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Distributional Dunkl Transform and Dunkl Convolution Operators.

JORGE J. BETANCOR

Sunto. – *In questo lavoro, che è diviso in due parti, studiamo la trasformata distribuzionale di Dunkl su* $\boldsymbol{R}$. *Nella prima parte studiamo la trasformata di Dunkl e gli operatori di convoluzione di Dunkl sulle distribuzioni temperate. Dimostriamo che le distribuzioni temperate che definiscono operatori di convoluzione di Dunkl sullo spazio di Schwartz* $\mathcal{S}$ *sono gli elementi di* $\mathcal{O}'_c$, *lo spazio degli operatori convoluzione usuali su S. Nella seconda parte definiamo la trasformata distribuzionale di Dunkl usando il metodo del nucleo. Introduciamo gli spazi funzione di Fréchet contenenti il nucleo della trasformata di Dunkl. Nella dimostrazione delle proprietà della trasformata distribuzionale di Dunkl, definita sugli spazi duali corrispondenti, alcune rappresentazioni degli elementi degli spazi duali giocheranno un ruolo importante. Queste rappresentazioni ci permettono di semplificare, in contrasto con i metodi usuali e precedenti (vedi, per esempio* [7] *e* [13]), *le sopracitate dimostrazioni. La nostra nuova procedura si applica anche ad altre trasformate distribuzionali integrali che sono state studiate da altri autori (trasformate di Hankel* ([7] *e* [13]), *fra le altre).*

Summary. – *In this paper, that is divided in two parts, we study the distributional Dunkl transform on* $\boldsymbol{R}$. *In the first part we investigate the Dunkl transform and the Dunkl convolution operators on tempered distributions. We prove that the tempered distributions defining Dunkl convolution operators on the Schwartz space* $\mathcal{S}$ *are the elements of* $\mathcal{O}'_c$, *the space of usual convolution operators on S. In the second part we define the distributional Dunkl transform by employing the kernel method. We introduce Fréchet function spaces containing the kernel of the Dunkl transform. In the proof of the properties of the distributional Dunkl transform, defined on the corresponding dual spaces, certain representations of the elements of the dual spaces will play an important role. These representations allows us to simplify, in contrast with the previous and usual methods* (*see, for instance* [7] *and* [13]), *the mentioned proofs. Our new procedure also applies to other distributional integral transforms that had been investigated by other authors* (*Hankel transforms* ([7] *and* [13]), *amongst others*).

1. – Introduction.

An extension of the notion of hypergroup of Jewett ([11]) was introduced by Roesler ([16]). She considered the concept of signed hypergroup by modifying the hypergroup axioms in several points, mainly in abandoning positivity and support continuity of the convolution. A special case of commutative signed hypergroup is that one is associated with Dunkl operators and closely connected with the Bessel-Kingman hypergroups ([12]).

The Dunkl operator Λ_a, $a \geq -1/2$, associated with the reflection group $\boldsymbol{Z}_2$ on $\boldsymbol{R}$ is defined by

$$\Lambda_a f(x) = \frac{df(x)}{dx} + \frac{2a+1}{2}\frac{f(x) - f(-x)}{x}, \qquad x \neq 0,$$

and $\Lambda_a f(0) = 2(a+1)\frac{df}{dx}(0)$. This operator was studied for the first time by Dunkl ([4]) in connection with a generalization of the classical theory of spherical harmonics.

Assume through this paper that $a \geq -1/2$. The initial value problem

$$\Lambda_a f(x) = \lambda f(x), \qquad f(0) = 1, \tag{1.1}$$

has, for every $\lambda \in \boldsymbol{C}$, a unique solution that is called Dunkl kernel and is denoted by $E_a(\lambda .)$, $\lambda \in \boldsymbol{C}$ ([17] and [19]). E_a is related with the Bessel function J_a of the first kind and order a as follows

$$E_a(z) = T_a(z) + \frac{z}{2(a+1)} T_{a+1}(z), \qquad z \in \boldsymbol{C},$$

where

$$T_a(z) = \Gamma(a+1) \sum_{n=0}^{\infty} \frac{z^{2n}}{2^{2n} n! \Gamma(n+a+1)}, \qquad z \in \boldsymbol{C}.$$

Note that $T_a(z) = 2^a \Gamma(a+1)(iz)^{-a} J_a(iz)$, $z \in \boldsymbol{C}$.

Associated with Dunkl kernels, integral transforms F_a are defined as follows

$$F_a(\phi)(x) = \int_{\boldsymbol{R}} E_a(-iyx)\phi(y) dw_a(y), \qquad x \in \boldsymbol{R},$$

where, for instance, ϕ is in the Lebesgue space $L_1(w_a)$ and

$$dw_a(x) = \frac{|x|^{2a+1}}{2^{a+1}\Gamma(a+1)} dx.$$

The classical Fourier transform on $\boldsymbol{R}$ coincides with $F_{-1/2}$.

By S we represent the Schwartz function space, that is endowed, as usual, with the topology generated by the family $\{\gamma_{n,k}\}_{n,k \in \boldsymbol{N}}$ of seminorms, where, for every $n, k \in \boldsymbol{N}$,

$$\gamma_{n,k}(\phi) = \sup_{x \in \boldsymbol{R}} \left| x^n \frac{d^k}{dx^k} \phi(x) \right|, \qquad \phi \in S.$$

The space of pointwise multipliers of S is constituted by all those functions f that are in $C^\infty(\boldsymbol{R})$ and for which, for every $k \in \boldsymbol{N}$ there exists $n_k \in \boldsymbol{N}$ such that

$$\sup_{x\in\boldsymbol{R}} (1+|x|)^{-n_k} \left| \frac{d^k}{dx^k} f(x) \right| < \infty.$$

The Dunkl transform F_a is an isomorphism from S into itself ([10, Corollary 4.22]). Moreover, the inverse mapping F_a^{-1} of F_a is given by, for each $\phi \in S$,

$$F_a^{-1}(\phi)(x) = \int_{\boldsymbol{R}} E_a(ixy)\phi(y)dw_a(y) = F_a(\phi)(-x), \qquad x \in \boldsymbol{R}.$$

For every $\phi \in S$, we have that

$$F_a(\Lambda_a\phi)(x) = ixF_a(\phi)(x), \qquad x \in \boldsymbol{R}.$$

The Dunkl transform F_a is defined on S', the dual space of S, by transposition. That is, if $T \in S'$, the generalized Dunkl transform $F_a'T$ of T is given by

$$\langle F_a'T, \phi\rangle = \langle T, F_a\phi\rangle, \qquad \phi \in S. \tag{1.2}$$

The convolution operation associated with Dunkl transform was investigated by Roesler ([17]). The Dunkl translation ${}_a\tau_x$, $x \in \boldsymbol{R}$, is defined through

$${}_a\tau_x(\phi)(y) = \int_{\boldsymbol{R}} f(z)d(\delta_x \#_a \delta_y)(z), \qquad y \in \boldsymbol{R},$$

where $d(\delta_x \#_a \delta_y)(z)$ is a signed measure given, for every $x, y \in \boldsymbol{R}$, by

$$d(\delta_x \#_a \delta_y)(z) = K_a(x,y,z)dw_a(z), \qquad x, y \in \boldsymbol{R} \setminus \{0\},$$

$d(\delta_0 \#_a \delta_y)(z) = \delta_y$, $y \in \boldsymbol{R}$, and $d(\delta_x \#_a \delta_0)(z) = \delta_x$, $x \in \boldsymbol{R}\setminus\{0\}$. Here δ_x represents the Dirac measure in x, for every $x \in \boldsymbol{R}$, and K_a is a function defined in [17] as follows

$$K_a(x,y,z) = (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x})\rho_a(|x|,|y|,|z|),$$

where

$$\sigma_{x,y,z} = \begin{cases} (x^2+y^2-z^2)/2xy, & \text{if } x, y \in \boldsymbol{R}\setminus\{0\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\rho_a(|x|,|y|,|z|) = \begin{cases} b_a \frac{(((|x|+|y|)^2-z^2)(z^2-(|x|-|y|)^2))^{a-1/2}}{|xyz|^{2a}}, & \text{if } |z| \in [||x|-|y||, |x|+|y|], \\ 0, & \text{otherwise} \end{cases}$$

being $b_a = 2^{1-a}(\Gamma(a+1))^2\sqrt{\pi}\Gamma(a+1/2)$.

The Dunkl convolution $f\#_a g$ of f and g in $L_1(dw_a)$ is defined by

$$(f\#_a g)(x) = \int_{\boldsymbol{R}} g(-y)_a\tau_x(f)(y)dw_a(y), \qquad x \in \boldsymbol{R}.$$

The following interchange formula relating to $\#_a$ and F_a

$$F_a(f\#_a g) = F_a(f)F_a(g) \tag{1.3}$$

holds for every $f, g \in L_1(w_a)$.

Throughout this paper, to simplify, we will write τ_x, $x \in \boldsymbol{R}$, K and $\#$, instead of ${}_a\tau_x$, $x \in \boldsymbol{R}$, K_a and $\#_a$.

This paper is divided in two parts.

Our objective in the first part of this paper is to investigate the Dunkl transforms, the Dunkl translations τ_x, $x \in \boldsymbol{R}$, and Dunkl convolution on the space S' of tempered distributions. Our study is inspired in the classical investigation of Schwartz ([18]) about the usual convolution operators and in the analysis of Betancor and Marrero ([3] and [14]) for the distributional Hankel convolution. We characterize the tempered distributions that defines Dunkl convolution operators in S and we prove that the space of Dunkl convolution operators on S coincides with the space $\mathcal{O}'_c$ ([18]) of usual convolution operators on S.

Dube and Pandey [7] and Koh and Zemanian [13] studied the distributional Hankel transforms by using a procedure usually called the kernel method. This procedure has been used by other authors to investigate different integral transformations (see, for instance, [1], [6] and [22]). In the monograph of Zemanian [23] the interested reader can to find the studies of several distributional integral transformations by employing the kernel method and other procedures. When an integral transform is defined in spaces of distributions by using the kernel method, the main property that must be proved is the inversion formula. Usually, this inversion theorem is very difficult to prove (see [7], [13] and [22]).

In the second part of this paper we propose to investigate the distributional Dunkl transform in two different spaces, inspired by [7] and [13], by using the kernel method. We introduce new ideas that allows us to simplify the proofs of the most of properties of the distributional Dunkl transforms considered (in contrast with the method of proofs in [7] and [13]). We remark that our ideas, supported by certain representations of the elements of the corresponding dual spaces, apply also for the Hankel transforms and they allow us to obtain new and simpler proofs of the properties of the distributional Hankel transforms established in [7] and [13].

We emphasize that our results complement the ones obtained by Ben Mohamed and Trimèche ([15]) about the distributional Dunkl transforms and convolutions in other spaces of distributions.

By C always we represent a suitable positive constant that can be changed from one line to another.

2. – Dunkl transformation and Dunkl convolution on tempered distributions.

2.1. – *Some new properties of the spaces S and S′.*

In this paragraph we present two properties of the spaces S and S' that will be useful in the sequel. We firstly describe the space S of Schwartz through the Dunkl operators.

PROPOSITION 2.1. – *We define, for every $n, k \in \boldsymbol{N}$,*

$$\gamma_{n,k}^{a}(\phi) = \sup_{x \in \boldsymbol{R}} |x^n \Lambda_a^k \phi(x)|, \qquad \phi \in S.$$

Then a function $\phi \in C^\infty(\boldsymbol{R})$ is in S if, and only if, $\gamma_{n,k}^{a}(\phi) < \infty$, for each $n, k \in \boldsymbol{N}$. Moreover the topology generated by the family $\{\gamma_{n,k}^{a}\}_{n,k \in \boldsymbol{N}}$ of seminorms on S agrees with the usual topology of S.

PROOF. – To prove this property it is sufficient to proceed as in the proof of [5, Proposition 4.4]. ∎

We now establish a new representation for the elements of S'. This representation can be proved in a standard way (see, for instance, [2] and [20]) by using the Hahn-Banach and Riesz representation theorems.

PROPOSITION 2.2. – *Let T be a functional on S. Then $T \in S'$ if, and only if, there exist $r \in \boldsymbol{N}$ and complex regular Borel measures $w_{n,k}$, $n, k \in \boldsymbol{N}$, $n, k \leq r$, on $\boldsymbol{R}$, such that*

$$\langle T, \phi \rangle = \sum_{n,k=0}^{r} \int_{\boldsymbol{R}} x^n \Lambda_a^k \phi(x) dw_{n,k}(x), \qquad \phi \in S. \tag{2.1}$$

∎

Suppose that f is a Lebesgue measurable function on $\boldsymbol{R}$ such that, for some $m \in \boldsymbol{N}$,

$$\int_{\boldsymbol{R}} (1 + |x|)^{-m} |f(x)| |x|^{2a+1} dx < \infty. \tag{2.2}$$

Then f generates an element, that we continue denoting by f, on S' through

$$\langle f, \phi \rangle = \int_{\boldsymbol{R}} f(x)\phi(x) \frac{|x|^{2a+1}}{2^{a+1}\Gamma(a+1)} dx, \qquad \phi \in S.$$

In particular, in this way, the space S and the space of pointwise multipliers of S can be seen as subspaces of S'.

As it was mentioned in the introduction, the Dunkl transform F_a is an isomorphism from S into itself ([10, Corollary 4.22]) and the Dunkl transform is defined in the the dual space S' of S by transposition.

2.2. – *Dunkl convolution operators on tempered distributions.*

In this section we analyze the behaviour of Dunkl translations and Dunkl convolutions on the spaces S and S'.

The Dunkl transform F_a and the Dunkl translations τ_x, $x \in \boldsymbol{R}$ are related by the following formula ([15, (2.25)])

$$F_a(\tau_x \phi) = E_a(ix.)F_a(\phi), \qquad \phi \in S \text{ and } x \in \boldsymbol{R}.$$

Then, if $x \in \boldsymbol{R}$, τ_x defines a continuous linear mapping from S into itself provided that $E_a(ix.)$ is a multiplier of S.

PROPOSITION 2.3. – *Let $x \in \boldsymbol{R}$. The Dunkl translation τ_x defines a continuous linear mapping from S into itself.*

PROOF. – We are going to see that $E_a(ix.)$ is a pointwise multiplier of S. It is sufficient to prove that, for every $k \in \boldsymbol{N}$ there exists $n_k \in \boldsymbol{N}$ for which

$$\sup_{t \in \boldsymbol{R}} (1+|t|)^{-n_k} \left| \frac{d^k}{dt^k} E_a(ixt) \right| < \infty.$$

This can be proved by using well-known properties of Bessel functions (see [21, p. 199] and [23, (7), p. 129]). ■

Proposition 2.3 allows us to define the Dunkl convolution $T \# \phi$ of $T \in S'$ and $\phi \in S$ as follows

$$(T \# \phi)(x) = \langle T(t), (\tau_x \phi)(-t) \rangle, \qquad x \in \boldsymbol{R}. \tag{2.3}$$

If $T \in S'$ and $\phi \in S$, it is not true always that $T \# \phi \in S$. Indeed, assume that T is the functional defined on S by

$$\langle T, \phi \rangle = \int_{\boldsymbol{R}} \phi(x)|x|^{2a+1} dx, \qquad \phi \in S.$$

It is clear that $T \in S'$. We have that, by taking into account the properties of the function K ([17]), for every $\phi \in S$,

$$\begin{aligned}(T\#\phi)(x) =& \langle T(t), \tau_x(\phi)(-t)\rangle \\ =& \int_{\boldsymbol{R}} (\tau_x\phi)(-t)|t|^{2a+1}dt \\ =& \int_{\boldsymbol{R}}\int_{\boldsymbol{R}} \phi(z)K(x,-t,z)\frac{|z|^{2a+1}}{2^{a+1}\Gamma(a+1)}dz|t|^{2a+1}dt \\ =& \int_{\boldsymbol{R}} \phi(z)\int_{\boldsymbol{R}} K(x,-t,z)|t|^{2a+1}dt\frac{|z|^{2a+1}}{2^{a+1}\Gamma(a+1)}dz \\ =& \int_{\boldsymbol{R}} \phi(z)\int_{\boldsymbol{R}} K(x,-z,t)|t|^{2a+1}dt\frac{|z|^{2a+1}}{2^{a+1}\Gamma(a+1)}dz \\ =& \int_{\boldsymbol{R}} \phi(z)|z|^{2a+1}dz, \qquad x \in \boldsymbol{R}.\end{aligned}$$

Hence, if $\phi \in S$ such that $\int_{\boldsymbol{R}}\phi(z)|z|^{2a+1}dz \neq 0$, then $T\#\phi \notin S$.

PROPOSITION 2.4. – *Let $T \in S'$ and $\phi \in S$. Then $T\#\phi$ is a pointwise multiplier of S.*

PROOF. – By Proposition 2.2 we can assume without loss of generality that

$$\langle T,\psi\rangle = \int_{\boldsymbol{R}} x^n \Lambda_a^k \psi(x)dw(x), \qquad \psi \in S,$$

where $n,k \in \boldsymbol{N}$ and w is a complex Borel regular measure on $\boldsymbol{R}$.

Then, we have

$$\begin{aligned}(T\#\phi)(x) =& \int_{\boldsymbol{R}} y^n \Lambda_{a,y}^k \tau_x(\phi)(-y)dw(y) \\ =& (-1)^k\int_{\boldsymbol{R}} y^n \tau_x(\Lambda_a^k\phi)(-y)dw(y) \\ =& (-1)^k\int_{\boldsymbol{R}} y^n F_a^{-1}(E_a(-ixt)F_a(\Lambda_a^k\phi)(-t))(y)dw(y) \\ =& (-1)^k\int_{\boldsymbol{R}} y^n F_a(E_a(-ixt)F_a(\Lambda_a^k\phi)(-t))(-y)dw(y), \qquad x \in \boldsymbol{R}.\end{aligned}$$

Moreover, for every $l \in \boldsymbol{N}$,

$$\frac{d^l}{dx^l}(T\#\phi)(x) = (-1)^k \int_{\boldsymbol{R}} y^n F_a\left(\frac{d^l}{dx^l}E_a(-ixt)F_a(\Lambda^k\phi)(-t)\right)(-y)dw(y)$$

$$= (-1)^k i^n \int_{\boldsymbol{R}} F_a\left(\Lambda^n_{a,t}\left(\frac{d^l}{dx^l}E_a(-ixt)F_a(\Lambda^k\phi)(-t)\right)\right)(-y)dw(y), \qquad x \in \boldsymbol{R}.$$

The derivation under the integral sign is justified because $E(ix.)$ is a pointwise multiplier of S (see the proof of Proposition 3.1).

A straightforward manipulation, by taking into account properties of the Bessel functions ([23, (7), p. 129] and the boundedness of the functions $z^{1/2}J_\upsilon(z)$ and $z^{-\upsilon}J_\upsilon(z)$ on $(0,\infty)$) and that the Dunkl transforms is an automorphism of S, allows us to obtain $r \in \boldsymbol{N}$ such that

$$\left|\frac{d^l}{dx^l}(T\#\phi)(x)\right| \leq C(1+|x|)^r, \qquad x \in \boldsymbol{R}.$$

Thus we establish that $T\#\phi$ is a pointwise multiplier of S. ■

REMARK 1. – *Note that in the proof of the above proposition we prove that if $T \in S'$, then for every $k \in \boldsymbol{N}$ there exists $n_k \in \boldsymbol{N}$ for which*

$$\sup_{x\in\boldsymbol{R}}(1+|x|)^{-n_k}\left|\frac{d^k}{dx^k}(T\#\phi)(x)\right| < \infty,$$

for each $\phi \in S$.

As a consequence of Proposition 2.4 we can also get the following.

PROPOSITION 2.5. – *Let $T \in S'$. For every $k \in \boldsymbol{N}$ there exists $n_k \in \boldsymbol{N}$ such that*

$$\sup_{x\in\boldsymbol{R}}(1+|x|)^{-n_k}|\Lambda^k_a(T\#\phi)(x)| < \infty,$$

for every $\phi \in S$.

PROOF. – By invoking Proposition 2.2 we can write

$$\langle T,\phi\rangle = \sum_{n,k=0}^{r}\int_{\boldsymbol{R}} x^n \Lambda^k_a\phi(x)dw_{n,k}(x), \qquad \phi \in S,$$

where $r \in \boldsymbol{N}$ and $w_{n,k}$, $n,k \in \boldsymbol{N}$, $n,k \leq r$, are complex Borel regular measures on $\boldsymbol{R}$.

Let $l \in \boldsymbol{N}$. By differentiation under the integral sign we obtain

$$\begin{aligned}\Lambda_a^l(T\#\phi)(x) &= \sum_{n,k=0}^{r} \int_{\boldsymbol{R}} y^n \Lambda_{a,y}^k \Lambda_{a,x}^l \tau_x(\phi)(-y)dw_{n,k}(y) \\ &= \sum_{n,k=0}^{r} \int_{\boldsymbol{R}} y^n \Lambda_a^k(\tau_x \Lambda_a^l \phi)(-y)dw_{n,k}(y) \\ &= (T\#\Lambda_a^l\phi)(x), \qquad x \in \boldsymbol{R} \text{ and } \phi \in S.\end{aligned}$$

Now Proposition 2.4 allows us to conclude the proof of this proposition. ■

REMARK 2. – *Along the proof of* Proposition 2.5 *we establish that*

$$\Lambda_a^k(T\#\phi) = T\#(\Lambda^k\phi),$$

for every $T \in S'$, $\phi \in S$ and $k \in \boldsymbol{N}$.

According to (1.3), [10, Corollary 4.22] and since the elements of S are pointwise multipliers of S, we can see that the $\#$ convolution defines a continuous linear mapping from $S \times S$ into S. Now we establish an associative property for the distributional $\#$-convolution.

PROPOSITION 2.6. – *Let $T \in S'$ and $\phi, \psi \in S$. Then*

$$T\#(\phi\#\psi) = (T\#\phi)\#\psi.$$

PROOF. – According to Proposition 2.2, without loss of generality, we can consider that T is defined by

$$\langle T, \psi \rangle = \int_{\boldsymbol{R}} x^n \Lambda_a^k \psi(x) dw(x), \qquad \psi \in S,$$

where $n, k \in \boldsymbol{N}$ and w is a complex Borel regular measure on $\boldsymbol{R}$.

Then

$$\begin{aligned}(T\#(\phi\#\psi))(x) &= \int_{\boldsymbol{R}} y^n \Lambda_a^k \tau_x(\phi\#\psi)(-y)dw(y) \\ &= \int_{\boldsymbol{R}} y^n \Lambda_a^k(\phi\#\tau_x\psi)(-y)dw(y) \\ &= (-1)^k \int_{\boldsymbol{R}} y^n((\Lambda_a^k\phi)\#(\tau_x\psi))(-y)dw(y) \\ &= (-1)^k \int_{\boldsymbol{R}}\int_{\boldsymbol{R}} \tau_{-y}(\Lambda_a^k\phi)(-z)(\tau_x\psi)(z)\frac{|z|^{2a+1}}{2^{a+1}\Gamma(a+1)}dzdw(y)\end{aligned}$$

$$=(-1)^k\int_{\boldsymbol{R}}\tau_x(\psi)(z)\int_{\boldsymbol{R}}y^n\tau_{-z}(\Lambda_a^k\phi)(-y)dw(y)\frac{|z|^{2a+1}}{2^{a+1}\Gamma(a+1)}dz$$

$$=\langle(T\#\phi)(y),(\tau_x\psi)(-y)\rangle,\qquad x\in\boldsymbol{R}.$$

The interchange of the order of integration is justified because w is a complex regular Borel measure on $\boldsymbol{R}$ and $\phi,\psi\in S$. ∎

From Proposition 2.6 we can deduce the following interchange formula for the distributional Dunkl transform.

PROPOSITION 2.7. – *Let $T\in S'$ and $\phi\in S$. Then*

$$F_a'(T\#S)=F_a'(T)F_a(\phi).$$

PROOF. – By Proposition 2.6 we can write, for $\psi\in S$,

$$\begin{aligned}\langle F_a'(T\#\phi),\psi\rangle&=\langle T\#\phi,F_a\psi\rangle\\&=\langle(T\#\phi)(y),F_a^{-1}(\psi)(-y)\rangle\\&=\langle T(y),(\phi\#F_a^{-1}\psi)(-y)\rangle\\&=\langle T,F_a^{-1}(F_a(\phi\#F_a^{-1}\psi))(-y)\rangle\\&=\langle T(y),F_a^{-1}((F_a\phi).\psi)(-y)\rangle\\&=\langle F_a'(T)F_a(\phi),\psi\rangle.\end{aligned}$$

∎

Let now $m\in\boldsymbol{Z}$. We define the space $A_{a,m}$ that consists of all these functions $C^\infty(\boldsymbol{R})$ such that, for every $k\in\boldsymbol{N}$,

$$\eta_k^{a,m}(\phi)=\sup_{x\in\boldsymbol{R}}(1+|x|)^m|\Lambda_a^k\phi(x)|<\infty.$$

$A_{a,m}$ is endowed with the topology associated to the family $\{\eta_k^{a,m}\}_{k\in\boldsymbol{N}}$ of seminorms.

Ben Mohamed and Trimèche ([15, Proposition 3.1]) established that, for every $m\in\boldsymbol{N}$, $A_{a,-m}$ does not depend on $a\geq-\frac{1}{2}$ and $A_{a,-m}=S_{-m}$, where S_{-m} is defined as follows (see [9]). Let $m\in\boldsymbol{N}$. A function $\phi\in C^\infty(\boldsymbol{R})$ is in S_{-m}, if and only if, for every $k\in\boldsymbol{N}$,

$$\beta_k^m(\phi)=\sup_{x\in\boldsymbol{R}}(1+|x|)^{-m}\left|\frac{d^k}{dx^k}\phi(x)\right|<\infty.$$

S_{-m} is endowed with the topology generated by $\{\beta_k^m\}_{k\in \boldsymbol{N}}$. It is clear that S is contained in S_{-m}. We define $\mathcal{S}_{-m}$ as the closure of S in S_{-m}. We have that $\mathcal{S}_{-m}$ is continuously contained in $\mathcal{S}_{-m-1}$. The union space $\mathcal{O}_c = \cup_{m\in \boldsymbol{N}} \mathcal{S}_{-m}$ is endowed with the inductive topology. $\mathcal{O}_c'$, the dual space of $\mathcal{O}_c$, is the subspace of S' constituted by the elements of S' that define usual convolution operators in S ([9] and [18]).

Our objective is to show that the elements of $\mathcal{O}_c'$ can be characterized also as the Dunkl convolution operators on S. First we need to prove the following useful representation result that can be proved in a standard way by using Hahn-Banach and Riesz representation theorems (see, for instance, [2] and [20]).

PROPOSITION 2.8. – *Let $T \in \mathcal{O}_c'$. For every $m \in \boldsymbol{Z}$, there exist $r \in \boldsymbol{N}$ and complex Borel regular measures w_0, w_1,..., w_r, on $\boldsymbol{R}$, such that*

$$\langle T, \phi \rangle = \sum_{k=0}^{r} \int_{\boldsymbol{R}} (1+|x|)^m \Delta_a^k \phi(x) dw_k(x), \qquad \phi \in S.$$

■

Next we establish the main result of this section.

PROPOSITION 2.9. – *Let $T \in S'$. The following conditions are equivalent.*

i) *$T \in \mathcal{O}_c'$.*

ii) *$F_a'(T)$ is a multiplier of S.*

iii) *For every $m \in \boldsymbol{N}$, there exist $r \in \boldsymbol{N}$ and continuous functions $f_0, f_1,..., f_r$, on $\boldsymbol{R}$, such that*

$$\langle T, \phi \rangle = \sum_{j=0}^{r} \int_{\boldsymbol{R}} f_j(x) \Delta_a^j \phi(x) |x|^{2a+1} dx, \qquad \phi \in S,$$

and $\sup_{y\in \boldsymbol{R}} |y^m f_j(x)| < \infty$, $j = 0, 1, ..., r$.

iv) *The mapping $\phi \to T \# \phi$ is continuous from S into itself.*

PROOF. – $(i) \Rightarrow (ii)$. Assume that $T \in \mathcal{O}_c'$. According to Proposition 2.8, for every $m \in \boldsymbol{Z}$, we can write

$$\langle T, \phi \rangle = \sum_{k=0}^{r} \int_{\boldsymbol{R}} (1+|x|)^m \Delta_a^k \phi(x) dw_k(x), \qquad \phi \in S,$$

where $r \in \boldsymbol{N}$ and w_0, w_1,..., w_r are complex Borel regular measures on $\boldsymbol{R}$.

Then, fixed $m \in \boldsymbol{Z}$, $m \leq 0$, that will be specified later, we have, for every $\phi \in S$,

$$\begin{aligned}\langle F'_a T, \phi\rangle &= \langle T, F_a \phi\rangle \\ &= \sum_{k=0}^{r} \int_{\boldsymbol{R}} (1+|x|)^m \Delta_a^k F_a(\phi)(x) dw_k(x) \\ &= \sum_{k=0}^{r} \int_{\boldsymbol{R}} (1+|x|)^m F_a^{-1}((iz)^k \phi(-z))(x) dw_k(x) \\ &= \sum_{k=0}^{r} \int_{\boldsymbol{R}} (1+|x|)^m \int_{\boldsymbol{R}} E(izx)(iz)^k \phi(-z) \frac{|z|^{2a+1}}{2^{a+1}\Gamma(a+1)} dz dw_k(x) \\ &= \int_{\boldsymbol{R}} \phi(z) \sum_{k=0}^{r} (-iz)^k \int_{\boldsymbol{R}} (1+|x|)^m E(-izx) dw_k(x) \frac{|z|^{2a+1}}{2^{a+1}\Gamma(a+1)} dz.\end{aligned}$$

Hence

$$F'_a(T)(z) = \sum_{k=0}^{r} (-iz)^k \int_{\boldsymbol{R}} (1+|x|)^m E(-izx) dw_k(x), \qquad z \in \boldsymbol{R}.$$

Let $l \in \boldsymbol{N}$. Leibniz's rule leads to

$$\frac{d^l}{dz^l} F'_a(T)(z) = \sum_{k=0}^{r} (-i)^k \sum_{j=0}^{l} \binom{l}{j} \frac{d^{l-j}}{dz^{l-j}}(z^k) \int_{\boldsymbol{R}} (1+|x|)^m \frac{d^j}{dz^j}(E_a(-izx)) dw_k(x), \quad z \in \boldsymbol{R},$$

provided that $m \leq -m_l$, for some m_l large enough. Then by using again [23, (7), Chapter 5] and by boundedness properties of the Bessel functions we conclude (by choosing m suitably) that

$$\sup_{z \in \boldsymbol{R}} (1+|z|)^{-n_l} \left| \frac{d^l}{dz^l} F'_a(T)(z) \right| < \infty,$$

for a certain $n_l \in \boldsymbol{N}$. Hence $F'_a T$ is a multiplier of S.

$(ii) \Rightarrow (iii)$ Suppose that $F'_a(T)$ is a multiplier of S, that is, for every $l \in \boldsymbol{N}$ there exists $n_l \in \boldsymbol{N}$ such that

$$\sup_{x \in \boldsymbol{R}} (1+|x|)^{-n_l} \left| \frac{d^l}{dx^l} F'_a(T)(x) \right| < \infty.$$

We define the function

$$G(x) = (1+x^2)^{-k} F'_a(T)(x), \qquad x \in \boldsymbol{R},$$

where $k \in \boldsymbol{N}$ will be specified later.

Note that, if $k > \frac{n_0}{2} + 2(\alpha + 1)$ then

$$\int_{\boldsymbol{R}} |G(x)|\,|x|^{2\alpha+1}dx \leq C\int_{\boldsymbol{R}} (1+x^2)^{-k+n_0/2}|x|^{2\alpha+1}dx < \infty.$$

Hence $(F'_\alpha)^{-1}G = F_\alpha^{-1}G$. Indeed, for every $\phi \in S$, we have

$$\begin{aligned}\langle (F'_\alpha)^{-1}G, \phi\rangle =& \langle G, F_\alpha^{-1}\phi\rangle \\ =& \int_{\boldsymbol{R}} G(x)(F_\alpha^{-1}\phi)(x)|x|^{2\alpha+1}dx \\ =& \int_{\boldsymbol{R}} F_\alpha^{-1}(G)(x)\phi(x)|x|^{2\alpha+1}dx.\end{aligned}$$

Thus we obtain that

$$\begin{aligned}T =& (F'_\alpha)^{-1}((1+x^2)^k G(x)) \\ =& \sum_{j=0}^{k}\binom{k}{j}(F'_\alpha)^{-1}(x^{2j}G(x)) \\ =& \sum_{j=0}^{k}\binom{k}{j}(-1)^j \Lambda_\alpha^{2j}(F'_\alpha)^{-1}(G) \\ =& \sum_{j=0}^{k}\binom{k}{j}(-1)^j \Lambda_\alpha^{2j}F_\alpha^{-1}(G),\end{aligned}$$

where Λ_α is here understood in a distributional sense.

Let $m \in \boldsymbol{N}$. If k is large enough we can write

$$\begin{aligned}(-iy)^m F_\alpha(G)(y) =& F_\alpha(\Lambda_\alpha^m(G))(-y) \\ =& \int_{\boldsymbol{R}} E_\alpha(ixy)\Lambda_\alpha^m G(x)|x|^{2\alpha+1}dx, \qquad y \in \boldsymbol{R}.\end{aligned}$$

Then, for a suitable $t \in \boldsymbol{N}$ and provided that k is large enough,

$$|y^m F_\alpha^{-1}(G)(y)| \leq C \max_{0\leq\delta,\varepsilon\leq t}\gamma_{\delta,\varepsilon}(G) < \infty, \qquad y \in \boldsymbol{R}.$$

Thus we prove that (ii) is true.

$(iii) \Rightarrow (i)$. Assume that (iii) is true. Let $m \in \boldsymbol{N}$. There exist $r \in \boldsymbol{N}$ and continuous functions f_j, $j = 0, 1, \ldots, r$, such that $T = \sum_{j=0}^{r}\Lambda_\alpha^j f_j$, where Λ_α is understood in a distributional sense, and $\sup_{y\in\boldsymbol{R}}(1+|y|)^{m+2\alpha+3}|f_j(y)| < \infty$. Then, we have that

$$\langle T, \phi\rangle = \sum_{j=0}^{r}\int_{\boldsymbol{R}} \Lambda_\alpha^j\phi(x)f_j(x)|x|^{2\alpha+1}dx, \qquad \phi \in S. \tag{2.4}$$

Hence

$$|\langle T, \phi\rangle| \leq C \sum_{j=0}^{r} \sup_{x\in \boldsymbol{R}} (1+|x|)^{-m} |\Lambda_a^j \phi(x)|$$

$$\cdot \sup_{x\in \boldsymbol{R}} (1+|x|)^{m+2a+3} |f_j(x)| \int_{\boldsymbol{R}} \frac{|x|^{2a+1}}{(1+|x|)^{2a+3}} dx, \qquad \phi \in S.$$

Thus we prove that T can be extended to $\mathcal{S}_{-m}$ as an element of $\mathcal{S}'_{-m}$ in a unique way given by (2.4).

$(ii) \Rightarrow (iv)$. By Proposition 2.7, for every $\phi \in S$, we can write

$$T \# \phi = F_a((F'_a)(T) F_a(\phi)).$$

Since, if (ii) is true, $F'_a(T)$ is a multiplier of S, the mapping $\phi \longrightarrow T\#\phi$ is continuous from S into itself.

$(iv) \Rightarrow (ii)$. Assume that (iv) holds. Then, according to [10, Corollary 4.22] and Proposition 2.7, the mapping $\phi \rightarrow \phi F'_a(T)$ is continuous from S into itself, where $F'_a(T) \in S'$. Then, by invoking the Fourier transform, we conclude that $F'_a(T)$ is a multiplier of S. ■

For every $T \in S'$ and $R \in \mathcal{O}'_c$ we define the Dunkl convolution $T\#R$ of T and R as the element of S' given by

$$\langle T\#R, \phi\rangle = \langle T, R\#\phi\rangle, \qquad \phi \in S. \tag{2.5}$$

Proposition 2.6 allows us to see that the definition (2.5) is an extension of definition (2.3) because each element of S is also a multiplier of S.

It is immediate to prove the following interchange formula for the distributional Dunkl transform.

PROPOSITION 2.10. – *Let $T \in S'$ and $R \in \mathcal{O}'_c$. Then*

$$F'_a(T\#R) = F'_a(T) F'_a(R).$$

■

The following algebraic properties of the distributional Dunkl convolution can be proved by using Propositions 2.9 and 2.10.

PROPOSITION 2.11. – *Let $T \in S'$ and $R, L \in \mathcal{O}'_c$. Then*

i) *$R\#L \in \mathcal{O}'_c$ and $R\#L = L\#R$.*

ii) *$T\#(R\#L) = (T\#R)\#L$.*

iii) *$T\#\delta = T$, where δ denotes the Dirac functional.*

iv) *$\Lambda_a(T\#R) = (\Lambda_a T)\#R = T\#(\Lambda_a R)$, where Λ_a is understood in a distributional sense.* ■

3. – Distributional Dunkl transform by using the kernel method.

In this section we define the distributional Dunkl transforms by employing the kernel method. In this method we need to define function spaces containing the kernel of the Dunkl transform. We will define the Dunkl transform in two different distribution spaces according to the ideas developed by Dube and Pandey [7] and Koh and Zemanian [13] to investigate the distributional Hankel transforms.

3.1. – *The function spaces $\mathcal{H}_\beta$ and their duals.*

We consider a function ξ that is continuous and zero free on $\boldsymbol{R}$ and such that $\xi(x) = O(|x|^\beta)$, as $|x| \to \infty$, where $\beta \in \boldsymbol{R}$.

The space $\mathcal{H}_\beta$ of functions is defined as follows. A smooth function ϕ on $\boldsymbol{R}$ is in $\mathcal{H}_\beta$ if, and only if, for every $m \in \boldsymbol{N}$, the quantity

$$w_m^\beta(\phi) = \sup_{x \in \boldsymbol{R}} |\xi(x) \Lambda_a^m \phi(x)|$$

is finite. $\mathcal{H}_\beta$ is endowed with the topology associated with the family $\{w_m^\beta\}_{m \in \boldsymbol{N}}$ of seminorms.

PROPOSITION 3.1. – *The space $\mathcal{H}_\beta$ is Fréchet.*

PROOF. – To see this property we can use Proposition 2.1 and that the space $C^\infty(\boldsymbol{R})$, equipped with the usual topology, is Montel ([9, p. 239]). Indeed, assume that $(\phi_n)_{n \in \boldsymbol{N}}$ is a Cauchy sequence in $\mathcal{H}_\beta$. It is not hard to see that, for every $m \in \boldsymbol{N}$, $(\frac{d^m}{dx^m}\phi_n)_{n \in \boldsymbol{N}}$ is a Cauchy sequence uniformly in every compact subset of $\boldsymbol{R} \setminus \{0\}$. We choose a function $\varphi \in C^\infty(\boldsymbol{R})$ such that $\varphi(x) = 0$, $|x| \geq 2$, and $\varphi(x) = 1$, $|x| \leq 1$. Then the function $\phi_n \varphi \in S$, $n \in \boldsymbol{N}$. According to Proposition 2.1 we have that, for every $m \in \boldsymbol{N}$, there exists $r \in \boldsymbol{N}$ and $C > 0$ for which

$$\begin{aligned}
\sup_{|x| \leq 1} \left| \frac{d^m}{dx^m} \phi_n(x) \right| &\leq \sup_{x \in \boldsymbol{R}} \left| \frac{d^m}{dx^m} (\phi_n(x)\varphi(x)) \right| \\
&\leq C \sum_{j=0}^{r} \sup_{x \in \boldsymbol{R}} |\Lambda_a^j(\phi_n \varphi)(x)| \\
&\leq C \sum_{j=0}^{r} \left(\sup_{|x| \leq 1} |\Lambda_a^j(\phi_n)(x)| + \sup_{1 \leq |x| \leq 2} |\Lambda_a^j(\phi_n \varphi)(x)| \right) \\
&\leq C \sum_{j=0}^{r} \left(\sup_{|x| \leq 1} |\Lambda_a^j(\phi_n)(x)| + \sup_{1 \leq |x| \leq 2} \left| \frac{d^j}{dx^j}(\phi_n \varphi)(x) \right| \right) \\
&\leq C, \qquad n \in \boldsymbol{N}.
\end{aligned}$$

Hence, the sequence $(\phi_n)_{n\in\boldsymbol{N}}$ is a bounded sequence in $C^\infty(\boldsymbol{R})$. Then, since $C^\infty(\boldsymbol{R})$ is a Montel space, there exists a subsequence $(\phi_{n_k})_{k\in\boldsymbol{N}}$ of $(\phi_n)_{n\in\boldsymbol{N}}$ and a function $\phi \in C^\infty(\boldsymbol{R})$ such that $\phi_{n_k} \to \phi$, as $k \to \infty$, in $C^\infty(\boldsymbol{R})$. Therefore, for every $m \in \boldsymbol{N}$ and $x \in \boldsymbol{R}$,

$$\Lambda_a^m \phi_{n_k}(x) \to \Lambda_a^m \phi(x), \qquad as \qquad k \to \infty. \tag{3.1}$$

Since $(\phi_n)_{n\in\boldsymbol{N}}$ is a Cauchy sequence in $\mathcal{H}_\beta$, standard arguments allow us to infer from (3.1) that $\phi_{n_k} \to \phi$, as $k \to \infty$, in $\mathcal{H}_\beta$, and then that $\phi_n \to \phi$, as $n \to \infty$, in $\mathcal{H}_\beta$. ∎

In Proposition 3.1 we also prove that $\mathcal{H}_\beta$ is continuously contained in $C^\infty(\boldsymbol{R})$. Moreover, since Λ_a is a continuous mapping from S into itself (Proposition 2.1), S is continuously contained in $\mathcal{H}_\beta$. Then $\mathcal{H}_\beta$ is dense in $C^\infty(\boldsymbol{R})$. However, if $\beta\langle 0$ the space S is not dense in $\mathcal{H}_\beta$. Indeed, let $\beta\langle 0$. It is clear that if S is a dense subspace of $\mathcal{H}_\beta$, then $\lim_{|x|\to\infty} \phi(x) = 0$, for every $\phi \in \mathcal{H}_\beta$. We define the function $\varphi \in C^\infty(\boldsymbol{R})$ such that

$$\varphi(x) = 1, \qquad |x| \leq 1, \text{ and } \varphi(x) = |x|^{-\beta}, \qquad |x| \geq 2.$$

Thus $\varphi(x)$ does not converge to 0, as $|x| \to \infty$. However, $\varphi \in \mathcal{H}_\beta$. Hence we conclude that S is not dense in $\mathcal{H}_\beta$, for every $\beta\langle 0$. Moreover, since Λ_a is a continuous mapping from S into itself, S is continuously contained in $\mathcal{H}_\beta$.

The dual space of $\mathcal{H}_\beta$ is denoted by $\mathcal{H}'_\beta$. By using again Hahn-Banach and Riesz representation theorems (see Propositions 2.2 and 2.8) we can obtain the following representation for the restriction to S of the elements of $\mathcal{H}'_\beta$.

PROPOSITION 3.2. – *Let T be a functional on $\mathcal{H}_\beta$. If $T \in \mathcal{H}'_\beta$, then there exist $r \in \boldsymbol{N}$ and complex regular Borel measure $w_0, w_1, \ldots, w_r$ on $\boldsymbol{R}$ such that*

$$\langle T, \phi \rangle = \sum_{k=0}^{r} \int_{\boldsymbol{R}} \xi(x) \Lambda_a^k \phi(x) dw_k(x), \qquad \phi \in S.$$

∎

3.2. – *The Dunkl transform on the space $\mathcal{H}'_\beta$.*

Assume now and in the sequel that $\beta < a + \frac{1}{2}$. Our objective is to define the Dunkl transform of the generalized functions in $\mathcal{H}'_\beta$. The main result of this section is Proposition 3.7 where we show an inversion formula for the distributional Dunkl transform introduced.

PROPOSITION 3.3. – *Let $y \in \boldsymbol{R}$. Then the function $E_a(-iy.)$ is in the closure of S into $\mathcal{H}_\beta$.*

PROOF. – We firstly prove that $E_a(-iy.) \in \mathcal{H}_\beta$. Let $m \in \boldsymbol{N}$. According to (1.1) we have that

$$\Lambda_a^m E_a(-ixy) = (-iy)^m E_a(-ixy), \qquad x \in \boldsymbol{R}.$$

Then, by taking into account the behaviour of the Bessel function at infinite ([21, p. 199], we obtain, when $|x|$ is large, that

$$|\xi(x)\Lambda_a^m E_a(-iyx)| \leq C|x|^{\beta-a-1/2}.$$

Hence, since E_a is a smooth function on $\boldsymbol{R}$,

$$w_m^\beta(E_a(-iy.)) < \infty.$$

Thus we show that $E_a(-iy.) \in \mathcal{H}_\beta$.

Let ϕ be an even and C^∞-function on $\boldsymbol{R}$ such that $\phi(x) = 1$, $|x| \leq 1$, and $\phi(x) = 0$, $|x| \geq 2$. We define, for every $n \in \boldsymbol{N}$, $\phi_n(x) = \phi(x/n)$, $x \in \boldsymbol{R}$. It is clear that $\phi_n E_a(-iy.) \in S$, for each $n \in \boldsymbol{N}$. We are going to see that $\phi_n E_a(-iy.) \to E_a(-iy.)$, as $n \to \infty$, in $\mathcal{H}_\beta$.

Let $m \in \boldsymbol{N}$. By invoking [5, Proposition 2.1, (iii)], for every $n \in \boldsymbol{N}$,

$$\begin{aligned} |\Lambda_a^m(\phi_n(x)E_a(-iyx) - E_a(-iyx))| \leq C\Bigg(&\left|\frac{d^m}{dx^m}(\phi_n(x)E_a(-iyx) - E_a(-iyx))\right| \\ &+ \sum_{j=0}^{m-1}\Bigg(\left|\frac{d^j}{dx^j}(\phi_n(x)E_a(-iyx) - E_a(-iyx))\right| \\ &+ \left|\frac{d^j}{dx^j}(\phi_n(-x)E_a(iyx) - E_a(iyx))\right|\Bigg)\Bigg), \quad |x| > 1. \end{aligned}$$

By taking into account several properties of the Bessel functions, namely: $z^{-\upsilon}J_\upsilon(z)$ is an entire function for every $\upsilon > -1/2$; the behaviour of $J_\upsilon(z)$ for large $|z|$ ([21, p. 199]), and the differentiability property ([23, (7), p. 129]), we can conclude that, for every $\varepsilon > 0$, there exists $x_0 > 0$ such that

$$|\xi(x)\Lambda_a^m(\phi_n(x)E_a(-iyx) - E_a(-iyx))| < \varepsilon, \qquad |x| \geq x_0 \text{ and } n \in \boldsymbol{N}.$$

On the other hand, if $n \in \boldsymbol{N}$ and $n \geq x_0$,

$$\xi(x)\Lambda_a^m(\phi_n(x)E_a(-iyx) - E_a(-iyx)) = 0, \qquad |x| < x_0.$$

Thus we establish that

$$w_m^\beta(\phi_n E_a(-iy.) - E_a(-iy.)) \to 0, \text{ as } n \to \infty.$$

Hence $\phi_n E_a(-iy.) \to E_a(-iy.)$, as $n \to \infty$, in $\mathcal{H}_\beta$, and $E_a(-iy.)$ is in the closure of S in $\mathcal{H}_\beta$. ■

If $T \in \mathcal{H}'_\beta$ we define the Dunkl transform $F_a(T)$ of T by

$$F_a(T)(y) = \langle T(x), E_a(-iyx)\rangle, \qquad y \in \boldsymbol{R}. \tag{3.2}$$

Assume that $T \in \mathcal{H}'_\beta$. According to Proposition 3.2, for certain complex regular Borel measures $w_0, w_1, ..., w_r$ on $\boldsymbol{R}$, we have

$$\langle T, \phi\rangle = \sum_{k=0}^{r} \int_{\boldsymbol{R}} \xi(x)\Lambda_a^k \phi(x) dw_k(x), \qquad \phi \in S. \tag{3.3}$$

Moreover, the right hand side of (3.3) defines an element of $\mathcal{H}'_\beta$. Indeed, for every $\phi \in \mathcal{H}_\beta$, we have that

$$\left|\sum_{k=0}^{r} \int_{\boldsymbol{R}} \xi(x)\Lambda_a^k \phi(x) dw_k(x)\right| \le \sup_{x\in\boldsymbol{R}} |\xi(x)\Lambda_a^k\phi(x)| \sum_{k=0}^{r} |w_k|(\boldsymbol{R}),$$

being $\sum_{k=0}^{r} |w_k|(\boldsymbol{R}) < \infty$.

Hence

$$\langle T, \phi\rangle = \sum_{k=0}^{r} \int_{\boldsymbol{R}} \xi(x)\Lambda_a^k \phi(x) dw_k(x),$$

for every ϕ being in the closure of S into $\mathcal{H}_\beta$.

Then, by Proposition 3.3, for each $y \in \boldsymbol{R}$, we can write

$$\begin{aligned} F_a(T)(y) &= \langle T(x), E_a(-iyx)\rangle \\ &= \sum_{k=0}^{r} \int_{\boldsymbol{R}} \xi(x)\Lambda_a^k E_a(-iyx)(x) dw_k(x) \\ &= \sum_{k=0}^{r} (-iy)^k \int_{\boldsymbol{R}} \xi(x)E_a(-iyx)dw_k(x), \qquad y \in \boldsymbol{R}. \end{aligned} \tag{3.4}$$

In the following we establish several properties of the generalized Dunkl transform. Note that our proofs are simpler than the ones for the corresponding properties for the distributional Hankel transforms in [7]. Representations (3.3) (and then (3.4)) will play a main role in our proofs.

PROPOSITION 3.4. – *Let $T \in \mathcal{H}'_\beta$. Then there exists $l \in \boldsymbol{N}$ and $C > 0$ for which*

$$|F_a(T)(y)| \le C(1 + |y|)^l, \qquad y \in \boldsymbol{R}.$$

PROOF. – It is sufficient to use (3.4) and to proced as in the proof of Proposition 3.3. ∎

PROPOSITION 3.5. – *Let $T \in \mathcal{H}'_\beta$. Then $F_a(T)$ is m-times continuously differentiable in $\boldsymbol{R}$, provided that $\beta \le a - m + 1/2$.*

PROOF. – According to (3.4) we can write that

$$F_a(T)(y) = \sum_{k=0}^{r} (-iy)^k \int_{\boldsymbol{R}} \xi(x) E_a(-iyx) dw_k(x), \qquad y \in \boldsymbol{R},$$

where $r \in \boldsymbol{N}$ and w_k is a complex regular Borel measure, for $k = 0, ..., r$.

Let $j \in \boldsymbol{N}$. A straightforward manipulation allows us to obtain that

$$\frac{d^j}{dy^j} = \sum_{i=1}^{j} c_{i,j} y^{d_{i,j}} \left(\frac{1}{y}\frac{d}{dy}\right)^i,$$

for certain $c_{i,j} \in \boldsymbol{R}$ and $d_{i,j} \in \boldsymbol{N}$, $i = 1, 2, ..., j$. Then, by invoking [23, (7), p. 129] we get

$$\begin{aligned}\frac{d^j}{dy^j} E_a(-iyx) &= \frac{d^j}{dy^j}\left(2^a \Gamma(a+1)((xy)^{-a} J_a(xy) - i(xy)^{-a} J_{a+1}(xy))\right) \\ &= -2^a \Gamma(a+1)\Bigg(\sum_{k=1}^{j} c_{k,j} y^{d_{k,j}} x^{2k} (xy)^{-a-k} J_{a+k}(xy) \\ &\quad + i \sum_{k=1}^{j} c_{k,j} y^{d_{k,j}} x^{2k} (xy)^{-a-k} J_{a+k+1}(xy) \\ &\quad + ijx \sum_{k=1}^{j-1} c_{k,j-1} y^{d_{k,j-1}} x^{2k} (xy)^{-a-k-1} J_{a+k+1}(xy)\Bigg), \qquad x, y \in \boldsymbol{R}.\end{aligned}$$

By using now [21, p. 199] we have

$$\left|\xi(x) \frac{d^j}{dy^j} E_a(-iyx)\right| \le \psi(|y|)|\xi(x)| \begin{cases} |x|^{-a+j-1/2}, & |x| > 1 \\ 1, & |x| \le 1 \end{cases}$$

where ψ is a polynomial with positive coefficients.

Then, differentiating under the integral sign we can see that $F_a(T)$ is m-times continuously differentiable provided that $\beta \le a - m + 1/2$. ■

Since S is continuously contained in $\mathcal{H}_\beta$, if $T \in \mathcal{H}'_\beta$, then the restriction of T to S is in S'. Hence we can define two distributional Dunkl transforms of T: by (1.2) as an elementof S' and by (3.2) as an element of $\mathcal{H}'_\beta$. We now establish that the two distributional transforms of T coincides as elements of S'.

PROPOSITION 3.6. – *Let $T \in \mathcal{H}'_\beta$. Then, $F_a(T)$ defines an element of S' and, for every $\phi \in S$,*

$$\langle F_a(T), \phi \rangle = \langle T, F_a(\phi) \rangle. \tag{3.5}$$

PROOF. – Note firstly that, by Proposition 3.4, $F_a(T)$ defines an element of S' according to (2.2). We have that, for every $\phi \in S$,

$$\int_{-\infty}^{\infty} F_a(T)(y)\phi(y)\frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)}dy = \langle T, F_a(\phi)\rangle.$$

By invoking Proposition 3.2 it is sufficient to prove the property when T takes the form

$$\langle T, \phi\rangle = \int_{\boldsymbol{R}} \xi(x)\Lambda_a^m \phi(x)dw(x), \qquad \phi \in S,$$

where $m \in \boldsymbol{N}$ and w is a complex regular Borel measure. Then

$$F_a(T)(y) = (-iy)^m \int_{\boldsymbol{R}} \xi(x)E_a(-iyx)dw(x), \qquad y \in \boldsymbol{R}.$$

Now, by interchanging the order of integration, that is justified by the properties of the Bessel functions and the measure w, and by (1.1) we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} F_a(T)(y)\phi(y)\frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)}dy &= \int_{-\infty}^{\infty} \phi(y)(-iy)^m \int_{\boldsymbol{R}} \xi(x)E_a(-iyx)dw(x)\frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)}dy \\ &= \int_{\boldsymbol{R}} \xi(x)\Lambda_a^m \int_{\boldsymbol{R}} \phi(y)E_a(-iyx)\frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)}dydw(x) \\ &= \langle T, F_a(\phi)\rangle, \qquad \phi \in S.\end{aligned}$$

■

The inversion formula for the distributional Dunkl transform on $\mathcal{H}'_\beta$ is proved in the folllowing.

PROPOSITION 3.7. – *Let $T \in \mathcal{H}'_\beta$. For every $r > 0$ we define*

$$T_r(x) = \int_{-r}^{r} F_a(T)(y)E_a(ixy)\frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)}dy, \qquad x \in \boldsymbol{R}. \tag{3.6}$$

*Then, $T_r \to T$, as $r \to \infty$ in the weak * topology of S'.*

PROOF. – Note firstly that according to Propositions 3.4 and 3.5 the integral defining $T_r(x)$ is absolutely convergent, for every $r > 0$ and $x \in \boldsymbol{R}$. Moreover, for each $r > 0$, T_r is bounded on $\boldsymbol{R}$ and then, it defines an element of S' by

$$\langle T_r, \phi\rangle = \int_{\boldsymbol{R}} T_r(x)\phi(x)\frac{|x|^{2a+1}}{2^{a+1}\Gamma(a+1)}dx, \qquad \phi \in S.$$

We are going to see that, for every $\phi \in S$,

$$\langle T_r, \phi \rangle \to \langle T, \phi \rangle, \text{ as } r \to \infty, \tag{3.7}$$

when T takes the form

$$\langle T, \phi \rangle = \int_{\boldsymbol{R}} \xi(x) \Lambda_a^m \phi(x) dw(x), \qquad \phi \in S,$$

where $m \in \boldsymbol{N}$ and w is a complex regular Borel measure. Then

$$F_a(T)(y) = (-iy)^m \int_{\boldsymbol{R}} \xi(x) E_a(-iyx) dw(x), \qquad y \in \boldsymbol{R}.$$

For every $r > 0$, we have

$$T_r(x) = \int_{-r}^{r} (-iy)^m E_a(iyx) \frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)} \int_{\boldsymbol{R}} \xi(x) E_a(-iyx) dw(x) dy,$$

and, for every $\phi \in S$,

$$\begin{aligned}\langle T_r, \phi \rangle &= \int_{\boldsymbol{R}} \phi(x) \frac{|x|^{2a+1}}{2^{a+1}\Gamma(a+1)} \int_{-r}^{r} (-iy)^m E_a(iyx) \frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)} \int_{\boldsymbol{R}} \xi(z) E_a(-iyz) dw(z) dy dx \\ &= \int_{\boldsymbol{R}} \xi(z) \int_{-r}^{r} (-iy)^m E_a(-iyz) \frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)} \int_{\boldsymbol{R}} \phi(x) E_a(iyx) \frac{|x|^{2a+1}}{2^{a+1}\Gamma(a+1)} dx dw(z) \\ &= \int_{\boldsymbol{R}} \xi(z) \int_{-r}^{r} (-iy)^m E_a(-iyz) \frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)} F_a^{-1}(\phi)(y) dy dw(z).\end{aligned}$$

On the other hand, we can write, for every $\phi \in S$,

$$\begin{aligned}\langle T, \phi \rangle &= \int_{\boldsymbol{R}} \xi(x) \Lambda_a^m \phi(x) dw(x) \\ &= \int_{\boldsymbol{R}} \xi(x) \Lambda_a^m \int_{\boldsymbol{R}} F_a^{-1}(\phi)(y) E_a(-iyx) \frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)} dy dw(x) \\ &= \int_{\boldsymbol{R}} \xi(x) \int_{\boldsymbol{R}} (-iy)^m F_a^{-1}(\phi)(y) E_a(-iyx) \frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)} dy dw(x).\end{aligned}$$

Then, (3.7) is proved when we see that, for every $\phi \in S$,

$$\begin{aligned}&\lim_{r \to \infty} \int_{\boldsymbol{R}} \xi(z) \int_{-r}^{r} (-iy)^m E_a(-iyz) \frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)} F_a^{-1}(\phi)(y) dy dw(z) \\ &= \int_{\boldsymbol{R}} \xi(x) \int_{\boldsymbol{R}} (-iy)^m E_a(-iyx) F_a^{-1}(\phi)(y) \frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)} dy dw(x).\end{aligned} \tag{3.8}$$

Let $\phi \in S$ and $r\rangle 0$. We can write, by proceeding as in the proof of Proposition 3.4,

$$\left|\int_{\boldsymbol{R}} \xi(x)\left(\int_{\boldsymbol{R}} - \int_{-r}^{r}\right)(-iy)^m F_a^{-1}(\phi)(y) E_a(-iyx)|y|^{2a+1} dy dw(x)\right|$$

$$\leq \int_{\boldsymbol{R}} |\xi(x)| \int_{|y|\geq r} |y|^m |F_a^{-1}(\phi)(y)|\,|E_a(-iyx)|\,|y|^{2a+1} dy d|w|(x)$$

$$\leq C \int_{\boldsymbol{R}} d|w|(x) \int_{|y|\geq r} |y|^l |F_a^{-1}(\phi)(y)| dy,$$

for some $l \in \boldsymbol{N}$. Hence, since $F_a^{-1}(\phi) \in S$ ([10, Corollary 4.22], (3.8) is established.

Thus the proof is finished. ∎

From Proposition 3.7 follows immediately the following uniqueness property.

PROPOSITION 3.8. – *Let* $T \in \mathcal{H}'_\beta$. *If* $F_a(T) = 0$, *then* $T = 0$ *on* S. ∎

By arguing as in [7] we can present some applications of our theory to some differential-difference equations involving the operator Λ_a.

3.3. – *The complex distributional Dunkl transform.*

Koh and Zemanian [13] investigated a complex version of the distributional Hankel transform by using the kernel method. In this paragraph we present an analogous study for the Dunkl transforms but now the properties can be proved, as in the previous case, by using the corresponding representation of the elements of the dual space and the distributional Dunkl transforms. Since the proofs of these properties are similar than to those presented in the previous sections we will omit here the proofs of the properties.

We firstly introduce new spaces of functions. Let $a > 0$. A function $\phi \in C^\infty(\boldsymbol{R})$ is in $\boldsymbol{M}_a$ if, and only if, for every $m \in \boldsymbol{N}$,

$$p_a^m(\phi) = \sup_{x\in\boldsymbol{R}} e^{-a|x|} |\Lambda_a^m \phi(x)| < \infty.$$

The space $\boldsymbol{M}_a$ is endowed with the topology associated with the family $\{p_a^m\}_{m\in\boldsymbol{N}}$ of seminorms. By proceeding as in the proof of Proposition 3.1 we can show that $\boldsymbol{M}_a$ is a Fréchet space. Moreover the inclusions $S \subset \boldsymbol{M}_a \subset C^\infty(\boldsymbol{R})$ are continuous.

By J'_a we denote the dual space of $\boldsymbol{M}_a$. A representation of the restriction to S of the elements of J'_a that, as we have mentioned, plays a crucial role in the following.

PROPOSITION 3.9. – *Let T be a functional on $\boldsymbol{M}_a$. Then, if $T \in \boldsymbol{M}'_a$, there exist $r \in \boldsymbol{N}$ and complex Borel regular measures $w_0, w_1, ..., w_r$ on $\boldsymbol{R}$ such that*

$$\langle T, \phi \rangle = \sum_{k=0}^{r} \int_{\boldsymbol{R}} e^{-a|x|} \Lambda_a^k \phi(x) dw_k(x), \qquad \phi \in S. \tag{3.9}$$

■

Note that the right hand side of (3.9) defines an element of $\boldsymbol{M}'_a$.

The kernel of the Dunkl transform is in the closure of the space S in $\boldsymbol{M}_a$ in a suitable strip of the complex plane.

PROPOSITION 3.10. – *If $y \in \boldsymbol{C}$ where $|Im\, y| < a$, then $E_a(-iy.)$ is in the closure of S in $\boldsymbol{M}_a$.*

PROOF. – It is sufficient by taking into account that

$$|(xz)^{-w} J_w(z)| \leq C e^{|x||Im\, z|}, \qquad x \in \boldsymbol{R} \text{ and } z \in \boldsymbol{C}$$

([13, p. 950]), and to proceed as in the proof of Proposition 3.3. ■

If $T \in \boldsymbol{M}'_a$ we define the Dunkl transform $F_a(T)$ of T by

$$F_a(T)(u) = \langle T(x), E_a(-iyx) \rangle, \qquad |Im\, y| \langle a.$$

By combining Propositions 3.9 and 3.10 and by (1.1) we obtain that, if $T \in \boldsymbol{M}'_a$, then

$$F_a(T)(y) = \sum_{k=0}^{r} (-yi)^k \int_{\boldsymbol{R}} e^{-a|x|} E_a(-iyx) dw_k(x), \qquad |Im\, y| < a,$$

where $r \in \boldsymbol{N}$ and $w_0, w_1, ..., w_r$ are complex regular Borel measures on $\boldsymbol{R}$.

We now establish the main properties of the Dunkl transform on $\boldsymbol{M}'_a$. Each of them can be proved as the corresponding one in the previous case on $\mathcal{H}'_\beta$.

PROPOSITION 3.11. – *Let $T \in \boldsymbol{M}'_a$. Then*

i) *There exist $r \in \boldsymbol{N}$ and $C \rangle 0$ such that*

$$|F_a(T)(y)| \leq C(1 + |y|)^r, \qquad |Im\, y| < a.$$

ii) *$F_a(T)$ is a holomorphic function in the strip $\{y \in \boldsymbol{C} : |Im\, y| < a\}$.*

iii) *For every $\phi \in S$ we have that*

$$\int_{\boldsymbol{R}} F_a(T)(y) \phi(y) \frac{|y|^{2a+1}}{2^{a+1} \Gamma(a+1)} dy = \langle T, F_a(\phi) \rangle.$$

iv) *If, for every $r > 0$, T_r is defined by*

$$T_r(x) = \int_{-r}^{r} F_a(T)(y) E_a(iyx) \frac{|y|^{2a+1}}{2^{a+1}\Gamma(a+1)} dy,$$

*then $T_r \to T$, as $r \to \infty$, in the weak * topology of S'.*

v) *If $T \in \boldsymbol{M}'_a$ and $F_a(T) = 0$, then $T = 0$ on S.* ■

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