used to prove the Theorem 27 can be used to obtain the homologous result for Tauberian linear relations, namely that if Y is a Banach space and $T \in LR(X, Y)$, then $D(T)$ has the *VIAR*-property if and only if the properties, T is semi-Tauberian, and , T is F_+ , are equivalent. The part «only if» of this last result was proved in [10, VIII. 8.6] with a different scheme of proof.

We end this Section with some examples of semi-Tauberian linear relations.

In [3] the well-known factorisation of Davis, Figiel, Johnson and Pelczynski [11] of bounded operators in Banach spaces is reformulated for unbounded operators acting between normed spaces as follows:

THEOREM 28. – *Let $T : D(T) \subseteq X \to Y$ be given. Then corresponding to each $1 < p < \infty$ there is a Banach space Z_p and a factorisation $A : D(T) \subseteq X \to Z_p$, $J_p : Z_p \to \widetilde{Y}$, $J_Y T = J_p A$ in which AG_T is bounded, $\widetilde{TG_T} = Jp(\widetilde{AG_T})$ and J_p coincides with the injection in the DFJP factorisation of $\widetilde{TG_T}$ corresponding to p.*

As an application of this result we obtain a procedure of construction of semi-Tauberian operators.

THEOREM 29. – *Let $T : D(T) \subseteq X \to Y$ be an operator. Then, for $1 < p < \infty$ the factorisation of $J_Y T$ produces a Banach spaces Z_p and a semi-Tauberian operator J_p. In general, J_p is not F_+.*

PROOF. – If $J_Y T$ is thin, then so is $\widetilde{TG_T}$ [3, 3.3] and by [22, II. C. 8], J_p is strictly singular. In consequence, J_p is not F_+ if $J_Y T$ is thin. ■

The notion of **-injection (see [22]) is generalised to linear relations in a natural manner.

DEFINITION 30. – *We say that a linear relation T is a **-injection if T'' is injective.*

Notice that T is a **-injection if and only if the single valued $Q_T T$ is a **-injection because $N(T'') = N((Q_T T)''$ by [10, VIII. 5.2].

Define $T : l_\infty \to l_2$ by $T(a_n) := (a_n/n)$, $(a_n) \in l_\infty$. Then T is a **-injection but not Tauberian by [22]. In contrast to this example we should note that there may not exist linear relations T for which T is a **-injection but not semi-Tauberian.

PROPOSITION 31. – *Let $T \in LR(X, Y)$ be a **-injection. Then T is semi-Tauberian.*

PROOF. – The proof is similar to that of corresponding result for bounded operators in Banach spaces [22, I. E. 27], with only minor changes.

It will be assumed without loss of generality that T is single valued (as was noted in Definition 30). Moreover, since $N(T') = R(T)^\perp$ [10, III. 1.4], T is a **-injection if and only if $R(T')$ is dense.

Let (x_n) be a bounded sequence in $D(T)$ for which (Tx_n) is $\sigma(Y, D(T'))$-Cauchy. Then the sequence $y'(Tx_n)$ converges for all $y' \in D(T')$. Let $x' \in D(T)'$ and $\varepsilon > 0$. Then $x' = \lim T'y'_n$ for some (y'_n) in $D(T')$ and thus we choose $y' \in D(T')$ such that $\|x' - T'y'\| < \varepsilon/3M$ (where $M := \sup\{\|x_n\| : n \in N\}$). Since $y'(Tx_n)$ is Cauchy, there is $p \in N$ such that $| T'y'(x_n - x_m) | < \varepsilon/3$ for all $n, m \geq p$. Hence $| x'(x_n - x_m) | \leq | (x' - T'y')(x_n - x_m) | + | T'y'(x_n - x_m) < \varepsilon$ for all $n, m \geq p$. Thus (x_n) is $\sigma(D(T), X')$-Cauchy. ■

REFERENCES

[1] T. ÁLVAREZ - V. M. ONIEVA, *A note on three-space ideals of Banach spaces*, Proceedings of the Tenth Spanish-Portuguese Conference on Mathematics, III (Murcia, 1985), Univ. Murcia (1985), 251-254.

[2] T. ÁLVAREZ - R. W. CROSS - A. I. GOUVEIA, *Adjoint characterisations of unbounded weakly compact, weakly completely continuous and unconditionally converging operators*, Studia Math., **113** (3) (1995), 293-298.

[3] T. ÁLVAREZ - R. W. CROSS - M. GONZÁLEZ, *Factorization of unbounded thin and cothin operators*, Quaestiones Math., **22** (1999), 519-529.

[4] J. P. AUBIN - A. CELLINA, *Differential Inclusions*, Springer-Verlag, New York, 1984.

[5] J. P. AUBIN - H. FRANKOWSKA, *Set Valued Analysis*, Birkhauser, Boston, 1990.

[6] F. BOMBAL - B. HERNANDO, *A double-dual characterisation of Rosenthal and semi-Tauberian operators*, Proc. Royal Irish Acad., Ser. A, **95** (1995), 69-75.

[7] F. H. CLARKE, *Optimization and Nonsmooth Analysis*, Wiley-Interscience Publication, Wiley and Sons, Toronto, 1983.

[8] R. W. CROSS, *Properties of Some Norm Related Functions of Unbounded Linear Operators*, Math. Z., **199** (1988), 285-303.

[9] R. W. CROSS, *A characterisation of almost reflexive normed spaces*, Proc. Royal Irish Acad., Ser. A **92** (1992), 225-228.
[10] R. W. CROSS, *Multivalued Linear Operators*, Monographs and Textbooks in Pure and Applied Mathematics, **213**, Marcel Dekker, New York, 1998.
[11] W. J. DAVIS - T. FIGIEL - W. B. JOHNSON - A. PELCZYNSKI, *Factoring weakly compact operators*, J. Funct. Anal., **17** (1974), 311-327.
[12] A. FAVINI - A. YAGI, *Multivalued linear operators and degenerate evolution equations*, Ann. Mat. Pura. Appl. (4) **163** (1993), 353-384.
[13] D. H. J. GARLING - A. WILANSKY, *On a summability theorem of Berg, Crawford and Whitley*, Math. Proc. Camb. Phil. Soc., **71** (1972), 495-497.
[14] M. GONZÁLEZ - V. M. ONIEVA, *Semi-Fredholm operators and semigroups associated with some classical operator ideals*, Proc. Royal Irish Acad., Ser. A 88 A (1988), 35-38.
[15] M. GONZÁLEZ, *Dual results of factorization for operators*, Ann. Acad. Sc. Fenn., Ser. A. I. Math. **18** (1993), 3-11.
[16] M. GROMOV, *Partial differential relations*, Springer-Verlag, Berlin, 1986.
[17] B. HERNANDO, *Some Properties of the Second Conjugate of a Tauberian operator*, J. Math. Anal. Appl., **228** (1998), 60-65.
[18] J. J. KALTON - A. WILANSKY, *Tauberian operators in Banach spaces*, Proc. Amer. Math. Soc., **57** (1976), 251-255.
[19] C. KURATOWSKI, *Topologie I*, Polska Akademia Nauk. Warsaw, 1952.
[20] J. LINDENSTRAUSS - L. TZAFRIRI, *Classical Banach spaces I*, Springer- Verlag, Berlin, 1977.
[21] E. MICHAEL, *Continuous selections I, II, III*, Annals of Math., **63**, 361-381; **64**, 562-580; **65**, 375-390.
[22] R. NEIDINGER, *Properties of Tauberian operators in Banach spaces*, Ph. D. Thesis, Univ. Texas, 1984.
[23] J. VON NEUMANN, *Functional Operators, Vol. 2: The Geometry of Orthogonal spaces*, Ann. Math. Stud., **22**, Princeton University Press, Princeton N. J., 1950.
[24] W. SCHACHERMAYER, *For a Banach space isomorphic to its square the Radon-Nikodym property and the Krein-Milman property are equivalent*, Studia Math., **81** (1985), 329-338.
[25] D. G. TACON, *Generalised semi-Fredholms transformations*, J. Austral. Math. Soc., A **34** (1983), 60-70.
[26] D. WILCOX, *Multivalued Semi-Fredholm Operators in Normed Linear Spaces*, Ph. D. THESIS, Univ. Cape Town, 2001.
[27] K. W. YANG, *The generalized Fredholm operators*, Trans. Amer. Math. Soc., **219** (1976), 313-326.

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