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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **9-B** (2006), n.1, p. 133–144.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2006_8_9B_1_133_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2006.

Bollettino U. M. I.
(8) **9**-B (2006), 133-144

Commutative Cancellative Semigroups and Rational Vector Spaces.

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Sunto. – *Rappresentando un semigruppo commutativo cancellativo subarchimedeo* S *come* $\mathbb{N}_i(G, I)$*, consideriamo* $\mathrm{Hom}\,(S, \mathbb{Q})$ *e* $\mathrm{Hom}\,(G, \mathbb{Q})$*, dove* $\mathbb{Q}$ *è il gruppo additivo dei numeri razionali. Questi insiemi possono essere muniti di una struttura di spazio vettoriale razionale. Si trovano convenienti copie isomorfe di questi spazi vettoriali con uso di funzioni in relazione a certe applicazioni introdotte da T. Tamura.*

Summary. – *Representing a commutative cancellative subarchimedean semigroup* S *as* $\mathbb{N}_i(G, I)$*, we consider* $\mathrm{Hom}\,(S, \mathbb{Q})$ *and* $\mathrm{Hom}\,(G, \mathbb{Q})$*, where* $\mathbb{Q}$ *is the additive group of rational numbers. These sets can be given the structure of rational vector spaces. Suitable isomorphic copies of these vector spaces are found by means of certain functions related to some mappings introduced by T. Tamura.*

1. – Introduction and summary.

Commutative cancellative semigroups are embeddable into abelian groups by the usual method of forming the group of quotients. We can perform tighter embeddings by imposing additional restrictions on semigroups. In particular, if the semigroups are also power cancellative, they can be embedded into the additive group of rational vector spaces. The conditions of being commutative cancellative and power cancellative are obviously also necessary for such an embedding.

In the study of the structure of commutative cancellative semigroups we encounter rational vector spaces when considering homomorphisms of such semigroups into the additive group of rational numbers. A particular case of seminal importance is when the semigroups are also subarchimedean, that is, when there exists $z \in S$ with the property that for every $a \in S$ there are a positive integer n and $x \in S$ such that $z^n = ax$. They admit a Tamura-like re-

(*) The authors were partially supported by DGI of Spain and FEDER, Project: BFM2001-2886. 2000 *Math. Subject Classification*: 20M10, 20M14.

presentation which can be used for studying their homomorphisms into the additive group of rational numbers. The set of such homomorphisms admits the structure of a rational vector space which facilitates their study.

This work is motivated by the desire to elucidate the structure of the algebra of homomorphisms of a commutative cancellative semigroup S into the additive group of rational numbers $\mathbb{Q}$. This algebra is a rational vector space and the obvious question is posed of its dimension relative to S. Here the rank of S comes into the picture. In order to delve deeper into the relationship of S and $\mathrm{Hom}\,(S, \mathbb{Q})$, one quickly realizes that further suitable conditions on S must be imposed in order to make such a program feasable. This is achived by the concept of subarchimedeaness which is a generalization of the familiar notion of archimedeaness of commutative semigroups. This way we come close to the work of T. Tamura with the essential difference that we consider homomorphisms of S into the additive group of rationals rather than reals. As in Tamura's case, we represent a commutative cancellative subarchimedean semigroup S by means of an abelian group G and a function $I : G \times G \to \mathbb{N}$, the nonnegative integers. The group G plays an essential role in our considerations since we compare $\mathrm{Hom}\,(S, \mathbb{Q})$ and $\mathrm{Hom}\,(G, \mathbb{Q})$. While the conceptual similarity with Tamura's work is obvious, the endresults are sufficiently different. In addition, Tamura's analysis is generally restricted to the idempotent-free case whereas our considerations are general.

Section 2 contains some preliminary material. In Section 3 we consider homomorphisms of a general commutative cancellative semigroup S into the additive group of rationals. Sections 4 and 5 are devoted to the study of homomorphisms of S, which is subarchimedean and is given a Tamura-like representation. We conclude in Section 6 by considering a special case. In all cases, we obtain formulas relating ranks of semigroups and groups and dimensions of rational vector spaces.

2. – Preliminaries.

We generally follow the notation and terminology of the book [4] where a discussion of our class of semigroups can be found. Throughout the paper, S *denotes a commutative cancellative semigroups. All our semigroups and groups are commutative*. Let $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{P} = \{1, 2, ...\}$.

We let $SQ = (S \times S)/ \sim$, where $\sim$ is defined by

$$(a, b) \sim (c, d) \Longleftrightarrow ad = bc$$

with multiplication of $\sim$-classes $[a, b][c, d] = [ac, bd]$. Then SQ is the *group of quotients of* S and the *canonical injection*

$$\delta : a \mapsto [a^2, a] \qquad (a \in S)$$

embeds S into SQ.

The semigroup S is *power cancellative* if for any $a, b \in S$ and $n \in \mathbb{P}$, $a^n = b^n$ implies that $a = b$. A congruence ρ on S is *power cancellative* if the quotient semigroup S/ρ is. As usual, congruences on S are ordered by inclusion as binary relations. We start with a simple but important auxiliary result.

LEMMA 2.1. – *On S define a relation τ by*

$$x \,\tau\, y \text{ if } x^n = y^n \text{ for some } n \in \mathbb{P}.$$

Then τ is the least power cancellative congruence on S.

PROOF. – Straightforward. □

In the sequel we shall use the notation τ as above, denote the quotient semigroup by $S^{\mathrm{T}} = S/\tau$ and by $\tau^\sharp : S \twoheadrightarrow S^{\mathrm{T}}$ the natural epimorphism.

All our vector spaces are over the field $\mathbb{Q}$ of rational numbers. The set of all homomorphisms of a semigroup S into a semigroup U is denoted by $\mathrm{Hom}\,(S, U)$. The set $\mathrm{Hom}\,(S, \mathbb{Q})$ is given the structure of a rational vector space by defining the sum $\varphi + \psi$ and the scalar product $r\varphi$ by

$$(\varphi + \psi)a = \varphi a + \psi a, \quad (r\varphi)a = r(\varphi a) \qquad (a \in S,\ r \in \mathbb{Q}).$$

3. – The general case.

This section pertains to homomorphisms of general commutative cancellative semigroups into the additive group of rational numbers.

LEMMA 3.1. – *Let V be a rational vector space and $\varphi : V \to \mathbb{Q}$ be a homomorphism into additive rationals. Then φ is linear (and thus $\varphi \in V^*$).*

The second lemma establishes an isomorphism of vector spaces $\mathrm{Hom}\,(S^{\mathrm{T}}, \mathbb{Q})$ and $\mathrm{Hom}\,(S, \mathbb{Q})$.

LEMMA 3.2. – *The mapping χ defined by*

$$\chi : \varphi \mapsto \varphi\tau^\sharp \qquad \big(\varphi \in \mathrm{Hom}\,(S^{\mathrm{T}}, \mathbb{Q})\big)$$

is an isomorphism of $\mathrm{Hom}\,(S^{\mathrm{T}}, \mathbb{Q})$ *onto* $\mathrm{Hom}\,(S, \mathbb{Q})$.

A subset $\{a_1, \cdots, a_k\}$ of S is *independent* if

$$a_1^{m_1} \cdots a_k^{m_k} = a_1^{n_1} \cdots a_k^{n_k}$$

for some $m_i, n_i \in \mathbb{P}$ implies that $m_i = n_i$ for $i = 1, \cdots, k$. An infinite subset of S is *independent* if all its finite subsets are independent. We observe that a subset of

S is independent if and only if it is (linearly) independent in the group of fractions of S (viewed as a $\mathbb{Z}$-module). The *rank* of S, denoted by $\operatorname{rank} S$, is the cardinality of any maximal independent subset of S. It is proved in ([2], Theorem 3.3) that this concept is well defined.

We are now ready for the only result of this section,

THEOREM 3.3. – *We have that* $\operatorname{rank} S$ *is finite if and only if* $\dim \operatorname{Hom}(S, \mathbb{Q})$ *is finite. In such a case*

$$\operatorname{rank} S = \dim \operatorname{Hom}(S, \mathbb{Q}).$$

PROOF. – By ([2], Lemma 3.2), we have that $\operatorname{rank} S = \operatorname{rank} S^{\mathrm{T}}$, and by Lemma 3.2 that $\operatorname{Hom}(S, \mathbb{Q}) \cong \operatorname{Hom}(S^{\mathrm{T}}, \mathbb{Q})$. Hence we may assume that S is power cancellative so that ([2], Theorem 3.3) implies that S can be embedded into (the additive group of) a rational vector space V generated by the image of S in V. We may identify S with its image. Since then S generates V, it contains a minimal generating set of V, that is a basis of V. It follows that every homomorphism from S into $\mathbb{Q}$ extends uniquely to a homomorphism of the additive group of V into $\mathbb{Q}$. Now Lemma 3.1 implies that

$$\operatorname{Hom}(S, \mathbb{Q}) \cong \operatorname{Hom}(V, \mathbb{Q}) = V^{*}.$$

By ([2], Theorem 3.3), we have $\operatorname{rank} S = \dim V$. Also $\dim V$ is finite if and only if $\dim V^{*}$ is finite. It follows that $\operatorname{rank} S$ is finite if and only if $\dim \operatorname{Hom}(S, \mathbb{Q})$ is finite, in which case

$$\operatorname{rank} S = \dim V = \dim V^{*} = \dim \operatorname{Hom}(S, \mathbb{Q}).$$

□

4. – The subarchimedean case.

Recall from [1] that a commutative semigroup T is *subarchimedean* if there exists $z \in T$ such that for any $a \in T$, $z^n = ax$ for some $n \in \mathbb{P}$ and $x \in T$. We consider here homomorphisms of subarchimedean (as usual commutative cancellative semigroup) S into the additive group of rational numbers. To this end, we first introduce the necessary notation for the Tamura-like representation of these semigroups and some related symbolism.

For a group G, we consider a function $I : G \times G \to \mathbb{N}$ satisfying the following conditions:

$$\begin{array}{lll} (A) & I(a,b) + I(ab,c) = I(a,bc) + I(b,c) & (a,b,c \in G), \\ (C) & I(a,b) = I(b,a) \quad (a,b \in G), & \\ (N_i) & I(e,e) = i & \end{array}$$

where e is the identity of G and $i = 0$ or $i = 1$. On the set $\mathbb{N} \times G$ define a multiplication by

$$(M) \qquad (m,a)(n,b) = (m+n+I(a,b), ab)\,.$$

We denote the resulting groupoid by $\mathbb{N}_i(G,I)$.

The importance of the above construction stems from the following statements, see ([5], Section 3) for part (*ii*) and ([1], Section 4) for both parts.

FACT 4.1.

(*i*) *Groupoid $\mathbb{N}_0(G,I)$ is a commutative cancellative subarchimedean nongroup monoid. Conversely, every semigroup with these properties is isomorphic to some $\mathbb{N}_0(G,I)$.*

(*ii*) *Groupoid $\mathbb{N}_1(G,I)$ is a commutative cancellative subarchimedean idempotent-free semigroup. Conversely, every semigroup with these properties is isomorphic to some $\mathbb{N}_1(G,I)$.*

The purpose of the next two sections is to clarify the relationship of $\operatorname{Hom}(S,\mathbb{Q})$ and $\operatorname{Hom}(G,\mathbb{Q})$ when $S=\mathbb{N}_i(G,I)$ for $i=0,1$. We shall need some more notation.

For G a group and a function $\varphi : G \to \mathbb{Q}$, we write

$$\varphi(a,b) = \varphi(a)+\varphi(b)-\varphi(ab) \qquad (a,b\in G)\,.$$

Now let $I : G\times G \to \mathbb{N}$ satisfy conditions (A), (C) and (N_i) for $i=0$ or $i=1$, and e be the identity of G. We introduce the notation

$$\operatorname{def} I = \{\varphi : G\to\mathbb{Q} \mid \varphi(a,b) = I(a,b)+1-i \text{ for all } a,b\in G\},$$

$$\operatorname{Def} I = \{\varphi : G\to\mathbb{Q} \mid \varphi(a,b) = (I(a,b)+1-i)(\varphi e) \text{ for all } a,b\in G\}.$$

We provide $\operatorname{Def} I$ with the structure of a rational vector space by defining a sum and a scalar product by

$$(\varphi+\psi)a = \varphi a+\psi a,\;\; (r\varphi)a = r(\varphi a) \qquad (\varphi,\psi\in \operatorname{Def} I,\; r\in\mathbb{Q},\; a\in G).$$

A close analogue of these functions was used by Tamura, see [6], as an alternative for the function I. He also studied objects similar to elements of $\operatorname{def} I$ using the notation $\operatorname{Dfn}_I(G,\mathbb{R}_+)$: functions $\varphi : G\to\mathbb{R}_+$ (positive real numbers) satisfying $\varphi(a,b)=I(a,b)$ for all $a,b\in G$. He used these functions to construct groupoids akin to $\mathbb{N}_1(G,I)$ serving essentially the same purpose; see also [5].

We shall require three lemmas for the proof of the sole result of this section in which we use the notation introduced above.

LEMMA 4.2. – *For any $a\in G$, we have $I(a,e)=I(e,e)$.*

PROOF. – Substituting $b=c=e$ in condition (A), we obtain

$$I(a,e)+I(a,e) = I(a,e)+I(e,e)$$

which proves the assertion. $\square$

LEMMA 4.3. – *For* $i = 0, 1$, *we have*

$$\begin{array}{ll} (1-i,e)^{m-1+i}(1-i,a) = (m,a) & \text{if } m > 0 \text{ or } i = 1, \\ (1,a) = (1,e)(0,a) & \text{if } m = 0 = i. \end{array}$$

PROOF. – If $m - 1 + i = 0$, then

$$(1-i,e)^0(1-i,a) = (1-i,a) = (m,a).$$

Consider $k = m - 1 + i > 0$; the argument is by induction. If $k = 1$, then

$$(1-i,e)(1-i,a) = (2-2i+I(e,a),a) = (2-i,a)$$

where $1 = k = m - 1 + i$ implies $m = 2 - i$ and the formula holds. Suppose the formula valid for $k = m - 1 + i$. Then

$$\begin{array}{rcl} (1-i,e)^{m-1+i+1}(1-i,a) & = & (1-i,e)(1-i,e)^k(1-i,a) \\ & = & (1-i,e)(m,a) \\ & = & (1-i+m+I(e,a),a) = (m+1,a), \end{array}$$

as required. The second case is obvious. □

The next lemma clarifies somewhat the relationship of def I and Def I.

LEMMA 4.4. – $\emptyset \neq \operatorname{def} I \subset \operatorname{Def} I$.

PROOF. – Set $S = \mathbb{N}_i(G,I)$ and recall that $\delta : S \to SQ$ is the canonical injection of S into its group of quotients. Since $(1-i,e)^n = (n-i,e)$ for $i = 0, 1$, we conclude that the element $(1-i,e)$ is of infinite order. Hence the mapping

$$\big(\delta(1-i,e)\big)^n \mapsto n \qquad (n \in \mathbb{Z})$$

is a homomorphism of the subgroup of SQ generated by $\delta(1-i,e)$ into $\mathbb{Q}$. Since $\mathbb{Q}$ is a divisible group, it has the homomorphism extension property, see ([3], Theorem 21.1), δ extends to a homomorphism $\beta : SQ \to \mathbb{Q}$. Let $\gamma = \beta\delta : S \to \mathbb{Q}$ and observe that $\gamma(1-i,e) = 1$.

We now define φ_0 by

$$\text{(1)} \qquad \varphi_0 : \; a \mapsto \gamma(1-i,a) \quad (a \in G),$$

so that $\varphi_0 : G \to \mathbb{Q}$ (with $i = 0$ or $i = 1$ fixed). For $a, b \in G$, we obtain

$$\begin{aligned} \varphi_0 a + \varphi_0 b &= \gamma(1-i,a) + \gamma(1-i,b) = \gamma((1-i,a)(1-i,b)) \\ &= \gamma(2-2i+I(a,b),ab) \overset{4.3}{=} \gamma\big((1-i,e)^{I(a,b)+1-i}(1-i,ab)\big) \\ &= (I(a,b)+1-i)\gamma(1-i,e) + \gamma(1-i,ab) = I(a,b)+1-i+\varphi_0(ab) \end{aligned}$$

whence

$$\varphi_0 a + \varphi_0 b - \varphi_0(ab) = I(a,b) + 1 - i$$

so that $\varphi_0 \in \operatorname{def} I$. Therefore $\operatorname{def} I \neq \emptyset$.

Let $\varphi \in \operatorname{def} I$. Then using (N_i) we get

$$\varphi e = \varphi(e,e) = I(e,e) + 1 - i = i + 1 - i = 1$$

and thus $\varphi \in \operatorname{Def} I$. With the above φ_0, letting $\psi = 2\varphi_0$, we get immediately that $\psi \in \operatorname{Def} I$ (since $\operatorname{Def} I$ is a rational vector space) but $\psi \notin \operatorname{def} I$ (since $\psi e = 2 \neq 1$). □

We are now ready for the result of this section.

THEOREM 4.5. – *Let $S = \mathbb{N}_i(G,I)$ where $i = 0$ or $i = 1$.*

(*i*) $\operatorname{Def} I$ *has* $\operatorname{Hom}(G,\mathbb{Q})$ *as a vector subspace of codimension* 1.

(*ii*) *For any $\psi \in \operatorname{def} I$, we have $\operatorname{def} I = \operatorname{Hom}(G,\mathbb{Q}) + \psi$. Hence $\operatorname{def} I$ is an affine subspace of $\operatorname{Def} I$ of codimension* 1.

(*iii*) *For every $\varphi \in \operatorname{Def} I$, define $\overline{\varphi}$ by*

$$\overline{\varphi}(m,a) = (m - 1 + i)(\varphi e) + \varphi a \qquad ((m,a) \in S),$$

and for every $\chi \in \operatorname{Hom}(S,\mathbb{Q})$, define $\widehat{\chi}$ by

$$\widehat{\chi} a = \chi(1 - i, a) \qquad (a \in G).$$

Then the mappings

$$\Gamma : \operatorname{Def} I \to \operatorname{Hom}(S,\mathbb{Q}), \qquad \varphi \mapsto \overline{\varphi},$$
$$\Delta : \operatorname{Hom}(S,\mathbb{Q}) \to \operatorname{Def} I, \qquad \chi \mapsto \widehat{\chi},$$

are mutually inverse isomorphisms of rational vector spaces.

(*iv*) $\Gamma|_{\operatorname{Hom}(S,\mathbb{Q})}$ *is an isomorphism of* $\operatorname{Hom}(G,\mathbb{Q})$ *onto the subspace*

$$\Sigma = \{\chi \in \operatorname{Hom}(S,\mathbb{Q}) \mid \chi(m,a) = \chi(n,a) \text{ for all } m,n \in \mathbb{N}, a \in G\}.$$

PROOF. – Set $H = \operatorname{Hom}(G,\mathbb{Q})$ and observe that

$$H = \{\varphi \in \operatorname{Def} I \mid \varphi(a,b) = 0 \text{ for all } a,b \in G\}.$$

(*i*) Straightforward verification will show that H is a subspace of $\operatorname{Def} I$. In order to prove that it is of codimension 1, we let φ_0 be the function (1) in the proof of Lemma 4.4. First note that $\varphi_0 \notin H$ since $\varphi_0 e = 1 \neq 0$. Let $\varphi \in \operatorname{Def} I$ be such that $\varphi e = 0$. The definition of $\operatorname{Def} I$ implies that $\varphi \in H$. Hence for every $\varphi \in \operatorname{Def} I$ such that $\varphi \notin H$, we have $\varphi e \neq 0$. Defining $\theta = \frac{1}{\varphi e}\varphi - \varphi_0$, we get

$$\theta e = \frac{1}{\varphi e}(\varphi e) - \varphi_0 e = 1 - 1 = 0$$

so that $\theta \in H$. It follows that $\varphi = (\varphi e)\theta + (\varphi e)\varphi_0$ which proves that codimension of H in Def I equals 1.

(*ii*) Let $\psi, \theta \in \operatorname{def} I$ and set $\varphi = \theta - \psi$. For any $a, b \in G$, we immediately get that $\varphi(a, b) = 0$ and thus $\varphi \in H$. Therefore

$$\theta = \varphi + \psi \in H + \psi$$

and thus $\operatorname{def} I \subseteq H + \psi$. Conversely, let $\theta = \varphi + \psi$ where $\varphi \in H$. For any $a, b \in G$, we have

$$\theta(a, b) = \varphi(a, b) + \psi(a, b) = \psi(a, b) = I(a, b) + 1 - i$$

so that $\theta \in \operatorname{def} I$. Consequently $H + \psi \subseteq \operatorname{def} I$ and equality prevails.

(*iii*) For $\varphi \in \operatorname{Def} I$ and $a \in G$, we get

$$\widehat{\overline{\varphi}} a = \overline{\varphi}(1 - i, a) = (1 - i - 1 + i)(\varphi e) + \varphi a = \varphi a$$

and thus $\widehat{\overline{\varphi}} = \varphi$, that is $\varDelta\varGamma$ is the identity mapping on Def I.

Let $\chi \in \operatorname{Hom}(S, \mathbb{Q})$ and $(m, a) \in S$. First

$$\begin{aligned} \overline{\widehat{\chi}}(m, a) &= (m - 1 + i)(\widehat{\chi} e) + \widehat{\chi} a \\ &= (m - 1 + i)\chi(1 - i, e) + \chi(1 - i, a). \end{aligned}$$

If $m > 0$ or $i = 1$, then by Lemma 4.3, we get

$$\overline{\widehat{\chi}}(m, a) = \chi\big((1 - i, e)^{m-1+i}(1 - i, a)\big) = \chi(m, a),$$

otherwise $m = i = 0$ so that

$$\overline{\widehat{\chi}}(0, a) + \chi(1, e) = \chi(1, a).$$

This together with

$$\chi(0, a) + \chi(1, e) = \chi\big((0, a)(1, e)\big) = \chi(1, a)$$

yields that $\overline{\widehat{\chi}}(0, a) = \chi(0, a)$. Therefore $\overline{\widehat{\chi}}(m, a) = \chi(m, a)$ in all cases and $\varGamma\varDelta$ is the identity mapping on Hom $(S, \mathbb{Q})$.

For $\chi, \chi' \in \operatorname{Hom}(S, \mathbb{Q})$, $r \in \mathbb{Q}$ and $a \in G$, we have

$$\begin{aligned} \big(\varDelta(\chi + \chi')\big)a &= (\chi + \chi')(1 - i, a) = \chi(1 - i, a) + \chi'(1 - i, a) \\ &= \widehat{\chi} a + \widehat{\chi'} a = (\varDelta\chi + \varDelta\chi')a, \\ \big((\varDelta(r\chi)\big)a &= (r\chi)(1 - i, a) = r(\chi(1 - i, a)) = r(\widehat{\chi} a) = \big(r(\varDelta\chi)\big)a \end{aligned}$$

which implies that $\varDelta$ is an isomorphism and hence $\varGamma = \varDelta^{-1}$ is too.

(*iv*) This is an immediate consequence of parts (i) and (iii). □

Note that

$$\varSigma = \{\chi \in \operatorname{Hom}(S, \mathbb{Q}) \mid \chi \text{ factors through the homomorphism } (m, a) \mapsto a\},$$

so no wonder that $\sum \cong \operatorname{Hom}(G, \mathbb{Q})$.

5. – An alternative approach.

We will now devise an isomorphic copy of $\operatorname{Def} I$ which ought to clarify somewhat the relationship of the objects we studied in the preceding section. We start with the needed notation.

Fix an element $\varphi_0 \in \operatorname{def} I$ (by Lemma 4.4 it exists), and define a sum and scalar product on $\operatorname{Def} I$ by

$$\varphi \oplus \psi = \varphi + \psi - \varphi_0, \quad r \cdot \varphi = r\varphi + (1-r)\varphi_0 \qquad (\varphi, \psi \in \operatorname{Def} I,\ r \in \mathbb{Q})$$

and denote the resulting structure by $(\operatorname{Def} I, \oplus)$.

Restricting the operations of $(\operatorname{Def} I, \oplus)$ to $\operatorname{def} I$, we denote the resulting structure by $(\operatorname{def} I, \oplus)$.

THEOREM 5.1. – *With the above notation, the following statements hold.*

(i) $(\operatorname{Def} I, \oplus)$ *is a rational vector space having* $(\operatorname{def} I, \oplus)$ *as a subspace of codimension* 1.

(ii) *The mapping*

$$\Phi : \varphi \mapsto \varphi + \varphi_0 \qquad (\varphi \in \operatorname{Def} I)$$

is a vector space isomorphism of $\operatorname{Def} I$ *onto* $(\operatorname{Def} I, \oplus)$.

(iii) $\Phi|_{\operatorname{Hom}(G,\mathbb{Q})}$ *is a vector space isomorphism of* $\operatorname{Hom}(G, \mathbb{Q})$ *onto* $(\operatorname{def} I, \oplus)$.

PROOF. – It is clear that Φ is a bijective map. Furthermore, for any $\varphi, \psi \in \operatorname{Def} I$ and $r \in \mathbb{Q}$, we get

$$\Phi(\varphi + \psi) = \varphi + \psi + \varphi_0 = (\varphi + \varphi_0) + (\psi + \varphi_0) - \varphi_0 = \Phi(\varphi) \oplus \Phi(\psi)$$

$$\Phi(r\varphi) = r\varphi + \varphi_0 = r(\varphi + \varphi_0) + (1-r)\varphi_0 = r \cdot \Phi(\varphi)$$

which proves that Φ is an isomorphism and thus $(\mathit{Def}\, I, \oplus)$ is a rational vector space.

We prove now that $\Phi\big(\operatorname{Hom}(G, \mathbb{Q})\big) = \operatorname{def} I$. Indeed, if $\theta \in \operatorname{Hom}(G, \mathbb{Q})$, then

$$(\Phi\theta)(a,b) = (\theta + \varphi_0)(a,b) = \theta(a,b) + \varphi_0(a,b) = 0 + I(a,b) + 1 - i = I(a,b) + 1 - i$$

and thus $\Phi\theta \in \operatorname{def} I$. Conversely, if $\varphi \in \operatorname{def} I$, then

$$(\varphi - \varphi_0)(a,b) = I(a,b) + 1 - i - I(a,b) - 1 + i = 0,$$

and therefore $\theta = \varphi - \varphi_0 \in \operatorname{Hom}(G, \mathbb{Q})$ and $\varphi = \Phi\theta \in \Phi\big(\operatorname{Hom}(G, \mathbb{Q})\big)$.

Since $\operatorname{Hom}(G, \mathbb{Q})$ is a subspace of codimension 1 in $\operatorname{Def} I$, according to Theorem 4.5(i), the corresponding statement follows for $(\operatorname{def} I, \oplus)$ relative to $(\operatorname{Def} I, \oplus)$. □

We now draw some consequences of the results obtained so far.

COROLLARY 5.2. –*Let* $S = \mathbb{N}_i(G, I)$ *where* $i = 0$ *or* $i = 1$. *Then* rank S, rank G, dim (Def $I, \oplus$), dim (def $I, \oplus$), dim Hom $(S, \mathbb{Q})$ *and* dim Hom $(G, \mathbb{Q})$ *are all finite or are all infinite. If they are finite, then*

$$\begin{aligned}\operatorname{rank} S &= \dim(\operatorname{Def} I, \oplus) = \dim \operatorname{Hom}(S, \mathbb{Q}) = \operatorname{rank} G + 1 \\ &= \dim \operatorname{Hom}(G, \mathbb{Q}) + 1 = \dim(\operatorname{def} I, \oplus) + 1 .\end{aligned}$$

PROOF. – All the assertions follow from Theorems 3.3, 4.5 and 5.1. □

We now illustrate the mappings in Theorems 4.5 and 5.1 by the following diagram.

$$\begin{array}{ccccc}
\operatorname{Hom}(S, \mathbb{Q}) & \xleftarrow{\ \varGamma\ } & \operatorname{Def} I & \xrightarrow{\ \varPhi\ } & (\operatorname{Def} I, \oplus) \\
{\scriptstyle in}\downarrow & & {\scriptstyle in}\downarrow & & {\scriptstyle in}\downarrow \\
\varSigma & \xleftarrow[\varGamma|_{\operatorname{Hom}(G,\mathbb{Q})}]{} & \operatorname{Hom}(G, \mathbb{Q}) & \xrightarrow[\varPhi|_{\operatorname{Hom}(G,\mathbb{Q})}]{} & (\operatorname{def} I, \oplus) .
\end{array}$$

Corollary 5.2 is related to ([6], Corollary 4.8) which represents a weaker statement but in the context of functions from the group G into positive real numbers. Much of our discussion is based on the existence of nontrivial homomorphisms from a commutative cancellative semigroup into $\mathbb{Q}$. As ([5], Example 4.10) shows, a commutative cancellative idempotent-free subarchimedean semigroup need not have homomorphisms into $\mathbb{R}_+$, the semigroup of all positive real numbers, and thus *a fortiori* into $\mathbb{Q}_+$. According to ([5], Theorem 4.1), for a commutative cancellative idempotent-free semigroup of finite rank S, we have $\operatorname{Hom}(S, \mathbb{Q}_+^0) \neq \{0\}$ where $\mathbb{Q}_+^0$ is the additive semigroup of nonnegative rationales. This subject was also studied in [7]. Hence the exact situation here is still to be explored.

6. – A special case.

We illustrate some of the results obtained by considering the (very) special case of semigroups of rank 1. For any set X, we denote by $|X|$ its cardinality.

THEOREM 6.1. – *Let G and I be given, with I satisfying conditions* (A), (C) *and* (N_0) *or* (N_1), *and let* $S \cong \mathbb{N}_i(G, I)$. *Then the following conditions are equivalent.*

(i) *rank* $S = 1$.
(ii) G *is periodic.*
(iii) $|\operatorname{Hom}(G, \mathbb{Q})| = 1$.
(iv) $|\operatorname{def} I| = 1$.
(v) $\dim \operatorname{Def} I = 1$.

In such a case, the following holds.

(a) *In part (iii), the unique homomorphism is the zero homomorphism.*

(b) *In part (iv), the unique* $\varphi \in \operatorname{def} I$ *is given by*

$$\varphi a = \frac{1}{n}\sum_{j=1}^{n} I(a, a^j) + 1 - i \qquad (a \in G)$$

where n is the order of a in G.

(c) *In part* (*v*), φ *generates* Def I.

Proof. – *(i) and (ii) are equivalent.* By Corollary 5.2(iii) we know that $\operatorname{rank} S = 1$ if and only if $\operatorname{rank} G = 0$. But the abelian group G is of rank zero if and only if it has no elements of infinite order, that is, if and only if it is a periodic group.

(*ii*) *and* (*iii*) *are equivalent*. This is a consequence of Theorem 3.3.

(*iii*) *implies* (*iv*). By Theorem 5.1(iii), we have $\operatorname{Hom}(G, \mathbb{Q}) \cong (\operatorname{def} I, \oplus)$.

(*iv*) *implies* (*v*). The hypothesis implies that $\dim(\operatorname{def} I, \oplus) = 0$ which by Theorem 5.1(i) yields that $\dim(\operatorname{Def} I, \oplus) = 1$. This by Theorem 5.1(ii) implies that $\dim \operatorname{Def} I = 1$.

(*v*) *implies* (*i*). By Corollary 5.2(i) and Theorem 5.1(ii), we have $\operatorname{rank} S = \dim(\operatorname{Def} I, \oplus) = \dim \operatorname{Def} I = 1$.

We now prove the additional assertions of the theorem.

(*a*) This is trivial.

(*b*) Let $\varphi \in \operatorname{def} I$ and $a \in G$. Then

$$\varphi a + \varphi a - \varphi a^2 = I(a, a) + 1 - i$$

and hence

$$\varphi a^2 = 2\varphi a - I(a, a) - 1 + i\,, \tag{2}$$

next

$$\varphi a + \varphi a^2 - \varphi a^3 = I(a, a^2) + 1 - i$$

so that

$$\begin{aligned}\varphi a^3 &= \varphi a + \varphi a^2 - I(a, a^2) - 1 + i \\ &= \varphi a + 2\varphi a - I(a, a) - 1 + i - I(a, a^2) - 1 + i \\ &= 3\varphi a - I(a, a) - I(a, a^2) - 2(1 - i)\,.\end{aligned}$$

For $k \leq n$, let $I_k = \sum_{j=1}^{k} I(a, a^j)$. Continuing this procedure, for $n > 0$ we get

$$\varphi a^n = n\varphi a - I_{n-1} a - (n-1)(1 - i)\,. \tag{3}$$

On the other hand, by (2) we get

$$\varphi e = \varphi e^2 = 2\varphi e - I(e,e) - 1 - i$$

and thus $\varphi e = 1$. Let n be the order of a. Then (3) implies that

$$1 = n\varphi a - I_{n-1}a - (n-1)(1-i)$$

whence

$$\varphi a = \frac{1}{n}(I_{n-1}a + i) + 1 - i = \frac{1}{n} I_n a + 1 - i\,,$$

since $i = I(e,e) = I(a, a^n)$.

(c) In particular, $\varphi e = \frac{1}{1} I(e,e) + 1 - i = 1$. Hence $0 \neq \varphi \in \operatorname{def} I \subset \operatorname{Def} I$ and thus φ generates $\operatorname{Def} I$ since $\dim \operatorname{Def} I = 1$. □

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Pervenuta in Redazione
il 25 luglio 2003 e in forma rivista il 15 maggio 2004