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Mathematical analysis of a two-phase parabolic free boundary problem derived from a Bingham-type model with visco-elastic core.

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Sunto. – *In questo lavoro vengono riassunti i risultati ottenuti nello studio di un problema unidimensionale a frontiera libera per l'equazione del calore. Tale problema scaturisce dallo studio di un modello matematico per il flusso di un fluido di Bingham con nucleo visco-elastico. La principale caratteristica del problema risiede nella particolare struttura della condizione imposta sulla frontiera, che non permette l'uso di tecniche classiche per la dimostrazione dei risultati di buona posizione. L'esistenza di soluzioni classiche è dimostrata mediante una tecnica di punto fisso basata sul teorema di Schauder. L'unicità viene invece provata con tecniche di formulazione debole.*

Summary. – *In this paper we consider a two-phase one-dimensional free boundary problem for the heat equation, arising from a mathematical model for a Bingham-like fluid with a visco-elastic core. The main feature of this problem consists in the very peculiar structure of the free boundary condition, not allowing to use classical tools to prove well posedness. Existence of classical solution is proved using a fixed point argument based on Schauder's theorem. Uniqueness is proved using a technique based on a weak formulation of the problem.*

1. – Introduction.

Bingham fluids are non-Newtonian fluids that undergo no deformation if the applied stress is below a certain threshold (called yield stress) and behave like viscous fluids if the stress exceeds that threshold. We refer the reader to [1] for the general 3D Bingham constitutive equation. In previous papers [4], [5] we extended the usual model allowing deformability for the material below threshold. In particular, in [4] we modeled it as a nonlinear neo-Hookean elastic material, whereas in [5] as a visco-elastic upper convected Maxwell fluid material. For both models a one dimensional flow has been studied and in both cases we formulated a free boundary problem.

In this note we summarize the analytical results of the mathematical problem arising from the model presented in [5]. Such a problem belongs to the broad class of two-phase 1D free boundary problems for the heat equations. However it cannot be included in any of the two-phase free boundary problems listed in [2]. The full problem will appear elsewhere [3].

2. – Description of the problem.

We assume that the system is confined between two parallel planes (channel flow) $y = L$ and $y = -L$ and that the flow is driven by a constant pressure gradient.

We assume that in the inner part of the channel $(-s(t) < y < s(t))$ the fluid behaves as a visco-elastic upper convected Maxwell fluid, whereas in $s(t) < y < L$ and in $-L < y < s(t)$ as a purely viscous fluid. Due to symmetry reasons we confine our study to the region $0 < y < L$. The velocity field is expressed by

$$\begin{aligned} \vec{v} =& v(y,t)\vec{e}_x, \qquad s(t) < y < L, \ \text{(viscous region)}, \\ \vec{u} =& u(y,t)\vec{e}_x, \qquad 0 < y < s(t), \ \text{(visco-elastic region)}. \end{aligned}$$

Following [5] the problem to be solved is (in non-dimensional form)

$$\text{(1)} \qquad \begin{cases} \chi_1 v_t - v_{yy} = 1, & s(t) < y < 1, \ t > 0, \\ v(y,0) = v_o(y), & s_o < y < 1, \\ v(1,t) = 0, & t > 0, \\ v_y(s(t),t) = -\chi_2 & t > 0, \\ \chi_3 u_{tt} + \chi_1 u_t - u_{yy} = 1, & 0 < y < s(t), \ t > 0, \\ u(s(t),t) = v(s(t),t), & t > 0, \\ u_y(s(t),t) + \chi_3 \dot{s} u_t = \chi_4 \dot{s} - \chi_2 & t > 0, \\ u(y,0) = u_o(y), & 0 < y < s_o, \\ u_t(y,0) = \chi_5, & 0 < y < s_o, \\ u_y(0,t) = 0, & t > 0, \\ s(0) = s_o, & 0 < s_o < 1, \end{cases}$$

where $v_o(y)$ and $u_o(y)$ are given functions and where χ_j, $j = 1, \ldots, 5$, are positive given constants that depend on data. Problem (1) is a free boundary problem involving a parabolic and a hyperbolic equation. Setting

$$\text{(2)} \qquad z(y,t) = v_y(y,t), \qquad w(y,t) = u(y,t) - t$$

we study the problem for z and w in the case $\chi_1 = 1$, $\chi_3 \ll 1$, that is

$$(3) \qquad \begin{cases} z_t - z_{yy} = 0, & s(t) < y < 1,\ t > 0, \\ z(y,0) = v_o'(y), & s_o < y < 1, \\ z_y(1,t) = -1, & t > 0, \\ z(s(t),t) = -\chi_2, & t > 0, \\ w_t - w_{yy} = 0, & 0 < y < s(t),\ t > 0, \\ w(y,0) = u_o(y), & 0 < y < s_o, \\ w_y(0,t) = 0, & t > 0, \\ w(s(t),t) = -\displaystyle\int_{s(t)}^{1} z(\xi,t)d\xi - t, & t > 0, \\ w_y(s(t),t) = \chi_4\dot{s} - \chi_2, & t > 0, \\ s(0) = s_o, & 0 < s_o < 1. \end{cases}$$

3. – Existence of a classical solution.

Local existence of problem (3) can be proved using Schauder's fixed point theorem. Applying Green's identity to $(3)_5$ yields

$$(4) \qquad \chi_4[s(t) - s_o] = \chi_2 t - \int_0^t w(s,\tau)\dot{s}(\tau)d\tau + \int_0^{s(t)} w(\xi,t)d\xi - \int_0^{s_o} u_o(\xi)d\xi,$$

which is the integral formulation of the free boundary condition. We select a function s in the set Γ of continuously differentiable functions with Hölder first derivative with exponent l and we solve problem $(3)_1$-$(3)_8$, $(3)_{10}$ getting w and z. Then we use (4) to get the new function $\widehat{s}$

$$(5) \qquad \chi_4[\widehat{s}(t) - s_o] = \chi_2 t - \int_0^t w(s,\tau)\dot{s}(\tau)d\tau + \int_0^{s(t)} w(\xi,t)d\xi - \int_0^{s_o} u_o(\xi)d\xi.$$

The above defines an operator $\Phi : \Gamma \to \Gamma$ such that $\Phi(s) = \widehat{s}$. It can be shown that Γ is compact, closed and convex in the Hölder norm and that Φ is continuous in the topology induced by such a norm. Thus, using Schauder's theorem, the existence of at least one fixed point, that is a classical solution to problem (3), is proved.

3.1 – *Remarks about the sign of solutions.*

Under the hypothesis $\chi_2 < 1$ and compatibility conditions $u_o'(0) = v_o(1) = 0$, $v_o''(1) = -1$, $v_o'(s_o) = -\chi_2$ and using the parabolic version of Hopf's lemma, we see that

$$z_y(s(t), t) < 0, \quad \forall\, t \in (0, T].$$

It can be shown that

$$w_t(s, t) = z_y(s, t) - \chi_4 \dot{s}^2 < 0 \quad \forall\, t \in (0, T], \tag{6}$$

yielding

$$w_{yy}(s, t) < 0 \quad \forall\, t \in (0, T]. \tag{7}$$

Setting $W = w_y$ we observe that W solves the problem

$$\begin{cases} W_t - W_{yy} = 0, & 0 < y < s(t), \quad t > 0, \\ W(y, 0) = u_o'(y), & 0 < y < s_o, \\ W(0, t) = 0, & t > 0, \\ W(s(t), t) = \chi_4 \dot{s} - \chi_2, & t > 0. \end{cases}$$

Exploiting again the parabolic version of Hopf's lemma we conclude that $W(s(t), t) < 0$ in $(0, T]$. Indeed if $W(s(t), t) \geq 0$ there would be a maximum on $y = s(t)$ and this would imply $W_y(s(t), t) = w_{yy}(s(t), t) > 0$ that contradicts (7). The maximum principle yields

$$w_{yy}(y, t) \leq 0,$$

or equivalently

$$w_t(y, t) \leq 0.$$

Further, using maximum principle we may also prove that

$$u(y, t) \geq 0 \text{ and } v(y, t) \geq 0,$$

in their respective domain of definition. Of course this is consistent with the physical problem of the flow in the channel.

4. – Uniqueness.

To prove uniqueness we follow [6] and [7]. We set

$$E(y, t) = \begin{cases} u(y, t) & (y, t) : 0 < y < s(t), \quad 0 < t < T, \\ \widehat{v}(y, t) = v(y, t) + \chi_4 & (y, t) : s(t) < y < 1, \quad 0 < t < T, \end{cases}$$

and we give a weak formulation of problem (1) in $D_T = \{0 < y < 1,\ 0 < t < T\}$, with a suitable test function φ such that:

- $\varphi(1,t) = 0,\ \ 0 \le t \le T.$
- $\varphi(y,T) = 0,\ \ 0 \le y \le 1.$

We thus have

$$\int_{D_T} (E_y \varphi_y - E\varphi_t) dydt = \int_{D_T} \varphi dydt.$$

So, assuming that E_1 and E_2 are two solutions, we consider the difference $(E_1 - E_2)$ getting

$$\int_{D_{s,T}} [(E_{1y} - E_{2y})\varphi_y - (E_1 - E_2)\varphi_t]\, dydt = 0. \tag{8}$$

If now, fixed $t_1 \in (0,T)$, we select φ such that

$$\varphi(y,t) = \begin{cases} 0 & 0 \le y \le 1, \quad t_1 \le t \le T, \\ \displaystyle\int_{t_1}^{t} (E_1(y,\theta) - E_2(y,\theta))d\theta, & 0 \le y \le 1, \quad 0 \le t \le t_1, \end{cases}$$

exploiting (8) we obtain

$$\int_0^1 \int_0^{t_1} \left\{ -\frac{1}{2}\frac{d}{dt}\left[\int_t^{t_1} (E_1 - E_2)_y\, d\theta\right]^2 + (E_1 - E_2)^2 \right\} dydt = 0\,,$$

yielding

$$\int_0^1 \int_0^{t_1} (E_1 - E_2)^2 dydt = 0\,,$$

that is uniqueness.

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