
BOLLETTINO UNIONE MATEMATICA ITALIANA

LORENZO MENCHERINI

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well-posedness of 2×2 differential systems with
time-dependent coefficients**

*Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. 9-B (2006),
n.1, p. 215–220.*

Unione Matematica Italiana

http://www.bdim.eu/item?id=BUMI_2006_8_9B_1_215_0

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A simple Necessary and Sufficient Condition for Well-Posedness of 2×2 Differential Systems with Time-dependent Coefficients.

LORENZO MENCHERINI

Sunto. – *Dato il Problema di Cauchy*

$$\partial_t u(x, t) + A(t)\partial_x u(x, t) = 0 \quad u(0, x) = u_0(x)$$

Nishitani [N], dopo aver effettuato, mediante una matrice di cambiamento di base costante, la trasformazione della matrice

$$A(t) = \begin{bmatrix} d(t) & a(t) \\ b(t) & -d(t) \end{bmatrix} \quad t \in [0, T]$$

reale, analitica e iperbolica, nella matrice complessa

$$A^\sharp(t) = \begin{bmatrix} c^\sharp(t) & a^\sharp(t) \\ \overline{a^\sharp(t)} & -c^\sharp(t) \end{bmatrix} = \begin{bmatrix} i\frac{a-b}{2} & \frac{a+b}{2} + id \\ \frac{a+b}{2} - id & -i\frac{a-b}{2} \end{bmatrix},$$

ha dimostrato che il Problema di Cauchy considerato è ben posto in C^∞ in un intorno di zero se e solo se vale la condizione

$$h|a^\sharp|^2 \geq Ct^2|D^\sharp|^2,$$

dove

$$D^\sharp = \dot{a}^\sharp c^\sharp - \dot{c}^\sharp a^\sharp \quad \text{e} \quad h = -\det A = |a^\sharp|^2 - |c^\sharp|^2.$$

In questo breve lavoro invece diamo una semplicissima condizione equivalente a quella di Nishitani (e quindi necessaria e sufficiente per la buona positura), in cui compaiono solamente gli elementi di $A(t)$ e non le loro derivate.

Summary. – *Given the Cauchy Problem*

$$\partial_t u(x, t) + A(t)\partial_x u(x, t) = 0 \quad u(0, x) = u_0(x) \quad x \in \mathbf{R}$$

Nishitani [N], by making use of a change of basis by a constant matrix, transformed the real, analytic, hyperbolic matrix

$$A(t) = \begin{bmatrix} d(t) & a(t) \\ b(t) & -d(t) \end{bmatrix} \quad t \in [0, T]$$

into the complex matrix

$$A^\sharp(t) = \begin{bmatrix} c^\sharp(t) & a^\sharp(t) \\ \overline{a^\sharp(t)} & -c^\sharp(t) \end{bmatrix} = \begin{bmatrix} i\frac{a-b}{2} & \frac{a+b}{2} + id \\ \frac{a+b}{2} - id & -i\frac{a-b}{2} \end{bmatrix},$$

and showed that the given Cauchy Problem is well posed in C^∞ in a neighborhood of zero if and only if (see also [MS]) the following condition

$$h|a^\sharp|^2 \geq Ct^2|D^\sharp|^2$$

is satisfied, where

$$D^\sharp = \dot{a}^\sharp c^\sharp - \dot{c}^\sharp a^\sharp \quad \text{and} \quad h = -\det A = |a^\sharp|^2 - |c^\sharp|^2.$$

In this short note, we give a very simple condition, which is equivalent to that of Nishitani (and then a necessary and sufficient for the Well-Posedness), but where only the elements of $A(t)$, appear and not their derivatives.

1. – The main result.

Since $A(t)$ is analytic, there exists an integer $\nu \geq 0$ so that $A(t) = t^\nu A_1(t)$ with $A_1(t) \neq 0$. Let us set

$$A_1(0) = \begin{bmatrix} d_\circ & a_\circ \\ b_\circ & -d_\circ \end{bmatrix}.$$

By the formula of Nishitani, we can see easily that the Cauchy Problem for the matrix $A(t)$ is well posed in C^∞ if and only if it is for the Cauchy Problem with the matrix $A_1(t)$. For the condition of Well-Posedness, one is automatically reduced to that of the matrix $A_1(t)$, so it is natural to denote the latter matrix again by $A(t)$. On this matrix the following result holds

THEOREM 1. – *If*

$$A(0) = \begin{bmatrix} d_\circ & a_\circ \\ b_\circ & -d_\circ \end{bmatrix} \neq 0,$$

then the following two conditions are equivalent

- (1)
$$h \geq C t^2 |D^\sharp|^2$$
- (2)
$$h \geq C |d_\circ(a + b) - (a_\circ + b_\circ)d|^2.$$

REMARK 1. – *If the determinant of $A(t)$ does not vanish at the origin, both the conditions are satisfied and then the constraint of determinant equal to zero at zero time does not constitute any loss of generality.*

REMARK 2. – *To avoid too much heavy formalism, from now on we shall use the following notation*

$$x(t) = \operatorname{Re}(a^\sharp(t)), \quad y(t) = \operatorname{Im}(a^\sharp(t)), \quad z(t) = \operatorname{Im}(c^\sharp(t)),$$

although we'll keep the symbols $a^\sharp(t)$ e $c^\sharp(t)$ in cases where these would be more significant. The three coordinate functions $x(t), y(t), z(t)$ characterize completely the matrix $A(t)$. Indeed

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a(t) \\ b(t) \\ d(t) \end{bmatrix}$$

and on the right hand side the 3×3 matrix is invertible. In this vectorial form, we'll indicate the matrix $A^\sharp(t)$ by $P(t)$. Besides, in order to avoid confusion between this triplet of functions and the generic point (x, y, z) of the euclidean reference space, we shall denote the latter by (X, Y, Z) .

The proof is split in three steps.

First Step: we show that the relation (1) is rotation-invariant for time-independent angles around the Z -axis in the real 3-dimensional space whose orthogonal axes are X, Y, Z . To this end it is sufficient to show that the quantities $h(t)$ e $|D^\sharp(t)|$ are rotation-invariant for constant angles about the Z -axis, in the sense of

$$h(T_{\mathcal{J}}P) = h(P) \quad \text{and} \quad |D^\sharp(T_{\mathcal{J}}P)| = |D^\sharp(P)|,$$

where we have denoted by $T_{\mathcal{J}}$ the rotation operator relative to an angle \mathcal{J} around to the Z -axis. The invariance of h follows by the fact that $h = x^2 + y^2 - z^2$, whereas the respective invariance of $x^2 + y^2$ and z^2 is trivial. As to $|D^\sharp|$, we have

$$D^\sharp(T_{\mathcal{J}}P) = (a^\sharp e^{i\mathcal{J}}) c^\sharp - \dot{c}^\sharp (a^\sharp e^{i\mathcal{J}}) = e^{i\mathcal{J}} D^\sharp(P)$$

whence

$$|D^\sharp(T_{\mathcal{J}}P)| = |D^\sharp(P)|.$$

Second Step: We recall that (see the example 5 in [N]) the

$$(3) \quad h \geq C d^2$$

imply (1). Conversely here we'll show that if a certain matrix satisfies the conditions

$$(4) \quad a_\circ \neq 0 \quad b_\circ = 0 = d_\circ,$$

at time zero, then also the opposite implication holds and then (3) and (1) are equivalent. We observe preliminarily that

$$|D^\sharp|^2 = \frac{1}{4} [\dot{d} (a - b) - d (\dot{a} - \dot{b})]^2 + \frac{1}{4} (ab - \dot{a}b)^2 \quad h = ab + d^2.$$

Let's suppose that (1) holds. Then we have

$$\begin{aligned} |D^\sharp| &\geq C_1 |(a - b) \dot{d} - (\dot{a} - \dot{b}) d| + C_1 |\dot{a}b - a\dot{b}| \\ &\geq C_1 \left| |(a - b) \dot{d}| - |(\dot{a} - \dot{b}) d| \right| + C_1 |\dot{a}b - a\dot{b}| \\ &\geq C_1 \left| |(a - b) \dot{d}| - |(\dot{a} - \dot{b}) d| \right|. \end{aligned}$$

for some constant $C_1 > 0$.

Notice then that

$$a_\circ \neq 0 \Rightarrow \exists C_2 > 0 : |a(t) - b(t)| \geq C_2 \Rightarrow |(a - b) \dot{d}| \geq C_2 |\dot{d}(t)|$$

and that, by the analyticity of $d(t)$, we have

$$d_\circ = 0 \Rightarrow \exists C_3 > 0 : |d(t)| \leq C_3 t \dot{d}(t) \Rightarrow |(\dot{a} - \dot{b}) d| \leq C_3 t |\dot{d}(t)|,$$

so that

$$|D^\sharp| \geq C_1 \left| |(a - b) \dot{d}| - |(\dot{a} - \dot{b}) d| \right| \geq C_2 |\dot{d}(t)|.$$

If we now apply the condition of Nishitani, we have

$$\sqrt{h} \geq C t |D^\sharp| \geq C C_2 t |\dot{d}| \geq C_4 |d| \quad \text{for some constant } C_4 > 0$$

that is,

$$h \geq C_6 d^2, \quad \text{where } C_6 = C_4^2,$$

which we can write as

$$h \geq C_6 \mathcal{I} m^2 a^\sharp.$$

Third Step. Suppose that for $t = 0$ we have $P(0) = P_\circ = (X_\circ, Y_\circ, Z_\circ)$ with $h(0) = X_\circ^2 + Y_\circ^2 - Z_\circ^2 = 0$. We now consider the rotation $T_{\mathcal{J}}$ of the whole space XYZ , around to the Z -axis, which transforms the generator OP_\circ of the cone $h = 0$ in the generator $O\tilde{P}_\circ$ whose equations are $\tilde{X} = \tilde{Z}$ and $\tilde{Y} = 0$. In this way, in the new variables $(\tilde{X}, \tilde{Y}, \tilde{Z}) = T_{\mathcal{J}}(X, Y, Z)$, the conditions (4) are satisfied, where we know that (1) and (3) are equivalent. By this rotation, the triple of functions $(x(t), y(t), z(t))$ goes to the correspondent triple $(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))$, so that the condition (1) of well-posedness (unchanged for the first step) for the second step is equivalent to the

$$(5) \quad h(t) \geq C \tilde{y}^2(t),$$

which becomes therefore a necessary and sufficient condition for well-posedness in the new variables. Thus, we have simplified considerably the condition (1) of Nishitani. Now we want to express (5) as a function of the elements of the starting matrix. Therefore we apply $T_{\mathcal{J}}^{-1} = T_{-\mathcal{J}}$ to (5). We have already observed that $h(t)$ is rotation-invariant for a time-independent angle. On the right hand

side of (5) we have $\tilde{y}^2(t)$: note that $|\tilde{Y}|$ is the distance of $(\tilde{X}, \tilde{Y}, \tilde{Z})$ from the plane $\pi : \tilde{Y} = 0$, determined by the point $\tilde{P}_o = (\tilde{X}_o, 0, \tilde{X}_o)$ and by the axis $\tilde{X} = 0 = \tilde{Y}$ of the cone $\tilde{X}^2 + \tilde{Y}^2 - \tilde{Z}^2 = 0$. As the rotations are isometries, this distance is unchanged under T_{-g} . Under this transformation the point $(\tilde{X}, \tilde{Y}, \tilde{Z})$ is changed into (X, Y, Z) and the plane $\pi : \tilde{Y} = 0$ is transformed into the plane

$$Y_o X - X_o Y = 0$$

because it is the plane through the axis $X = 0 = Y$ and the initial point $P_o = (X_o, Y_o, Z_o)$. Finally we have the assertion simply by using the formula of the distance point from the plane, which in this particular case becomes:

$$dist(P, \pi) = \frac{|Y_o X - X_o Y|}{\sqrt{X_o^2 + Y_o^2}}.$$

We observe that the denominator of the previous expression does not vanish because $A(0) \neq 0$ e $h(0) = 0$. Going back from the pointwise notation to the functional notation we have

$$h(t) \geq C|Y_o x(t) - X_o y(t)|^2$$

which in the variables of the original matrix $A(t)$ becomes

$$h(t) \geq C|d_o(a(t) + b(t)) - (a_o + b_o)d(t)|^2. \quad \square$$

COROLLARY 1. – *The Cauchy Problem*

$$\partial_t u(x, t) + A(t)\partial_x u(x, t) = 0 \quad u(0, x) = u_o(x) \quad x \in \mathbf{R}$$

where

$$A(t) = t^\nu \begin{bmatrix} d(t) & a(t) \\ b(t) & -d(t) \end{bmatrix}, \quad t \in [0, T]$$

is a real, analytic, hyperbolic matrix, and $\nu \in \mathbf{N}$ is such that $\|t^{-\nu}A(0)\| \neq 0$, is Well-Posed in C^∞ in a neighborhood of zero if and only if the condition

$$a(t) b(t) + d^2(t) \geq C |d(0) [a(t) + b(t)] - [a(0) + b(0)] d(t)|^2$$

is satisfied.

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Dipartimento di Matematica, Università di Pisa,
Largo B. Pontecorvo n. 5, 56127, Pisa
E-mail: mencheri@mail.dm.unipi.it

Pervenuta in Redazione
il 25 settembre 2003