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Numerical Treatment of a Time Dependent Inverse Problem in Photon Transport.

S. Pieraccini - R. Riganti - A. Belleni-Morante

Sunto. – *Si studia un problema inverso unidimensionale per un'equazione integro-differenziale del trasporto di fotoni, e si determinano le proprietà di una sorgente di fotoni ultravioletti immersa in una nube interstellare. Un procedimento iterativo che si basa sulla discretizzazione spazio-temporale del problema diretto, porta alla determinazione della intensità della sorgente e delle sue variazioni al crescere del tempo.*

Summary. – *The time-dependent intensity of a UV-photon source, located inside an interstellar cloud, is determined by formulating and solving an inverse problem for the integro-differential transport equation of photons in a one-dimensional slab. Starting from a discretizazion of the forward problem, an iterative procedure is used to compute the values of the source intensity at increasing values of the time.*

In the framework of astrophysical applications, a great interest is devoted to inverse problems allowing to evaluate some physical and geometrical properties of UV-photon sources, that are located somewhere inside interstellar clouds.

Based on known model equations regarding both the photon transport theory [1, 7, 10] and the nature of interstellar medium [5], contributions in this field are given, among several others, in [6, 8] and more recently in [3] where time-independent inverse problems were considered, related to a source $q(x)$ emitting UV-photons inside an interstellar cloud. These problems were studied by using a stationary version of the photon transport equation, where the time is not present or is treated as a parameter [4].

On the other hand, the more general inverse problem dealing with the time evolution of some physical properties of the photon source was recently studied in [2]. Here, by assuming that the source $q(\boldsymbol{x}, t)$ is spatially homogeneous within a region $V_0 \subset V \in \mathbb{R}^3$ and that a time series of "far field" measurements of the photon density is known, it was proved that it is possible to identify the time behaviour of the source by means of a suitable time-discretization procedure of the photon transport equation, leading to explicit formulae for the source intensity $q(t)$ at discrete times t_j, $j = 0, 1, \ldots, J$.

In this note the following one-dimensional problem is studied. We consider a slab $V = \{x : -L \leq x \leq L\}$ "representing" an interstellar cloud where photons, produced by a source $q(x,t) > 0$, $x \in (-L_0, L_0) \subset (-L, L)$, $t \in [t_0, t_J]$, undergo capture and isotropic scattering processes with constant total and scattering cross sections σ and σ_s respectively, with $0 < \sigma_s < \sigma$. Let $\vec{v} = c\vec{u}$ be the velocity of photons (c is the speed of light) and $\mu = \vec{u} \cdot \vec{i} = \cos\vartheta$. Let us further assume non-reentry conditions for the photons outgoing the slab. Then by rescaling the time independent variable with $t^\star = ct$, the photon transport equation, the boundary condition and the initial condition have the form

$$(1) \qquad \frac{\partial}{\partial t^\star} M(x,\mu,t^\star) = -\mu \frac{\partial M}{\partial x} - \sigma M + \frac{\sigma_s}{2} \int_{-1}^{1} M(x,\mu',t^\star) d\mu' + Q(x,t^\star)$$

$$(2) \qquad M(-L,\mu,t^\star) = 0, \ \mu \in (0,1]; \quad M(L,\mu,t^\star) = 0, \ \mu \in [-1,0)$$

$$(3) \qquad M(x,\mu,0) = M_0(x,\mu)$$

where $Q(x,t^\star) = q(x,t^\star/c) = q(x,t)$ and $M(x,\mu,t^\star)$ is the number density of photons having a velocity component $\vec{v} \cdot \vec{i} = c\mu$ with $\mu \in [-1,1]$ and which at time $t = t^\star/c$ are at $x \in [-L, L]$.

Assume now that the values $\hat{M}_j = M(L,\hat{\mu},t_j^\star)$ of the photon density with $x = L$ and $\hat{\mu} > 0$ are measured at the instants $t_j^\star = j\tau$, $j = 0,1,\ldots,J$ where τ is a constant time interval. In fact, the measurements are made at some $\hat{x}$ "far from the slab", i.e. the values $M(\hat{x},\hat{\mu},t_j^\star + \hat{t}^\star)$ with $\hat{t}^\star = (\hat{x} - L)/\hat{\mu}$ are measured. However, since $M(\hat{x},\hat{\mu},t_j^\star + \hat{t}^\star) = M(L,\hat{\mu},t_j^\star)$, we may assume that the values $\hat{M}_j = M(L,\hat{\mu},t_j^\star)$ are known.

A suitable discretization of (1) in the Banach space $X = L^1([-L,L] \times [-1,1])$, endowed with the norm $\|f\| = \int_{-L}^{L} dx \int_{-1}^{+1} |f(x,\mu)| d\mu$, allows to prove the existence of an explicit solution of the inverse problem in terms of a sequence of values $Q_j = Q(x,t_j^\star)$ given by a formula which, however, tends to become extremely ill-conditioned when the time step τ is decreased.

Hence, in order to numerically solve the inverse problem, we use a different strategy, based on an iterative approach: given an initial estimate $Q_j^{(0)}$ for the value Q_j, we solve the forward problem integrating equation (1)-(3) from $t_j^\star$ to $t_j^\star + \Delta T$, where ΔT is such that at time $t^\star + \Delta T$ the contribution of the source at time $t^\star$ has reached the boundary of the cloud. Let s be such that $t_j^\star + \Delta T = t_{j+s}^\star$. Then, we compare the photon number density computed on the boundary at time $t_{j+s}^\star$ with the set of measures. Let us denote by Δn_{j+s} the difference between the computed value and the measure. If $|\Delta n_{j+s}|$ is smaller then a given tolerance, we accept the estimate for the source, otherwise we properly correct the estimate by assigning a new value $Q_j^{(1)}$ and we repeat the process. When a estimate $Q_j^{(k)}$ is accepted, we proceed with the next time step, starting again the process from $t_{j+1}^\star$.

As far as the forward problem is concerned, it is solved as follows. First, we discretize the velocity field, considering N_v values μ_r in the interval $[-1, 1]$. The integral in (1) is approximated by gaussian quadrature formulae, hence the values μ_r, $r = 1, ..., N_v$, correspond to the gaussian nodes on the interval $[-1, 1]$, which we assume labeled from the largest (μ_1) to the smallest (μ_{N_v}). Let us denote by a_r the corresponding weights. Further, let us denote by $n(x, \mu_r, t^\star)$ the approximation of $M(x, \mu_r, t^\star)$. Equation (1) is hence approximated by the following set of coupled equations:

$$(4) \quad \frac{\partial}{\partial t^\star} n(x, \mu_r, t^\star) = -\frac{\partial \mu_r n(x, \mu_r, t^\star)}{\partial x} - \sigma n(x, \mu_r, t^\star) + \\ + \frac{\sigma_s}{2} \sum_{p=1}^{N_v} a_p n(x, \mu_p, t^\star) + Q(x, t^\star)$$

for $r = 1, ..., N_v$.

Next, we consider a spatial discretization. Let us introduce the grid points $x_{i+1/2} = -L + i\Delta x$, for $i = 0, ..., N_x$ and $\Delta x = 2L/N_x$. Let $x_i = (x_{i-1/2} + x_{i+1/2})/2$ denote the cell centers. By using a WENO reconstruction procedure [9], equations (4) are approximated by

$$\frac{\mathrm{d}n(x_i, \mu_r, t^\star)}{\mathrm{d}t^\star} = \frac{1}{\Delta x}\left(\tilde{f}_{i+1/2} - \tilde{f}_{i-1/2}\right) - \sigma n(x_i, \mu_r, t^\star) \\ + \frac{\sigma_s}{2} \sum_{p=1}^{N_v} a_p n(x_i, \mu_p, t^\star) + Q(x_i, t^\star),$$

for $i = 0, ..., N_x$ and $r = 1, ..., N_v$, where the numerical flux $\tilde{f}_{i+1/2}$ is obtained by a fifth order WENO scheme.

Let $\bar{n}^r(t^\star), Q(t^\star) \in R^{N_x}$ denote the vectors whose elements are $n(x_i, \mu_r, t^\star)$ and $Q(x_i, t^\star)$, respectively, and let $\bar{n}(t^\star)$ denote the matrix with entries $\bar{n}_i^r(t^\star)$. After the approximations in space and velocity previously introduced, equation (1) is approximated by a set of coupled systems of ordinary differential equations which can be written in a compact form as

$$(5) \qquad \frac{\mathrm{d}\bar{n}^r(t^\star)}{\mathrm{d}t^\star} = g_r(t^\star, \bar{n}(t^\star); Q(t^\star)), \qquad r = 1, ..., N_v,$$

where

$$g_r(t^\star, \bar{n}(t^\star); Q(t^\star)) = \frac{1}{\Delta x}\left(\tilde{f}_{i+1/2} - \tilde{f}_{i-1/2}\right) - \sigma \bar{n}^r(t^\star) + \frac{\sigma_s}{2} \sum_{p=1}^{N_v} a_p \bar{n}^p(t^\star) + Q(t^\star).$$

Finally, each system of differential equations is solved using the MATLAB function ODE45.

The way in which at each iteration we correct the source estimate is the following:

$$Q_j^{(k+1)} = Q_j^{(k)} - \frac{Q_j^{(k)}}{\hat{M}_{j+s}} \Delta n_{j+s}, \qquad k = 0, 1, ... \tag{6}$$

The rule is based on the following consideration: if $\Delta n_{j+s} > 0$ this means that the source at time $t_j^\star$ was over-estimated and consequently we reduce $Q_j^{(k)}$; on the other hand, if $\Delta n_{j+s} < 0$ the source was under-estimated and $Q_j^{(k)}$ is therefore increased.

The overall numerical scheme is sketched as follows.

Numerical scheme.

1. Given $n_{ir}^0 = M_0(x_i, \mu_r)$ for $i = 0, 1, ..., N_x$ and $r = 1, ..., N_v$; $Q_0^{(0)}$; $\hat{M}_j$ for $j = 0, 1, ...$; ε
2. for $j = 0, 1, .., J - s$
   2.1. for $k = 0, 1, ...$
      2.1.1. Compute the solution of (5) with $g_r(t^\star, \bar{n}(t); Q_j^{(k)})$ on the interval $[t_j^\star, t_{j+s}^\star]$ for $r = 1, ..., N_v$
      2.1.2. Compute $\Delta n_{j+s} = n(L, \hat{\mu}, t_{j+s}^\star) - \hat{M}_{j+s}$; if $|\Delta n_{j+s}| \leq \varepsilon \hat{M}_{j+s}$ set $Q_j = Q_j^{(k)}$, else update $Q_j^{(k)}$ according to (6).

We performed some preliminary numerical experiments in different situations. In order to have an "exact" solution for computing the errors, we first solved the forward problem with a given source on a time interval $[T_1, T_2]$; than we considered two times T_0 and T_f such that $T_1 < T_0 < T_f < T_2$ and solved the inverse problem on the time interval $[T_0, T_f]$; the initial data $n_{jr}^0 = M(x_j, \mu_r, T_0)$ and the set of measures $\hat{M}_j$ are obtained from the solution of the forward problem.

Several test cases were considered, by assuming several kinds of time behaviour for the source, for example also considering a case in which sharp peaks are superimposed to a smooth periodic trend. For this test we summarize here some results.

We used two initial guesses, $Q_0^{(0)} = 0.1$ (initial guess A) and $Q_0^{(0)} = 0.01$ (initial guess B); further, at each time step $j \geq 1$ we started from $Q_j^{(0)} = Q_{j-1}$. A maximum number of 150 iterations was imposed. We remark that in our experiments if such a number was reached, failure was not declared, but the final extimate was accepted and the iteration proceeded. The given tolerance was $\varepsilon = 10^{-3}$. Further, we set $\sigma = 5$, $\sigma_s = 0.2$, $L = 1$, $L_0 = 0.1$. We assumed $\hat{\mu} = \mu_1$, $N_v = 8$ and $N_x = 100$. The time step was given by $\tau = 0.9 \Delta x / \mu_1$.

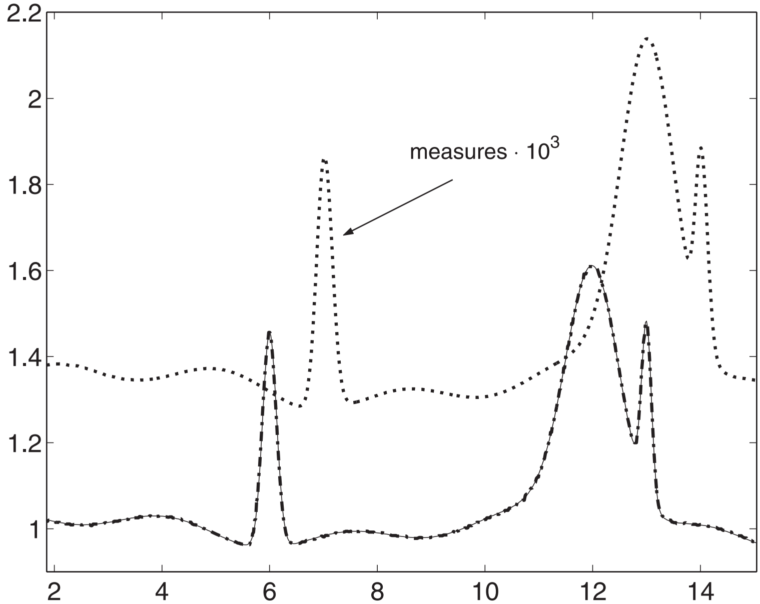

Fig. 1. – Computed solution (dash-dot line) and exact solution (continuous line); dotted line: measured data $\hat{M}_j$ amplified by 10^3.

The results obtained are summarized in figure 1, in which we plot (versus $t^\star$) the results obtained for Q_j with initial guess B (dash-dot line); the computed solution is compared with the exact one (continuous line). In the same figure we plot the corresponding set of measures $\hat{M}_j$ used (dotted line), amplified by a factor of 10^3.

From the figure it is clear that the qualitative behaviour of the computed solution is quite satisfactory. In order to estimate the results, we compared the computed and the exact solutions by analyzing the relative errors

$$(e_r)_j = \frac{|Q_j - Q(t_j^\star)|}{Q(t_j^\star)}, \qquad j = 0, 1, ..., J.$$

The values obtained are the following: the mean value of the relative error was $\bar{e}_r = 1.45 \cdot 10^{-3}$ with initial guess A and $\bar{e}_r = 1.47 \cdot 10^{-3}$ with initial guess B; the maximum relative error was $1.85 \cdot 10^{-2}$ with initial guess A and $1.92 \cdot 10^{-2}$ with initial guess B. We point out that in both cases the mean relative error has the same order of magnitude of the relative tolerance used in the stopping criterion.

Finally, in figure 2 we summarize the computational cost in terms of the number of iterations needed at each step in order to satisfy the stopping cri-

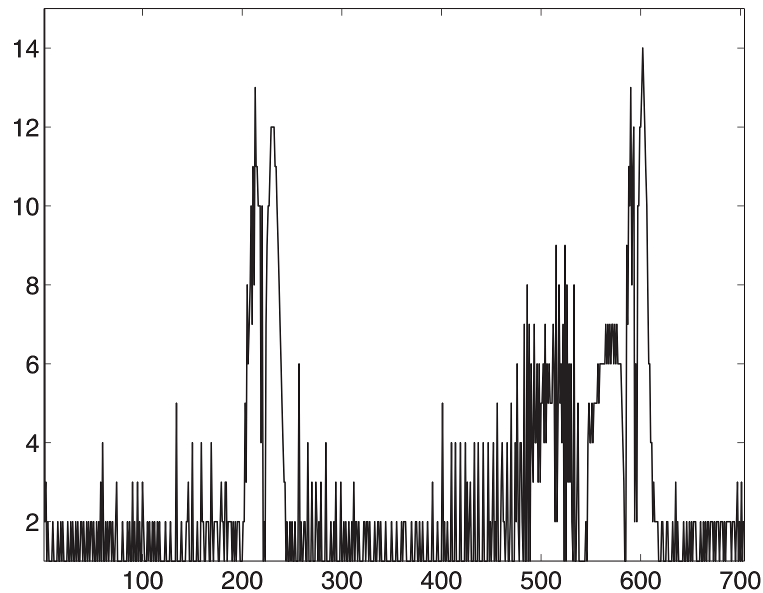

Fig. 2. – Number of iterations versus the step counter j (initial guess A).

terion. Let us denote by $K(j)$ the number of iterations performed at step j, in such a way that $Q_j = Q_j^{K(j)}$. Figure 2 show $K(j)$ versus j, starting with initial guess A. Similar results are obtained by using the initial guess B. Concerning the first time step, we remark that, using the initial guess A, 34 iterations are needed in order to satisfy the stopping criterion, while 58 iterations are needed when using the initial guess B. In the subsequent steps, the initial guess at each step is good enough that a moderate computational task is required in order to compute the next extimate Q_j of the solution, except for those regions in which the solution is steeper.

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