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On Simple and Stable Homogeneous Bundles.

Simona Faini

Sunto. – *Nell' articolo abbiamo voluto analizzare il rapporto tra i concetti di stabilità e semplicità per un fibrato vettoriale omogeneo su una varietà proiettiva.*
Il teorema principale mostra come un fibrato omogeneo non sia destabilizzato dai suoi sottofibrati omogenei se e solo se esso è il prodotto tensoriale fra un fibrato omogeneo stabile ed una rappresentazione irriducibile.
Daremo quindi un esempio di un fibrato omogeneo, che risulta semplice, ma non stabile.

Summary. – *In this work we will analyze the relation between the stability and the simplicity of a homogeneous vector bundle on a projective variety.*
Our main theorem shows that a homogeneous bundle is not destabilized by its homogeneous subbundles if and only if it is the tensor product of a stable homogeneous bundle and an irreducible representation.
Then we give an example of a homogeneous bundle, which is simple, but not stable.

1. – Introduction.

In this article we want to examine the relation between the concepts of stability and simplicity for a homogeneous vector bundle.

We start considering a homogeneous rational variety $X := G/P$, with G complex simple Lie group and P a parabolic subgroup; a homogeneous vector bundle E on G/P will be then, as we will see later, given by a representation ρ of P; thus we'll can write $E = E_\rho$ in the just specified sense.

We will define the notions of simplicity and H-stability in section 2; now we show the results we obtained.

The first result for homogeneous bundles on $X = G/P$ is the Ramanan theorem: if ρ is an irreducible representation of P, then E_ρ is a stable bundle.

From this, we have that symmetric powers of $T\mathbb{P}^n$, or symmetric powers of universal and quotient bundle on grassmannians are stable, because for a representation ρ of P, we have $E_{S^m\rho} \simeq S^m E_\rho$, for all $m \geq 0$.

More generally, in [9] Rohmfeld establishes the following semistability-criterion for homogeneous bundles:

Rohmfeld criterion for semistability: (1) E_ρ is H-semistable $\iff \mu_H(F) \leq \mu_H(E_\rho)$ for every homogeneous subbundle F induced by a subrepresentation of ρ;

(2) If E_ρ is undecomposable and $\mu_H(F) < \mu_H(E_\rho)$ for every homogeneous subbundle F of E_ρ induced by a subrepresentation of ρ, then E_ρ is stable.

Actually Rohmfeld states his theorem in a slightly different form, which is not suitably written: indeed, the Euler sequence provides a counterexample to the last statement of [9].

However, the reader can easily check that what really Rohmfeld proved is the just stated criterion.

Our first result in this work is therefore the next theorem, which is a refinement of the preceding criterion:

THEOREM 1. – *(Main theorem) Let E be a homogeneous bundle on the homogeneous rational variety G/P (with the preceding notations); then the two following conditions are equivalent:*

(i) *For every F, subbundle of E induced by a subrepresentation of ρ, we have*

$$\mu_H(F) < \mu_H(E);$$

(ii) *there exist an irreducible representation W of G and a stable homogeneous subbundle F_0 of E, such that*

$$E \simeq W \otimes F_0.$$

We will see later not only the proof of this theorem, but also an application for the next problem.

It is well known that for every vector bundle, H-stability $\Rightarrow$ simplicity, but for rank ≥ 3 the viceversa is not true: in [5] a counterexample is constructed (the simplest one has rk 3 on $\mathbb{CP}^2$).

Although the notion of homogeneity for vector bundles is a strong hypothesis, in the end of this work *we will give an example of a rk* 15 *homogeneous bundle on* $\mathbb{CP}^2$*, such that it is simple, but not stable*: doing this, we will use the main theorem we told before.

We finally remark that all simple homogeneous bundles on $\mathbb{CP}^2$ of $rk \leq 14$ are stable: this is the content of the conclusive tables.

2. – Notations and preliminaries.

Let $X := G/P$ a homogeneous rational variety, with G complex Lie group and P its parabolic subgroup.

We will give now two definitions:

DEFINITION 1. – *Let E be a vector bundle on the homogeneous rational variety $X := G/P$, of $dimX = d$. Fixed $H \in Pic(X)$, H ample, E is said to be H-stable (respectively semistable) if for all subsheaves F of E, $0 \neq F \subsetneq E$, it holds*

$$\mu_H(F) < \mu_H(E) \qquad \text{(respectively} \leq \text{)},$$

where

$$\mu_H(F) := \frac{H^{d-1} \cdot c_1(F)}{rk(F)}$$

is the slope of F with respect to H.

EXAMPLE. – If $G/P = \mathbb{CP}^2$, we will take $H = \mathcal{O}_{\mathbb{CP}^2}(1) = \mathcal{O}(1)$; then, for any vector bundle E on $\mathbb{CP}^2$, by identifying $\mathbb{Z} \simeq \mathrm{H}^2(\mathbb{P}^2, \mathbb{Z}) \ni c_1(\mathrm{E})$, we have

$$\mu(E) := \mu_H(E) = \frac{c_1(E)}{rk(E)}$$

DEFINITION 2. – *A vector bundle E on a homogeneous rational variety $X = G/P$ is said to be simple, if*

$$h^0(E \otimes E^*) = 1$$

REMARK 1. – *E is simple $\Longleftrightarrow End(E) = \{homotheties\ of\ E\}$.*

In this article, we will work essentially with a particular class of vector bundles on X: the homogeneous bundles.

To define these, we need to introduce before the following construction.

DEFINITION 3. – *Let $\rho : P \longrightarrow GL(r, \mathbb{C})$ a representation of P. We define the vector bundle E_ρ on G/P as the quotient of $G \times \mathbb{C}^r$, with respect to the equivalence relation $\sim$, given by*

$$(g, v) \sim (g', v') \Leftrightarrow \text{ there exist } p \in P : \; g = g'p \text{ and } v = \rho(p^{-1})v'$$

REMARK 2.

$$E_{\rho_1} \oplus E_{\rho_2} \simeq E_{\rho_1 \oplus \rho_2};$$

$$E_{\wedge^k(\rho)} \simeq \wedge^k E_\rho;$$

$$E_{\rho_1} \otimes E_{\rho_2} \simeq E_{\rho_1 \otimes \rho_2};$$

$$E_{S^m \rho} \simeq S^m E_\rho;$$

$$E_{\rho^*} \simeq (E_\rho)^*.$$

Now, the previous definition allows us to introduce the concept of homogeneity for a vector bundle:

DEFINITION 4. – *A vector bundle of* $rk = r$ *on* G/P, *E*, *is homogeneous, if there exists a representation* $\rho : P \longrightarrow GL(r, \mathbb{C})$ *s.t.* $E = E_\rho$.

REMARK 3. – *If E,F are homogeneous bundles on* G/P, *then* $E \oplus F$, $E \otimes F$ *and* E^* *are homogeneous too.*

EXAMPLE. – $\mathcal{O}_{\mathbb{CP}^n}(t)$, $T\mathbb{CP}^n(t)$, $S^m(T\mathbb{CP}^n)$ are homogeneous vector bundles, for all $t \in \mathbb{Z}$, for all $m \in \mathbb{N}$.

Now we begin by introducing some notations, which we will use in the fourth section: there, we will construct an example of homogeneous vector bundle on $\mathbb{CP}^2$, which is simple, but not stable.

In that case we will have thus $G/P = \mathbb{CP}^2$, i.e. $G = SL(3, \mathbb{C})$ and

$$P := \left\{ \left[\begin{array}{c|cc} det A^{-1} & A & y \\ \hline & & \\ 0 & & A \end{array} \right] \quad | \, A \in GL(2, \mathbb{C}), (x, y) \in \mathbb{C}^2 \right\}$$

By the definition of the homogeneity of a vector bundle, we are naturally interested in studying the indecomposable (otherwise, the induced vector bundle is decomposable, hence automatically not simple $\Rightarrow$ not stable) representations of P.

At first, we observe that $P \simeq GL(2, \mathbb{C}) \ltimes \mathbb{C}^2$, where the structure of semi-direct product on $GL(2, \mathbb{C}) \times \mathbb{C}^2$ is defined by, for $(A \ltimes a), (B \ltimes \beta) \in GL(2, \mathbb{C}) \ltimes \mathbb{C}^2$,

$$(A \ltimes a) \cdot (B \ltimes \beta) := (A \cdot B \ltimes (\hat{B}a) + \beta),$$

Here "$A \cdot B$" indicates the usual row-columns product and $\hat{B}a := detB \cdot a \cdot B$.

So we can find the representations of *P*, by combining representations of $GL(2, \mathbb{C})$ and of $\mathbb{C}^2$.

Now, the irreducible representations of $GL(2, \mathbb{C})$ are, for all $m \in \mathbb{N}$ and $l \in \mathbb{Z}$,

$$\rho_m^l : GL(2, \mathbb{C}) \longrightarrow GL(m + 1, \mathbb{C})$$

$$A \longmapsto (det A)^l \cdot S^m A$$

and thus, because of the complete reducibility of $GL(2, \mathbb{C})$, if $\psi : P \longrightarrow GL(r, \mathbb{C}) =$ Aut(V) is a representation of P , then

$$\psi|_{GL(2,\mathbb{C}) \ltimes 0} = \bigoplus_{i=1}^{j} \rho_{m_i}^{l_i}.$$

From all these considerations, we can deduce the following theorem (see [10] for more details):

THEOREM 2. – *Let* $\psi : P \longrightarrow Aut(V)$ *be a representation, such that*

$$\psi|_{GL(2,\mathbb{C})\ltimes 0} = \bigoplus_{i=1}^{j} \rho_{m_i}^{l_i}.$$

Then there exist a P-invariant flag

$$0 \subset V_1 \subset V_2 \ldots \subset V_j = V$$

such that

$$\rho_{m_i}^{l^i} \simeq V_i/V_{i-1} =: gr_i V. \quad (1)$$

Moreover, $\psi|_{Id_2 \ltimes \mathbb{C}^2} =: \pi$ *induces, for* $1 \le r \le s \le j$, *operators*

$$\pi_{rs} : \mathbb{C}^2 \longrightarrow Hom(gr_s V, gr_r V)$$

and, for $1 \le s < r \le j$, *operators* $\pi_{rs} \equiv 0$ *(* $*$ *).*

Finally, ψ *is completely reducible* $\Leftrightarrow \pi \equiv 0$.

DEFINITION 5. – *(fundamental!) Let* ψ *be as in the preceding theorem. Then* $(\rho_{m_1}^{l_1}, \ldots, \rho_{m_j}^{l_j})$ *is the type of* ψ, *and* j *is the index.*

THEOREM 3. – *Define* $Q = T\mathbb{P}^2(-1)$. *If* ρ_m^l *is an irreducible representation of* P, *then*

$$(1) E_{\rho_m^l} \simeq S^m T\mathbb{P}^2(l-m) \simeq S^m Q(l)$$

$$(2) (\rho_m^l)^* \simeq \rho_m^{-l-m}.$$

REMARK 4. – *The type of a representation of P doesn't determine uniquely the P-invariant flag of theorem* 2, *in general: this depends on the fact that, when we have three or more irreducible components of* $\psi|_{GL(2,\mathbb{C}\ltimes 0)}$ ($\Leftrightarrow j \ge 3$), $\Rightarrow$ *we can arrange on the matrix associated to* ψ *the correspondent diagonal blocks in many different ways (provided we still have a representation, i.e.* $\pi_{rs} \equiv 0$ *for all* $1 \le s < r \le j$, *as we said before in* $(*)$*).*

Now, we will display some results, which will help us to write the matrix-form for a representation of P; the first is the

THEOREM 4. – (*see* [10]) *Let* $\psi : P \longrightarrow Aut\,(V)$ *be a representation of P, of type*

$(\rho^{l_1}_{m_1}, \rho^{l_2}_{m_2})$; *if* ψ *is indecomposable, then necessarily we have*

$$|m_2 - m_1| = 1 \text{ and } \begin{cases} l_2 = l_1 + 1, & \text{if } m_1 < m_2; \\ l_2 = l_1 + 2, & \text{if } m_2 < m_1. \end{cases}$$

Moreover, the operator π_{12} *is uniquely determined, up to a scalar factor* $\lambda \in \mathbb{C}^*$, *as follows: for* $(x, y) \in \mathbb{C}^2$

(i) *if* $m_1 < m_2$, *we call* $m = m_1$ *and*

$$\Rightarrow \pi_{12}(x, y) = \begin{bmatrix} (m+1)\cdot x & y & & & \\ & mx & 2y & & 0 \\ & 0 & \ddots & \ddots & \\ & & & x & (m+1)\cdot y \end{bmatrix} =: D^{m+1}$$

is a $(m+1) \times (m+2)$ *matrix;*

(ii) *if* $m_2 < m_1$, *we call now* $m = m_2$

$$\Rightarrow \pi_{12}(x, y) = \begin{bmatrix} y & & & \\ -x & y & & 0 \\ & -x & \ddots & \\ & 0 & \ddots & y \\ & & & -x \end{bmatrix} =: I^{m+1}$$

is a $(m+1) \times (m+2)$ *matrix.*

As a consequence of this theorem, one can verify that in both previous cases it holds

$$\mu(E_{\rho^{l_2}_{m_2}}) = \mu(E_{\rho^{l_1}_{m_1}}) + \frac{3}{2}.$$

DEFINITION 6. – *In theorem 4 (where* ψ *is indecomposable!),* π_{12} *is said a connection-operator.*

We reported this theorem, because it is the point of departure to prove the following more general results:

PROPOSITION 1. – (*see* [10]) *Let* ψ *be an indecomposable representation of* P, *of type* $(\rho^{l_1}_{m_1}, \rho^{l_2}_{m_2}, \ldots, \rho^{l_t}_{m_t})$; *we call* $n := min\{m_i \mid i = 1, \ldots, t\}$ *and* $N := max\{m_i \mid i = 1, \ldots, t\}$. *Then*

(i) *for each irreducible component* ρ^l_m *of* ψ, *with* $m \neq n, N$, *there exist* $h, k \in \{1, \ldots, t\}$ *s.t.*

$$m_h = m - 1 \qquad \text{and} \qquad m_k = m + 1;$$

(ii) *for* $m = n, N$, *there exist* $h, k \in \{1, ..., t\}$ *s.t.*

$$m_h = n+1 \qquad \text{and} \qquad m_k = N-1.$$

In conclusion, the proposition says that the set $\{ m_i \mid i = 1, ..., t \}$ is connected.

THEOREM 5. – (*see* [10]) *Let* ψ *be an indecomposable representation of P, of index t. Then there exists an uniquely determined filtration*

$$0 \subset V_1 \subset V_2 \subset ... \subset V_k$$

with the following properties:

(1) $H_i := V_i/V_{i-1} = \oplus_{j \in M_i} \rho_{m_j}^{l_j}$, *where* $M_i \subset \{1, ..., t\}$ *is such that*

$$\mu(H_i) := \mu(\rho_{m_j}^{l_j}) = constant \ (with\ respect\ to\ j)$$

for all $j \in M_i$;

(2) $\mu(H_i) = \mu(H_{i-1}) + \frac{3}{2}$ *for all* $i \in \{2, ..., k\}$;

(3) $\psi|_{Id_2 \ltimes \mathbb{C}^2}$ *induces, for each* $i \in \{1, ..., k-1\}$, *a homogeneous not-trivial operator of degree* 1

$$\Theta_i : \mathbb{C}^2 \longrightarrow Hom\,(H_{i+1}, H_i).$$

DEFINITION 7. – *In the same hypothesis of theorem* 5, *we call* $\psi \leftrightsquigarrow (H_1, H_2, ..., H_k)$ *the* μ*-filtration of the representation* ψ.

COROLLARY 1. – *Let* ψ *be an indecomposable representation of P, with* μ*-filtration* $(H_1, H_2, ..., H_k)$ *and respective operators* $\{\Theta_i\}_{i \in \{1,...,k-1\}}$. *Then*

$$exp\left(\begin{bmatrix} 0 & \Theta_1 & & & \\ & 0 & \Theta_2 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & 0 & \Theta_{k-1} \\ & & & & 0 \end{bmatrix}\right) = \psi|_{Id_2 \ltimes \mathbb{C}^2}$$

Here and in the following we will intend V as in $\mathbb{CP}^2 = \mathbb{P}(V)$; now let $\Gamma^{p,q}V$ be the irreducible representation of $SL(V)$ corresponding to the Young diagram with p boxes in the first row, and q boxes in the second one.

This is all what we need to know to compute the cohomology groups we said; in fact, we have the following well known results:

PROPOSITION 2. – *Let* ρ_m^l *be the irreducible representation of P defining*

$S^mQ(l)$; then, if we identify ρ^l_m with the homogeneous bundle it induces,

$$(1)\ c_1(\rho^l_m) = \left(\frac{1}{2}m + l\right) \cdot (m+1);$$

$$(\Downarrow)$$

$$(2)\ \mu(\rho^l_m) = \left(\frac{1}{2}m + l\right).$$

THEOREM 6. – *The first cohomology-groups of $E_{\rho^l_m}$ are*

$$H^0(E_{\rho^l_m}) \simeq \Gamma^{m+l,l}V;$$

$$H^1(E_{\rho^l_m}) \simeq \Gamma^{m-1,l+m+1}V;$$

$$H^2(E_{\rho^l_m}) \simeq \Gamma^{-l-3,-l-m-3}V.$$

3. – Stability of homogeneous vector bundles.

The goal of this section is the proof of the main theorem (see Introduction): however, we need before the Ramanan construction of «CS-subbundle» ([9]).

It allows us to verify the failure of H-stability of a homogeneous vector bundle on its homogeneous subbundles, instead of on all its subsheaves (thus we call «CS-subbundles» those ones which are contradicting stability).

We're going now to introduce some results, which the reader could find in [9] in a more detailed way:

DEFINITION 8. – *Let E be a not-stable homogeneous vector bundle. A coherent subsheaf $0 \neq F \subsetneq E$ is said to be SCS (i.e., «strong contradicting stability») in E, if the two following conditions are fulfilled:*

(i) *F is H-stable and E/F is torsion-free;*

(ii) *for all coherent subsheaf Q, with $0 \neq Q \subsetneq E/F$, we have*

$$\mu_H(Q) \leq \mu_H(F).$$

LEMMA 1. – *Let $0 \neq U_1, U_2 \subsetneq E$ be two coherent subsheaves of E, with E/U_1 and E/U_2 torsion-free. If U_1 is H-stable and U_2 satisfies condition (ii) of the preceding definition, then*

$$U_1 \cap U_2 \neq 0 \qquad \text{and} \qquad U_1 \not\subseteq U_2 \implies \mu_H(U_1) < \mu_H(U_2)$$

LEMMA 2. – *Let E be a vector bundle on G/P and*

$$M := \{ \ c_1(F) \mid 0 \neq F \subsetneq E, F \ coherent \ subheaf \ of \ E\}$$

Then $\sup M < +\infty$ *and* $\sup M = \max M$.

PROPOSITION 3. – (*Existence of SCS-subsheaves*) *Let E be a H-not stable vector bundle; then E contains a SCS-subsheaf F.*

THEOREM 7. – *(Uniqueness of SCS-sheaves) Let E be a H-not stable vector bundle on G/P, and let* $0 \neq U_1, U_2 \subsetneq E$ *be two SCS-subsheaves of E, s.t.* $U_1 \cap U_2 \neq 0$. *Then it is* $U_1 = U_2$.

With the assumption that E is not H-stable, we now define the not-empty (by proposition 3) set

$$M_0 := \{\text{SCS} - \text{subsheaves of } E, \text{ with minimal rank } =: r_0 \text{ and maximal slope } \mu_0\}.$$

Proposition 3 and theorem 7 say us that M_0 contains SCS-subsheaves of E, for which every two intersect only trivially ($*$).

Now we call

$$\bar{F} := \oplus_{i \in I} F_i,$$

where I is the maximal set of indices for elements of M_0, which form a direct sum; since rkE is finite and by ($*$), then I is finite too.

Now, if $M \in M_0 - \{ \ F_i \mid i \in I \ \}$, then it is $M \cap \bar{F} \neq 0$; in fact, if $M \cap \bar{F} = 0$, then we have to add a new element for M to I, because of the maximality of I with respect to this property ($\oplus$).

Moreover, the sheaf $\bar{F}$ has maximal slope $\mu_H(\bar{F}) = \mu_0$; therefore $\bar{F}$ satisfies condition (ii) of the definition of SCS-subsheaf. By lemma 1, we have $M \subset \bar{F}$ and thus the subsheaf $\bar{F}$ is uniquely determined.

Finally, we have only to show the properties which characterize our $\bar{F}$: in particular, in the second of the following theorems we'll see that $\bar{F}$ is a homogeneous H-semistable subbundle of E. Once again, here we quote some results from [9]:

PROPOSITION 4. – *If* $\bar{F}$ *is defined as above and* $A \in M_0$, *then there exists* $i \in I$ *s.t.* $A \simeq F_i$.

THEOREM 8. – *If* $\bar{F}$ *is as above, then* $\bar{F}$ *is a homogeneous H-semistable vector subbundle of E, with* $\mu_H(\bar{F}) \geq \mu_H(E)$.

DEFINITION 9. – *Let* $\bar{F}$ *be as above; we call it the CS-subbundle of E.*

From this construction of $\bar{F}$, it follows the next

COROLLARY 2. – *Let H be an ample fixed element in Pic(G/P). If E_ρ is a H-not stable homogeneous vector bundle on G/P, then E_ρ contains a homogeneous CS-subbundle; i.e., there exists a homogeneous subbundle $\bar{F} = \oplus_{i \in I} F_i$ induced by a subrepresentation of ρ, s.t.*

$$\mu_H(\bar{F}) \geq \mu_H(E_\rho),$$

where the F_i's are homogeneous subbundles of E_ρ, H-stable and with the same slope and rank.

COROLLARY 3. – (*see* [6]) $\mu_H(\bar{F}) \geq \mu_H(E_\rho)$ *for every F, homogeneous subbundle induced by a subrepresentation of $\rho \Leftrightarrow E_\rho$ is H-semistable.*

We are now ready to prove the next

THEOREM 9. – *(Main theorem) Let $E = E_\rho$ be a homogeneous vector bundle on G/P; the following conditions are equivalent:*

(i) *For every homogeneous subbundle F given by a subrepresentation of ρ, we have $\mu_H(F) < \mu_H(E)$;*

(ii) *There exist an irreducible representation W of G and a homogeneous H-stable ($\Rightarrow$ simple) bundle (not necessarily a homogeneous subbundle) F_0 of E, s.t.*

$$E = W \otimes F_0.$$

PROOF:

($(i) \Rightarrow (ii)$) We only have the two following possibilities:

(a) E is H-stable, and we have already finished, because $E = E \otimes \mathbb{C}$;

(b) Otherwise E is H-not stable, and therefore, by hypothesis (i) and corollary 2, it is necessarily $\bar{F} = E$, where $\bar{F}$ is as in the same corollary.

Now, $\bar{F} = \oplus_{i \in I} F_i$ and in this direct sum we can group the F_i's which result isomorphic; thus we get

$$\bar{F} = \oplus_i W_i \otimes F_i,$$

with W_i vector spaces and F_i pairwise not isomorphic. Now, by using the H-semistability of $\bar{F}$, we will prove that there is only one summand in the above direct sum.

In fact, if there were at least two distinct $F_i \otimes W_i$ in $\bar{F}$, then each of these would be a homogeneous subbundle of E (using for this the same Rohmfeld's argument, with the ρ-invariance and the Krull-Schmidt theorem's application which is in [2]), of rank $< rk(E)$, but with $\mu_H(F_i \otimes W_i) = \mu_H(F_i) = \mu_H(E)$, in opposition to the assumption.

Hence $E = \bar{F} = F_0 \otimes W$, where F_0 is one fixed of the F_i's.

The stability of F_0 is directly given by corollary 2; therefore we only have to show that W is an irreducible representation of G.

• To prove that W is a representation, we need at first to define an action of G on W: but we can do this in a natural way, after we observed that

$$Hom\,(F_0, \bar{F}) = Hom\,(F_0, W \otimes F_0) = W \otimes Hom\,(F_0, F_0) \simeq W$$

because of the simplicity of F_0.

The first term of this chain is $Hom(F_0, \bar{F}) \simeq H^0(F_0^* \otimes \bar{F})$, on which there is already a natural action of G; therefore W is a representation.

• Now, by contradiction, if W wasn't irreducible, then it would be decomposable, that is we would have $W = W_1 \oplus W_2$, with W_i not-trivial subbundles.

But so it were also $E = F_0 \otimes W = F_0 \otimes W_1 \;\oplus\; F_0 \otimes W_2$, where the $F_0 \otimes W_i$'s are both homogeneous (again by [2]) proper subbundles of E, with $\mu_H(F_0 \otimes W_i) = \mu_H(E)$. Hence we found a contradiction to the hypothesis and therefore (i) $\Rightarrow$ (ii) also in the H-not stable case.

($(i) \Leftarrow (ii)$) We can suppose E not H-stable, because otherwise the thesis is obvious. Hence, let E be not-stable.

By (ii), we have immediately the H-semistability of E; so, it suffices to show that the only homogeneous subbundle of E, given by a subrepresentation of ρ, with slope $\mu_H(E)$ is E itself.

Hence, let E' be another subbundle of E, induced by a subrepresentation of ρ, with $\mu_H(E) = \mu_H(E')$: we can assume that rk E' is minimal with respect to the subbundles of E with the same properties.

Thus E' satisfies (i) and so, just by applying the first part of the proof, we know that $E' = W' \otimes F'_0$, with F'_0 homogeneous and stable. Now, from the morphism

$$i : W' \otimes F'_0 \hookrightarrow F_0 \otimes W$$

(induced by the inclusion $E' \hookrightarrow E$), it follows the existence of a not-zero $\varphi \in Hom\,(F'_0, F_0)$.

But $Hom\,(F'_0, F_0)$ is one dimensional, because both F_0 and F'_0 are stable bundles, with the same slope; thus φ itself is an isomorphism, so that

$$Hom\,(F'_0, F_0) \simeq Hom\,(F'_0, F_0)^G \simeq \mathbb{C},$$

by the stability ($\Rightarrow$ simplicity) of F_0; here by $Hom\,(F'_0, F_0)^G$ we mean the G-invariant subspace.

Finally, coming back to the morphism i, we can say that it induces a G-invariant map $W' \to W$ and this implies $W' = W$, by using the Schur's lemma.

Thus we've got

$$E' = F_0 \otimes W' = F_0 \otimes W = E,$$

which is our thesis.

4. – A simple homogeneous bundle, which is not-stable.

In this final section, we will discuss the key-example of a homogeneous, simple, but not-stable vector bundle; we will construct this bundle on $G/P = \mathbb{CP}^2$, so that in this case we will have $G = SL(3, \mathbb{C})$ and

$$P := \left\{ \left[\begin{array}{c|cc} det A^{-1} & x & y \\ \hline & & \\ 0 & \multicolumn{2}{c}{A} \end{array} \right] \quad | \, A \in GL(2, \mathbb{C}), (x, y) \in \mathbb{C}^2 \right\}$$

With the notations introduced in section 2, we consider the representation ψ of P of type $(\rho_1^{-2}, \rho_0^0 \oplus \rho_2^{-1} \oplus \rho_4^{-2}, \rho_3^0)$.

We want to write the matrix-form of this ψ, determined by its type: by the results we indicated in second section, we obtain a first matrix

$$A := \left[\begin{array}{c|c|c|c|c} \rho_1^{-2} & I_1 & D_2 & 0 & 0 \\ \hline 0 & \rho_0^0 & 0 & 0 & 0 \\ \hline 0 & 0 & \rho_2^{-1} & 0 & D_3 \\ \hline 0 & 0 & 0 & \rho_4^{-2} & I_4 \\ \hline 0 & 0 & 0 & 0 & \rho_3^0 \end{array} \right]$$

Hence the matrix-form of ψ is (by abuse of notation)

$$\psi = \exp A = \left[\begin{array}{c|c|c|c|c} \rho_1^{-2} & I_1 & D_2 & 0 & \frac{1}{2} D_2 D_3 \\ \hline 0 & \rho_0^0 & 0 & 0 & 0 \\ \hline 0 & 0 & \rho_2^{-1} & 0 & D_3 \\ \hline 0 & 0 & 0 & \rho_4^{-2} & I_4 \\ \hline 0 & 0 & 0 & 0 & \rho_3^0 \end{array} \right] \tag{2}$$

This matrix is important, because it allows us to investigate the not-stability of E: in fact, by corollary 2, if E is not-stable, then it contains the CS-subbundle, which is induced by a subrepresentation of ψ. So, examining all the subrepresentation of ψ and calculating the slope of each of these (or better, the slope of each bundle by these induced), we can find a destabilizing subbundle of E.

Recalling (see [10]) that two similar matrices associated to representations of P induce the same bundle, we are able to find a destabilizing subbundle of E, considering the following matrix, which is similar to ψ

$$(3)\qquad \left[\begin{array}{c|c|c|c|c} \rho_1^{-2} & D_2 & 0 & \frac{1}{2}D_2D_3 & I_1 \\ \hline 0 & \rho_2^{-1} & 0 & D_3 & 0 \\ \hline 0 & 0 & \rho_4^{-2} & I_4 & 0 \\ \hline 0 & 0 & 0 & \rho_3^0 & 0 \\ \hline 0 & 0 & 0 & 0 & \rho_0^0 \end{array}\right]$$

and its submatrix

$$(4)\qquad \rho := \left[\begin{array}{c|c|c|c} \rho_1^{-2} & D_2 & 0 & \frac{1}{2}D_2D_3 \\ \hline 0 & \rho_2^{-1} & 0 & D_3 \\ \hline 0 & 0 & \rho_4^{-2} & I_4 \\ \hline 0 & 0 & 0 & \rho_3^0 \end{array}\right]$$

Now, if $\bar{F} := E_\rho$, then

$$\mu(\bar{F}) = \frac{3}{14} > \frac{1}{5} = \mu(E)$$

and we conclude the not-stability of E.

Finally, we have to show the simplicity of E; we start with the following exact sequence, obtained out of the filtration of ψ:

$$(5)\qquad 0 \longrightarrow \bar{F} \longrightarrow E \longrightarrow \mathcal{O} \longrightarrow 0$$

We want to compute $H^0(E)$; hence we need before some information about the first cohomology groups of $\bar{F}$.

Let F' and F'' be the homogeneous subbundles of $\bar{F}$, given by subrepresentations of type $(\rho_1^{-2}, \rho_2^{-1})$ and (ρ_4^{-2}, ρ_3^0) respectively ($F' \leftrightarrow (\rho_1^{-2}, \rho_2^{-1})$ and $F'' \leftrightarrow (\rho_4^{-2}, \rho_3^0)$); then we have

$$(6)\qquad 0 \longrightarrow F' \longrightarrow \bar{F} \longrightarrow F'' \longrightarrow 0$$

and now we have to compute $H^0, H^1(F'), H^0, H^1(F'')$.

(a) F':

$$0 \longrightarrow \rho_1^{-2} \longrightarrow F' \longrightarrow \rho_2^{-1} \longrightarrow 0$$

$$\Rightarrow 0 \longrightarrow 0 \longrightarrow H^0(F') \longrightarrow 0 \longrightarrow \mathbb{C} \longrightarrow H^1(F') \longrightarrow 0 \longrightarrow H^2(F') \longrightarrow 0$$

$$\Rightarrow H^0(F') = 0,\ H^1(F') \simeq \mathbb{C}, H^2(F') = 0$$

(b) F'':

$$0 \longrightarrow \rho_4^{-2} \longrightarrow F'' \longrightarrow \rho_3^0 \longrightarrow 0$$

$$\Rightarrow 0 \longrightarrow 0 \longrightarrow H^0(F'') \xrightarrow{\beta} \Gamma^{3,3}V \xrightarrow{\gamma} \Gamma^{3,3}V \xrightarrow{\alpha} H^1(F'') \longrightarrow 0 \tag{7}$$

By Schur's lemma, α is $\equiv 0$, or it is an isomorphism.

If α is an isomorphism, then we have $\gamma \equiv 0$ and hence β is surjective; therefore β is an isomorphism, i.e. $H^0(F'') \simeq \Gamma^{3,3}V$.

Substituting this in the cohomology sequence associated to (6), we get

$$0 \longrightarrow 0 \longrightarrow H^0(\bar{F}) \xrightarrow{\delta} \Gamma^{3,3}V \xrightarrow{\sigma} \mathbb{C} \xrightarrow{\tau} H^1(F) \xrightarrow[\varepsilon]{} \Gamma^{3,3}V \longrightarrow 0$$

By the Schur's lemma ε must be an isomorphism; hence $H^1(\bar{F}) \simeq \Gamma^{3,3}V$ and then $\tau \equiv 0$: this is a contradiction, because $\sigma \equiv 0$, by the same lemma.

Hence α isn't an isomorphism, but $\alpha \equiv 0$.

This implies $H^1(F'') = 0$, because of the surjectivity of α, and thus sequence (7) becomes

$$0 \longrightarrow 0 \longrightarrow H^0(F'') \xrightarrow{\beta} \Gamma^{3,3}V \xrightarrow{\gamma} \Gamma^{3,3}V \xrightarrow{\alpha} 0$$

$\Rightarrow \gamma$ is an isomorphism, $\beta \equiv 0$ and $H^0(F'') = 0$.

Coming back now to the cohomology sequence of (6),

$$0 \rightarrow H^0(\bar{F}) \rightarrow 0 \rightarrow \mathbb{C} \rightarrow H^1(F) \rightarrow 0,$$

we finally obtain $H^1(\bar{F}) \simeq \mathbb{C}$ and $H^0(\bar{F}) = 0$.

Hence, with this results the cohomology sequence of (5) is

$$0 \rightarrow H^0(E) \xrightarrow{\mu} \mathbb{C} \xrightarrow{\theta} \mathbb{C} \xrightarrow{\upsilon} H^1(E) \longrightarrow 0$$

Now, since $\theta \neq 0, \theta$ must be an isomorphism. $\Rightarrow \upsilon \equiv 0, \mu \equiv 0$ and $H^0(E) = 0$.

We will use this information later, to compute $H^0(E \otimes E^*)$.

Now we examine the bundle $\bar{F} \leftrightarrow (\rho_1^{-2}, \rho_2^{-1}, \rho_4^{-2}, \rho_3^0)$, induced by matrix (4): with the same method exposed before to search for the subrepresentations of ψ, we can find all subbundles of $\bar{F}$ given by sub-representations, and verify that they all have slope $< \mu(\bar{F})$:

a) *Index* 3: (1) $G_1 \leftrightarrow (\rho_1^{-2}, \rho_2^{-1}, \rho_4^{-2})$, which is decomposable.

$$\Rightarrow \mu(G_1) = \frac{-3}{10} < \frac{3}{14} = \mu(\bar{F})$$

b) *Index* 2: (1) $G_2 \leftrightarrow (\rho_1^{-2}, \rho_2^{-1})$. Then

$$\mu(G_2) = -\frac{3}{5} < \frac{3}{14} = \mu(\bar{F});$$

(2) $G_3 \leftrightarrow (\rho_1^{-2}, \rho_4^{-2})$. Then

$$\mu(G_3) = -\frac{3}{7} < \frac{3}{14} = \mu(\bar{F});$$

c) *Index* 1: (1) $G_4 = E_{\rho_1^{-2}} = Q(-2)$. Therefore

$$\mu(G_4) = -\frac{3}{2} < \frac{3}{14} = \mu(\bar{F});$$

(2) $G_5 = E_{\rho_4^{-2}} = S^4Q(-2)$. Then

$$\mu(G_5) = 0 < \frac{3}{14} = \mu(\bar{F}).$$

This computation tells us by corollary 3 that $\bar{F}$ is semistable. Now, just by using the main theorem we can conclude the stability ($\Rightarrow$ simplicity) of $\bar{F}$: by contradiction, if $\bar{F}$ isn't stable, then by the main theorem in the expression $\bar{F} = W \otimes F_0$, F_0 is a proper homogeneous subbundle of $\bar{F}$, because F_0 is stable, while $\bar{F}$ is not by assumption.

Therefore $rk(\bar{F}) = 14 = rkW \cdot rk(F_0)$ and we have only three possibilities:

(1) $rk(W) = 2$ and $rk(F_0) = 7$; but so

$$\mu(F_0) = \mu(\bar{F}) = \frac{3}{14} \Leftrightarrow \mathbb{Z} \ni c_1(F_0) = \frac{3}{2}$$

Thus this possibility leads to a contradiction;

(2) $rk(W) = 7$ and $rk(F_0) = 2$; in this case

$$\mu(F_0) = \mu(\bar{F}) = \frac{3}{14} \Leftrightarrow \mathbb{Z} \ni c_1(F_0) = \frac{3}{7}$$

and, as above, this case isn't possible;

(3) $rk(W) = 14$ and $rk(F_0) = 1$; but

$$\mu(F_0) = \mu(\bar{F}) = \frac{3}{14} \Leftrightarrow \mathbb{Z} \ni c_1(F_0) = \frac{3}{14}$$

But this is another contradiction, and hence $\bar{F}$ is stable.

We are now finally ready to compute $H^0(End\,(E))$.

Starting from (5), and tensoring it with E^*, we get

$$0 \longrightarrow \bar{F} \otimes E^* \longrightarrow End\,(E) \longrightarrow E^* \longrightarrow 0, \tag{8}$$

Thus we need to study (i) $\bar{F} \otimes E^*$ and (ii) E^*:

(i) If we take the dual of (5) and afterwards we tensor by $\bar{F}$, we obtain

$$0 \longrightarrow \bar{F} \longrightarrow E^* \otimes \bar{F} \longrightarrow End\,(\bar{F}) \longrightarrow 0$$

and its cohomology sequence

$$0 \longrightarrow 0 \longrightarrow H^0(\bar{F} \otimes E^*) \xrightarrow{\sigma} \mathbb{C} \xrightarrow{\tau} \mathbb{C} \longrightarrow \dots \tag{9}$$

where we used the simplicity of $\bar{F}, H^0(\bar{F}) = 0$ and $H^1(\bar{F}) \simeq \mathbb{C}$. But τ is an isomorphism; $\Rightarrow \sigma \equiv 0$ and, by its injectivity, $H^0(\bar{F} \otimes E^*) = 0$.

(ii) The dual of (5) is

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathrm{E}^* \longrightarrow \mathrm{F}^* \longrightarrow 0 \tag{10}$$

Hence, to estimate $H^0, H^1(E^*)$, we need some information about the first cohomology groups of $\bar{F}^*$.

With the same techniques used for $H^0, H^1(\bar{F})$, we can compute $H^0(\bar{F}^*) = 0$.

Finally, coming back to the cohomology sequence of (10), we have

$$0 \to \mathbb{C} \to H^0(E^*) \to 0$$

$$\Rightarrow \quad H^0(E^*) \simeq \mathbb{C}.$$

$\Rightarrow$ from (8) we see that $H^0(End(E)) \simeq \mathbb{C}$, i.e. E is simple.

As conclusion to the article, we report some lists, in which we display the results we obtained in all cases of homogeneous vector bundles on $\mathbb{CP}^2$, of $rk \leq 15$:

Homogeneous bundles of index 3

μ-filtration	Stable	Simple
$(\rho^l_m, \rho^{l+1}_{m+1}, \rho^{l+2}_{m+2})$ for $m > 0$	yes	yes
$(\rho^{l-2}_{m+2}, \rho^l_{m+1}, \rho^{l+2}_m)$ for $m > 0$	yes	yes
$(\rho^l_m \oplus \rho^{l-1}_{m+2}, \rho^{l+1}_{m+1})$	no, but it is semistable $\Leftarrow$	no
$(\rho^l_{m+1}, \rho^{l+2}_m \oplus \rho^{l+1}_{m+2})$	no, but it is semistable $\Leftarrow$	no

for $l \in \mathbb{Z}$ and $m \in \mathbb{N}$.

Homogeneous bundles of index 4 and rank 10

μ -filtration	Stable	Simple
$(\rho^{-1}_0, \rho^0_1 \oplus \rho^{-1}_3, \rho^1_2)$	yes	yes
$(\rho^0_0 \oplus \rho^{-1}_2, \rho^1_1 \oplus \rho^0_3)$	yes	yes
$(\rho^{-1}_1, \rho^1_0 \oplus \rho^0_2, \rho^1_3)$	no	no

Homogeneous bundles of index 5 and rank 15

μ-filtration	Stable	Simple
$(\rho_0^{-2}, \rho_1^{-1}, \rho_2^0 \oplus \rho_4^{-1}, \rho_3^1)$	no	yes
$(\rho_0^{-1}, \rho_1^0 \oplus \rho_3^{-1}, \rho_2^1 \oplus \rho_4^0)$	no	?
$(\rho_0^0 \oplus \rho_2^{-1}, \rho_1^1 \oplus \rho_3^0, \rho_4^1)$	no	no
$(\rho_1^{-2}, \rho_0^0 \oplus \rho_2^{-1}, \rho_3^0, \rho_4^1)$	no	no
$(\rho_1^{-2}, \rho_0^0 \oplus \rho_2^{-1} \oplus \rho_4^{-2}, \rho_3^0)$	no	yes
$(\rho_0^0 \oplus \rho_4^{-2}, \rho_1^1 \oplus \rho_3^0, \rho_2^2)$	yes	yes
$(\rho_0^1 \oplus \rho_2^0 \oplus \rho_4^{-1}, \rho_1^2 \oplus \rho_3^1)$	yes	yes

REFERENCES

[1] V. Ancona, *Fibrati vettoriali su varietà razionali omogenee*, Boll. UMI, **7** (1988), 299-317.

[2] M. Atiyah, *On the Krull-Schmidt theorem with application to sheaves*, Bull. Soc. Math. France, 84 (1956), 306-317.

[3] S. Faini, *Stabilità e semplicità dei fibrati omogenei su* $\mathbb{CP}^2$, tesi di laurea, Firenze, 2002.

[4] L. Migliorini, *Stability of homogeneous vector bundles*, Boll. UMI, Ser. VII, B 10, nr. 4 (1996), 963-990.

[5] C. Okonek - M. Schneider - H. Spindler, *Vector bundles on complex projective spaces*, Birkhäuser, Boston, 1980.

[6] G. Ottaviani, *Rational homogeneous varieties*, notes for the SMI-course in Cortona, 1995 (available at http://www.math.unifi.it/ ~ottavian).

[7] R. Paoletti, *Stability of a class of homogeneous vector bundles on* $\mathbb{CP}^n$, Boll. UMI, Ser. VII, A 9, nr. 2 (1995), 329-343.

[8] S. Ramanan, *Holomorphic vector bundles on homogeneous spaces*, Topology, Vol. 5, pp. 159-177, Pergamon Press, 1966.

[9] R. Rohmfeld, *Stability of homogeneous vector bundles on* $\mathbb{CP}^n$, Geometriae Dedicata, 38 (1991), 159-166.

[10] R. Rohmfeld, *Stabile homogenen Vektorbündel über der komplexen projectiven Ebene*, PHD-thesis, Erlangen, 1989.

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