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María F. Natale, Domingo A. Tarzia

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Explicit Solutions for a One-phase Stefan Problem with Temperature-dependent Thermal Conductivity.

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Sunto. – *Si studia un problema di Stefan a una fase per un materiale semi-infinito con un coefficiente di conduttività termica dipendente dalla temperatura e con una condizione di temperatura costante o un flusso di calore del tipo* $-q_0/\sqrt{t}$ $(q_0 > 0)$ *sulla faccia fissa* $x = 0$. *Si ottengono, in entrambi i casi, condizioni sufficienti per i dati in modo da avere una rappresentazione parametrica della soluzione di tipo similarità per* $t \geqslant t_0 > 0$ *con* t_0 *un tempo positivo arbitrario. Queste soluzioni esplicite sono ottenute attraverso l'unica soluzione di una equazione integrale dove il tempo è un parametro.*

Summary. – *We study a one-phase Stefan problem for a semi-infinite material with temperature-dependent thermal conductivity with a constant temperature or a heat flux condition of the type* $-q_0/\sqrt{t}$ $(q_0 > 0)$ *at the fixed face* $x = 0$. *We obtain in both cases sufficient conditions for data in order to have a parametric representation of the solution of the similarity type for* $t \geqslant t_0 > 0$ *with* t_0 *an arbitrary positive time. These explicit solutions are obtained through the unique solution of an integral equation with the time as a parameter*

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I. – Introduction.

We will consider a phase-change problem (Stefan problem) for a non-linear heat conduction equation for a semi-infinite region $x > 0$ with a nonlinear thermal conductivity $k(\theta)$ given by

$$k(\theta) = \frac{\varrho c}{(a + b\theta)^2} \tag{1}$$

and phase change temperature θ_f. This kind of thermal conductivity or diffusion coefficient was considered in [4, 5, 7, 8, 16, 18, 21, 24, 26, 29, 32]. The modeling of this type of systems is a great mathematical and industrial significance problem. Phase-change problems appear frequently in industrial processes and other problems of technological interest [1, 2, 9, 10, 11, 13, 14, 15, 17, 19, 20]. A recent large bibliography on the subject was given recently in [31].

The mathematical formulation of our free boundary (fusion process) problem consists in determining the evolution of the moving phase separation $x = s(t)$ and the temperature distribution $\theta = \theta(x, t)$ satisfying the conditions

$$\varrho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta) \frac{\partial \theta}{\partial x}\right), \qquad 0 < x < s(t), \; t > 0 \tag{2}$$

$$k(\theta(0, t)) \frac{\partial \theta}{\partial x}(0, t) = -\frac{q_0}{\sqrt{t}}, \qquad q_0 > 0, \; t > 0 \tag{3}$$

$$k(\theta(s(t), t)) \frac{\partial \theta}{\partial x}(s(t), t) = -\varrho l \, \dot{s}(t), \; t > 0 \tag{4}$$

$$\theta(s(t), t) = \theta_f, \; t > 0 \tag{5}$$

$$s(0) = 0 \tag{6}$$

where $a + b\theta_f > 0$, in order to guarantee that k is well defined. Here $-q_0/\sqrt{t}$ denotes the prescribed flux on the boundary $x = 0$ which is of the type imposed in [30]; a constant temperature boundary condition on $x = 0$ of the type (52) will be considered later. In [30] it was proven that the heat flux condition (3) on the fixed face $x = 0$ is equivalent to the constant temperature boundary condition (52) for the two phase Stefan problem for a semi-infinite material with constant thermal coefficient in both phases. This kind of heat flux condition (3) was also considered in numerous papers, e.g. [3, 12, 25]. Other problems in this subject are [6, 22, 26, 27].

The free boundary problem (2)-(6) with $k(\theta)$ defined by (1) is the particular case of one studied in [23, 28] by taking the parameter $d = 0$ for the following equation

$$\varrho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta) \frac{\partial \theta}{\partial x}\right) - v(\theta) \frac{\partial \theta}{\partial x}, \qquad 0 < x < s(t), \qquad t > 0 \tag{7}$$

where the thermal conductivity $k(\theta)$ and the velocity term $v(\theta)$ are given by (1) and

$$v(\theta) = \varrho c \frac{d}{2(a + b\theta)^2} \tag{8}$$

respectively, and c, ϱ and l are the specific heat, the density and the latent heat of fusion of the medium respectively, all of them are assumed to be constant with positive parameters a, b and d.

In those papers temperature and flux type conditions on the fixed face $x = 0$ were studied. Furthermore, necessary and sufficient conditions for the existence of an explicit solution was found in [23]. Here we study the case without the velocity term, i.e. $d = 0$ in the differential equation (7) which cannot be obtained from what it was previously done in [23, 28] for the case $d \neq 0$. In those papers it was defined the transformation

$$y = \frac{2}{d}\left[(1 + dx)^{1/2} - 1\right] \tag{9}$$

which is the identity if we take $d \to 0$ since

$$\lim_{d \to 0} \frac{2}{d}\left[(1 + dx)^{1/2} - 1\right] = x\,, \qquad \forall x > 0\,.$$

Then, the case $d = 0$ must be solved by using other techniques which will be the goal of this study.

In Section II we prove the existence and uniqueness of an explicit solution of the similarity type of the free boundary problem (2)-(6) for $t \geqslant t_0 > 0$ with t_0 an arbitrary positive time when data satisfy condition $a + b\theta_f \geqslant bl/c$. The solution is explicitly given by (41)-(47), and by (50)-(86) for the cases $a + b\theta_f > bl/c$ and $a + b\theta_f = bl/c$ respectively. The explicit solution for the two cases is obtained through the unique solution of an integral equation in which time is a parameter.

Besides, there does not exist any solution of the similarity type to the free boundary problem (2)-(6) for the case $a + b\theta_f < bl/c$.

In Section III we replace the flux condition (3) for a constant temperature boundary condition on the fixed face $x = 0$, given by (52). We prove existence and uniqueness of an explicit solution of the similarity type of the problem (2), (4)-(6) and (52) for $t \geqslant t_0 > 0$ with t_0 an arbitrary positive time when data verifies condition $a + b\theta_f \geqslant bl/c$. The solution is explicitly given by (76)-(82), and by (84)-(86) for the cases $a + b\theta_f > bl/c$ and $a + b\theta_f = bl/c$ respectively. The explicit solution for the two cases is also obtained through the unique solution of an integral equation in which the time is a parameter.

II. – Existence and uniqueness of solution of the free boundary problem with flux boundary condition on the fixed face.

We consider the free boundary problem (2)-(6) with the parameters a, b and the coefficients l, c satisfy the following condition

$$a + b\theta_f > \frac{bl}{c}\,. \tag{10}$$

If we define

$$\Theta = \frac{1}{a + b\theta}\,, \tag{11}$$

the problem (2)-(6) becomes

$$\frac{\partial \Theta}{\partial t} = \Theta^2 \frac{\partial^2 \Theta}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0 \tag{12}$$

$$\frac{\partial \Theta}{\partial x}(0, t) = \frac{w}{\sqrt{t}}, \quad t > 0 \tag{13}$$

$$\frac{\partial \Theta}{\partial x}(s(t), t) = \frac{bl}{c} \dot{s}(t), \quad t > 0 \tag{14}$$

$$\Theta(s(t), t) = \frac{1}{a + b\theta_f}, \quad t > 0 \tag{15}$$

$$s(0) = 0 \tag{16}$$

where w is a constant defined by

$$w = \frac{bq_0}{\varrho c}. \tag{17}$$

Let us perform the transformation

$$\chi(x, t) = \int_0^x \frac{d\eta}{\Theta(\eta, t)} \qquad \Psi(\chi, t) = \Theta(x, t) \tag{18}$$

and

$$S(t) = \chi(s(t), t). \tag{19}$$

The problem (12)-(16) becomes

$$\frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial \chi^2} - \frac{w}{\sqrt{t}} \frac{\partial \Psi}{\partial \chi}, \quad 0 < \chi < S(t), \quad t > 0 \tag{20}$$

$$\frac{\partial \Psi}{\partial \chi}(0, t) = \frac{w}{\sqrt{t}} \Psi(0, t), \quad t > 0 \tag{21}$$

$$\frac{\partial \Psi}{\partial \chi}(S(t), t) = \frac{1}{(a + b\theta_f)\left(\frac{c}{bl}(a + b\theta_f) - 1\right)} \left(\dot{S}(t) - \frac{w}{\sqrt{t}}\right), \quad t > 0 \tag{22}$$

$$\Psi(S(t), t) = \frac{1}{a + b\theta_f}, \quad t > 0 \tag{23}$$

$$S(0) = 0 \tag{24}$$

where

(25) $$\dot{S}(t) = \left(a + b\theta_f - \frac{bl}{c}\right) \dot{s}(t) + \frac{w}{\sqrt{t}} .$$

If we introduce the similarity variable

(26) $$\xi = \frac{\chi}{2\sqrt{t}} ,$$

and the solution is sought of type

(27) $$\Psi(\chi, t) = \varphi(\xi) = \varphi\left(\frac{\chi}{2\sqrt{t}}\right)$$

then the free boundary $S(t)$ of the problem (20)-(24) must be of the type

(28) $$S(t) = 2\Lambda_1\sqrt{t}, \quad t > 0$$

with $\Lambda_1 > 0$ an unknown coefficient to be determined and the problem (20)-(24) yields

(29) $$\varphi''(\xi) + 2\varphi'(\xi)(\xi - w) = 0, \qquad 0 < \xi < \Lambda_1$$

(30) $$\varphi'(0) = 2w\varphi(0)$$

(31) $$\varphi(\Lambda_1) = \frac{1}{a + b\theta_f}$$

(32) $$\varphi'(\Lambda_1) = \frac{2}{(a + b\theta_f)\left(\frac{c}{bl}(a + b\theta_f) - 1\right)}(\Lambda_1 - w) .$$

Taking into account the expression (25) we have

(33) $$s(t) = 2\lambda_1\sqrt{t}$$

with

(34) $$\lambda_1 = \frac{\Lambda_1 - w}{a + b\theta_f - \frac{bl}{c}} .$$

If we integrate (29) we obtain

(35) $$\varphi(\xi) = D_1 \operatorname{erf}(\xi - w) + C_1$$

where D_1 and C_1 are two constants of integration which can be determined from (30) and (31)

$$D_1 = \frac{\sqrt{\pi}w \exp(w^2)}{(a+b\theta_f)[1+\sqrt{\pi}w\exp(w^2)(\operatorname{erf}(\Lambda_1 - w)+\operatorname{erf}(w))]} \tag{36}$$

$$C_1 = \frac{1+\sqrt{\pi}w \exp(w^2)\operatorname{erf}(w)}{(a+b\theta_f)(1+\sqrt{\pi}w\exp(w^2)(\operatorname{erf}(\Lambda_1 - w)+\operatorname{erf} w))} \tag{37}$$

Now, we have to consider here the condition (32) which implies that Λ_1 must be the solution of the following equation

$$W_1(x) = W_2(x), \qquad x > w \tag{38}$$

where

$$W_1(x) = \frac{w\exp(w^2)\exp[-(x-w)^2]}{1+w\exp(w^2)\sqrt{\pi}(\operatorname{erf}(x-w)+\operatorname{erf}(w))} \tag{39}$$

and

$$W_2(x) = \frac{bl}{c(a+b\theta_f)-bl}(x-w). \tag{40}$$

It is easy to prove that $W_1(0) = w > 0$, $W_1(+\infty) = 0$, and W_1 is a decreasing function, and $W_2(w) = 0$, $W_2(+\infty) = +\infty$ and W_2 is an increasing function because condition (10). So, there exists a unique solution Λ_1 of the equation (38) and then we have the following theorem.

THEOREM 1. – Let us consider the hypothesis (10).

(i) If (Θ, s) is a solution of the free boundary problem (12)-(16) then $\Theta = \Theta(x,t)$ is a solution, in variable x, of the integral equation:

$$\Theta(x,t) = C_1 + D_1 \operatorname{erf}\left(\frac{\int_0^x \frac{d\eta}{\Theta(\eta,t)}}{2\sqrt{t}} - w\right), \qquad 0 \leqslant x \leqslant s(t), \tag{41}$$

where $t > 0$ is a parameter and w, D_1 and C_1 are defined by (17), (36) and (37) respectively, and $s(t)$ is given by (33) and Λ_1 is the unique solution of the Eq. (38). Moreover, function $Y(x,t)$ defined by

$$Y(x,t) = \frac{1}{2\sqrt{t}}\int_0^x \frac{d\eta}{\Theta(\eta,t)} - w, \qquad 0 \leqslant x \leqslant s(t), \quad t > 0 \tag{42}$$

satisfies the conditions

$$\frac{\partial Y}{\partial x}(x,\,t)=\frac{1}{2\sqrt{t}}\,\frac{1}{\Theta(x,\,t)}\,,\qquad 0<x<s(t),\qquad t>0 \tag{43}$$

$$Y(0,\,t)=-w,\qquad t>0 \tag{44}$$

$$\frac{\partial Y}{\partial t}(x,t)=-\frac{1}{2t}\left(Y(x,t)+\frac{D_1}{\sqrt{\pi}}\,\frac{\exp(-Y^2(x,t))}{\Theta(x,t)}\right),\quad 0<x<s(t),\ t>0 \tag{45}$$

$$Y(s(t),\,t)=\Lambda_1-w,\qquad t>0 \tag{46}$$

(ii) Conversely, if Θ is a solution of the integral equation (41) with s given by (33) and function Y, defined by (42) satisfies the conditions (43)-(46), and w, D_1 and C_1 are defined by (17), (36) and (37) respectively, and Λ_1 is the unique solution of the Eq. (38) then $(\Theta,\,s)$ is a solution of the free boundary problem (12)-(16).

(iii) The integral equation (41) has a unique solution for $t\geqslant t_0>0$ with t_0 is an arbitrary positive time.

(iv) The free boundary problem (2)-(6) satisfying the hypothesis (10) has a unique similarity type solution $(\theta,\,s)$ for $t\geqslant t_0>0$ (with t_0 an arbitrary positive time) which is given by

$$\theta(x,\,t)=\frac{1}{b}\left[\frac{1}{\Theta(x,\,t)}-a\right],\qquad 0<x<s(t),\quad t\geqslant t_0>0 \tag{47}$$

$$s(t)=\frac{2(\Lambda_1-w)}{a+b\theta_f-\dfrac{bl}{c}}\sqrt{t},\qquad t\geqslant t_0>0 \tag{48}$$

where Θ is the unique solution of the integral Eq. (41) where Λ_1 is the unique solution of the Eq. (38), and w, D_1 and C_1 are defined by (17), (36) and (37) respectively.

PROOF.

(i) From the previous computation we have

$$\Theta(x,\,t)=\varphi(\xi)=C_1+D_1\,\mathrm{erf}\,(\xi-w)=C_1+D_1\,\mathrm{erf}\left(\frac{\displaystyle\int_0^x\frac{d\eta}{\Theta(\eta,\,t)}}{2\sqrt{t}}-w\right)$$

that is Θ is a solution of the integral equation (41). Function Y, defined by (42), satisfies the conditions (43), (44) by elementary computations, and

$$\frac{\partial Y}{\partial t}(x, t) = -\frac{1}{4t\sqrt{t}}\int_0^x \frac{d\eta}{\Theta(\eta, t)} - \frac{1}{2\sqrt{t}}\int_0^x \Theta_{xx}(\eta, t)\, d\eta =$$

$$-\frac{1}{2t}(Y(x, t) + w) - \frac{1}{2\sqrt{t}}(\Theta_x(x, t) - \Theta_x(0, t)) =$$

$$-\frac{1}{2\sqrt{t}}\left(\frac{Y(x, t)}{\sqrt{t}} + \Theta_x(x, t)\right) = -\frac{1}{2\sqrt{t}}\left(\frac{Y(x, t)}{\sqrt{t}} + \frac{D_1}{\sqrt{\pi t}}\frac{\exp(-Y^2(x, t))}{\Theta(x, t)}\right)$$

that is (45). Finally we get

$$Y(s(t), t) = \frac{1}{2\sqrt{t}}\int_0^{s(t)} \frac{d\eta}{\Theta(\eta, t)} - w = \frac{\chi(s(t), t)}{2\sqrt{t}} - w = \frac{S(t)}{2\sqrt{t}} - w = \Lambda_1 - w$$

that is (46).

(ii) In order to proof that (Θ, s) is a solution of the free boundary problem (12)-(16) we get:

a)

$$\Theta_{xx}(x, t) = \left(\frac{D_1}{\sqrt{\pi t}}\frac{\exp(-Y^2(x, t))}{\Theta(x, t)}\right)_x =$$

$$-\frac{D_1}{\sqrt{\pi t}}\frac{\exp(-Y^2(x, t))}{\Theta^2(x, t)}\left(Y(x, t) + \frac{D_1}{\sqrt{\pi}}\frac{\exp(-Y^2(x, t))}{\Theta(x, t)}\right);$$

b)

$$\Theta_t(x, t) = \frac{2D_1}{\sqrt{\pi}}\exp(-Y^2(x, t))\, Y_t(x, t) =$$

$$= -(D_1/\sqrt{\pi}t)\exp(-Y^2(x, t))\left(Y(x, t) + \frac{D_1}{\sqrt{\pi}}\frac{\exp(-Y^2(x, t))}{\Theta(x, t)}\right)$$

that is Eq. (12);

c)

$$\Theta(0, t) = C_1 - D_1\operatorname{erf}(w) = \frac{D_1}{\sqrt{\pi}w\exp(w^2)};$$

d)

$$\Theta_x(0, t) = \frac{D_1}{\sqrt{\pi t}}\frac{\exp(-Y^2(0, t))}{\Theta(0, t)} = \frac{w}{\sqrt{t}}, \quad \text{that is (13);}$$

e)

$$\Theta(s(t), t) = C_1 + D_1 \operatorname{erf}(\Lambda_1 - w) = \frac{1}{a + b\theta_f}, \quad \text{that is (15);}$$

f)

$$\Theta_x(s(t), t) = \frac{D_1}{\sqrt{\pi t}} \frac{\exp(-Y^2(s(t), t))}{\Theta(s(t), t)} = \frac{(a + b\theta_f) D_1}{\sqrt{\pi t}} \exp(-(\Lambda_1 - w)^2) =$$

$$\frac{1}{\sqrt{t}} W_1(\Lambda_1) = \frac{1}{\sqrt{t}} W_2(\Lambda_1) =$$

$$\frac{1}{\sqrt{t}} \frac{bl}{c(a + b\theta_f) - bl} (\Lambda_1 - w) = \frac{bl\lambda_1}{c\sqrt{t}} = \frac{bl}{c} \dot{s}(t), \quad \text{that is (14)}$$

(iii) Now in order to complete the proof, we just have to proof the existence of a solution of the integral equation (41). If we define $Y(x, t)$ by (42) then, Eq. (41) is equivalent to the following Cauchy differential problem

$$\text{(49)} \qquad \begin{cases} \dfrac{\partial Y}{\partial x}(x, t) = \dfrac{1}{2\sqrt{t}} \dfrac{1}{(C_1 + D_1 \operatorname{erf}(Y(x, t)))} \equiv G_1(x, t, Y(x, t)), \\ 0 < x < s(t), \quad t > 0, \quad Y(0, t) = -w, \end{cases}$$

with a positive parameter $t > 0$. We have $\left| \dfrac{\partial G_1}{\partial Y} \right| \leqslant \dfrac{D_1}{C_1^2 \sqrt{\pi t}}$ which is bounded for all $t \geqslant t_0 > 0$, $0 \leqslant x \leqslant s(t)$, for an arbitrary positive time t_0. Then, problem (49) (i.e. the integral Eq. (41)) has a unique solution for $t \geqslant t_0 > 0$, for an arbitrary positive time t_0.

(iv) It follows from elementary but tedious computation. ■

REMARK 1. – $Y(x, t)$ dos not possess a limit at $(0, 0)$ because $Y(0, t) = -w = -\dfrac{bq_0}{\varrho c} < 0$ for $t > 0$ and $\lim_{t \mapsto 0} Y(s(t), t) = \Lambda_1 - w > 0$ for all $t > 0$.

If Θ is the solution of the integral equation (41) then Θ is strictly monotone in variable x. We obtain that $\theta(x, t) = (1/\Theta(x, t) - a)/b$ does not have limit when $(x, t) \to (0, 0)$ but $\theta(x, t)$ is bounded in a neighborhood of $(0, 0)$ checking that

$$\theta_f = \lim_{(\eta, \tau) \mapsto (0, 0)} \inf \theta(\eta, \tau) \leqslant \theta(x, t) \leqslant \lim_{(\eta, \tau) \mapsto (0, 0)} \sup \theta(\eta, \tau) =$$

$$= \theta_f + \frac{a + b\theta_f}{b} \sqrt{\pi} w \exp(w^2)(\operatorname{erf}(w) + \operatorname{erf}(\Lambda_1 - w)).$$

When the hypothesis (10) is not satisfied we can follow an analogous method to the one described before in order to obtain the following result.

THEOREM 2. – (i) The result of the Theorem 1 is also true if we replace the condition (10) by $a + b\theta_f = \dfrac{bl}{c}$. Furthermore, in this case, the solution of the free boundary problem (2)-(6) is given by

$$\theta(x, t) = \frac{1}{b}\left[\frac{1}{\Theta(x, t)} - a\right], \qquad s(t) = 2D_0\sqrt{\frac{t}{\pi}} \tag{50}$$

where Θ is the unique solution of the following integral equation

$$\Theta(x, t) = D_0 \operatorname{erf}\left(\frac{\displaystyle\int_0^x \frac{d\eta}{\Theta(\eta, t)}}{2\sqrt{t}} - w\right) + \frac{c}{bl}, \qquad 0 \leqslant x \leqslant s(t), \tag{51}$$

with

$$D_0 = \frac{q_0\sqrt{\pi}\exp(w^2)}{\varrho l(1 + \sqrt{\pi}w\exp(w^2)\operatorname{erf}(w))}$$

for $t \geqslant t_0 > 0$, $0 \leqslant x \leqslant s(t)$ for any arbitrary positive time t_0 and w defined by (17).

(ii) There does not exist any solution to the free boundary problem (2)-(6) for the case $a + b\theta_f < \dfrac{bl}{c}$.

PROOF. – (i) It follows by using a similar method to the one developed for Theorem 1.

(ii) The non existence of any solution of the similarity type is due to the non existence of real solution of the Eq. (38).

III. – Existence and uniqueness of solution of the free boundary problem with temperature boundary condition on the fixed face.

In this section we consider the free boundary problem given by (2), (4)-(6) and the temperature boundary condition

$$\theta(0, t) = \theta_0, \quad t > 0 \qquad (\theta_0 > \theta_f) \tag{52}$$

instead of the heat flux boundary condition (3) on the fixed face $x = 0$ where the nonlinear thermal conductivity is given by (1). We also suppose that condition (10) is verified. If we define Θ as in (11), the free boundary problem (2), (4)-(6) and (52) transforms to (12), (14)-(16) and

$$\Theta(0,t)=\frac{1}{a+b\theta_0}. \tag{53}$$

We also define (18) and (19) and we obtain (23), (24) and

$$\frac{\partial\Psi}{\partial t}(x,t)=\frac{\partial^2\Psi}{\partial\chi^2}(x,t)-(a+b\theta_0)\frac{\partial\Psi}{\partial\chi}(x,t)\frac{\partial\Psi}{\partial\chi}(0,t),\ 0<\chi<S(t),\ t>0 \tag{54}$$

$$\Psi(0,t)=\frac{1}{a+b\theta_0} \tag{55}$$

$$\frac{\partial\Psi}{\partial\chi}(S(t),t)=\frac{1}{(a+b\theta_f)\left[\dfrac{c}{bl}(a+b\theta_f)-1\right]}\left(\dot{S}(t)-(a+b\theta_0)\frac{\partial\Psi}{\partial\chi}(0,t)\right) \tag{56}$$

and

$$\dot{S}(t)=\dot{s}(t)(a+b\theta_f-\frac{bl}{c})+(a+b\theta_0)\frac{\partial\Psi}{\partial\chi}(0,t). \tag{57}$$

Now, introducing (26) and (27) we get (28) and (31) with coefficient Λ_2 instead of Λ_1 and

$$\varphi''(\xi)+2\varphi'(\xi)\left(\xi-\frac{\varphi'(0)}{2}(a+b\theta_0)\right)=0,\quad 0<\xi<\Lambda_2 \tag{58}$$

$$\varphi(0)=\frac{1}{a+b\theta_0} \tag{59}$$

$$\varphi'(\Lambda)=\frac{2}{(a+b\theta_f)\left[\dfrac{c}{bl}(a+b\theta_f)-1\right]}\left(\Lambda_2-\frac{\varphi'(0)}{2}(a+b\theta_0)\right).$$

Taking into account (28) and (57) we have

$$s(t)=2\lambda_2\sqrt{t} \tag{61}$$

with

$$\lambda_2=\frac{\Lambda_2-r}{a+b\theta_f-\dfrac{bl}{c}} \tag{62}$$

where

(63) $$r = \frac{\varphi'(0)(a+b\theta_0)}{2}\,.$$

If we integrate (58) we obtain

(64) $$\varphi(\xi) = D_2 \operatorname{erf}(\xi - r) + C_2$$

where D_2 and C_2 are two constant of integration to be determined later. By considering (31) with Λ_2 instead of Λ_1 and (59) we get

(65) $$D_2 = \frac{b(\theta_0 - \theta_f)}{(a+b\theta_f)(a+b\theta_0)(\operatorname{erf}(r) + \operatorname{erf}(\Lambda_2 - r))}\,,$$

(66) $$C_2 = \frac{1}{a+b\theta_f}\left(1 - \frac{b(\theta_0-\theta_f)\operatorname{erf}(\Lambda_2 - r)}{(a+b\theta_0)(\operatorname{erf}(r)+\operatorname{erf}(\Lambda_2 - r))}\right).$$

Then function φ is given by the following expression

(67) $$\varphi(\xi) = \frac{1}{a+b\theta_f}\left[1 + \frac{b(\theta_0-\theta_f)(\operatorname{erf}(\xi - r) - \operatorname{erf}(\Lambda_2 - r))}{(a+b\theta_0)(\operatorname{erf}(r)+\operatorname{erf}(\Lambda_2 - r))}\right],$$

where Λ_2 and r are unknowns which must be obtained. From (60) and (63) we have

(68) $$\operatorname{erf}(\Lambda_2 - r) = F(r)$$

where

(69) $$F(x) = -\operatorname{erf}(x) + \frac{1}{\sqrt{\pi}}\,\frac{b(\theta_0-\theta_f)}{(a+b\theta_f)}\,\frac{\exp(-x^2)}{x}\,, \quad x > 0\,.$$

From [6] we known that $F(0^+) = +\infty$, $F(+\infty) = -1$ and F is a decreasing function, then there exist $r_0 = F^{-1}(0) > r_1 = F^{-1}(1) > 0$ such as $F(r) \in (-1, 1)$ for all $r > r_1$, that is

(70) $$\Lambda_2 - r = \operatorname{erf}^{-1}[\mathrm{F(r)}], \quad \text{with } r > r_1 = F^{-1}(1)\,.$$

Furthermore, taking into account (60) and (67) we obtain that

(71) $$\Lambda_2 - r = \left(\frac{c}{bl}(a+b\theta_f) - 1\right)\frac{(a+b\theta_f)\, r \exp(r^2)\exp(-(\Lambda_2 - r)^2)}{(a+b\theta_0)}\,,$$

where r must be a solution of the following equation

(72) $$W_3(x) = W_4(x)\,, \qquad x > r_1 = F^{-1}(1)$$

and functions W_3 and W_4 are defined by:

(73) $$W_3(x) = \operatorname{erf}^{-1}(F(x)) \exp((\operatorname{erf}^{-1}(F(x)))^2)$$

(74) $$W_4(x) = \left(\frac{c}{bl}(a+b\theta_f) - 1\right) \frac{(a+b\theta_f)\, x \exp(x^2)}{(a+b\theta_0)}.$$

It's easy to see that $W_3(r_1) = +\infty$, $W_3(+\infty) = -\infty$ and W_3 is a decreasing function for all $x > r_1$. Furthermore, from (10) we have $W_4(0^+) = 0$, $W_4(+\infty) = +\infty$ and W_4 is an increasing function. Therefore there exists a unique solution $r \in (r_1, r_0)$ of the equation (72) and then

(75) $$\Lambda_2 = r + \operatorname{erf}^{-1}\left[-\operatorname{erf}(r) + \frac{1}{\sqrt{\pi}} \frac{b(\theta_0 - \theta_f)}{(a+b\theta_f)} \frac{\exp(-r^2)}{r}\right] > r,$$

and we obtain the following theorem.

THEOREM 3. – Let us consider the hypothesis (10).

(i) If (Θ, s) is a solution of the free boundary problem (2), (4)-(6) and (52) then $\Theta = \Theta(x, t)$ is a solution, in variable x, of the integral equation:

(76) $$\begin{cases} \Theta(x,t) = \dfrac{1}{a+b\theta_f}\left[1 - \dfrac{b(\theta_0-\theta_f)}{(a+b\theta_0)} \dfrac{\left(\operatorname{erf}(\Lambda_2 - r) - \operatorname{erf}\left(\dfrac{\int_0^x \frac{d\eta}{\Theta(\eta,t)}}{2\sqrt{t}} - r\right)\right)}{\operatorname{erf}(r) + \operatorname{erf}(\Lambda_2 - r)}\right], \\ 0 \leqslant x \leqslant s(t), \end{cases}$$

where $t > 0$ is a parameter and $s(t)$ is given by (61), Λ_2 is the unique solution of the Eq. (75) and $r \in (F^{-1}(1), F^{-1}(0))$ is the unique solution of Eq. (72) where the function F is defined by (69), and function $Y(x, t)$ defined by

(77) $$Y(x,t) = \frac{1}{2\sqrt{t}} \int_0^x \frac{d\eta}{\Theta(\eta, t)} - r, \quad 0 \leqslant x \leqslant s(t), \quad t > 0$$

satisfies the conditions

(78) $$\frac{\partial Y}{\partial x}(x,t) = -\frac{1}{2\sqrt{t}} \frac{1}{\Theta(x,t)}; \quad 0 < x < s(t), \quad t > 0$$

(79) $$Y(0,t) = -r, \quad t > 0$$

(80) $$\frac{\partial Y}{\partial t}(x,t) = -\frac{1}{2t}\left(Y(x,t) + \frac{D_2}{\sqrt{\pi}} \frac{\exp(-Y^2(x,t))}{\Theta(x,t)}\right), \quad 0 \leqslant x \leqslant s(t), \quad t > 0$$

(81) $$Y(s(t), t) = \Lambda_2 - r, \quad t > 0$$

where D_2 is defined by (65).

(ii) Conversely, if Θ is a solution of the integral equation (76) with s given by (61) and function Y, defined by (77) satisfies the conditions (80)-(81), r is the unique solution of Eq. (72), D_2 are defined by (65), and Λ_2 is the unique solution of the Eq. (75) then (Θ, s) is a solution of the free boundary problem (2), (4)-(6) and (52).

(iii) The integral equation (76) has a unique solution for $t \geqslant t_0 > 0$ with t_0 is an arbitrary positive time.

(iv) The free boundary problem (2), (4)-(6) and (50) satisfying the hypothesis (10) has a unique similarity type solution (θ, s) for $t \geqslant t_0 > 0$ (with t_0 an arbitrary positive time) which is given by

$$\theta(x, t) = \frac{1}{b}\left[\frac{1}{\Theta(x, t)} - a\right], \qquad s(t) = \frac{2(\Lambda_2 - r)\sqrt{t}}{a + b\theta_f - \dfrac{bl}{c}} \tag{82}$$

where Θ is the unique solution of the integral Eq. (76) and Λ_2 is the unique solution of the Eq. (75) and r is the unique solution of Eq. (72).

PROOF. – (i) From the previous computation we have

$$\Theta(x, t) = \varphi(\xi) = C_2 + D_2 \operatorname{erf}(\xi - r) = C_2 + D_2 \operatorname{erf}\left(\int_0^x \frac{d\eta}{\Theta(\eta, t)} 2\sqrt{t} - r\right)$$

that is Θ is a solution of the integral equation (76). Function Y, defined by (77), satisfies the conditions (78), (79) by elementary computations, and

$$\frac{\partial Y}{\partial t}(x, t) = -\frac{1}{4t\sqrt{t}}\int_0^x \frac{d\eta}{\Theta(\eta, t)} - \frac{1}{2\sqrt{t}}\int_0^x \Theta_{xx}(\eta, t)\, d\eta =$$

$$-\frac{1}{2t}(Y(x, t) + r) - \frac{1}{2\sqrt{t}}(\Theta_x(x, t) - \Theta_x(0, t))$$

$$-\frac{1}{2\sqrt{t}}\left(\frac{Y(x, t)}{\sqrt{t}} + \Theta_x(x, t)\right) = -\frac{1}{2\sqrt{t}}\left(\frac{Y(x, t)}{\sqrt{t}} + \frac{D_2}{\sqrt{\pi t}}\frac{\exp(-Y^2(x, t))}{\Theta(x, t)}\right)$$

that is (80). Finally we get

$$Y(s(t), t) = \frac{1}{2\sqrt{t}}\int_0^{s(t)} \frac{d\eta}{\Theta(\eta, t)} - r = \frac{\chi(s(t), t)}{2\sqrt{t}} - r = \frac{S(t)}{2\sqrt{t}} - r = \Lambda_2 - r$$

that is (81).

(ii) In order to prove that (Θ, s) is a solution of the free boundary problem (10), (2), (4)-(6) and (52) we get:

a)

$$\Theta_{xx}(x,\,t) = \left(\frac{D_2}{\sqrt{\pi t}} \frac{\exp(-Y^2(x,\,t))}{\Theta(x,\,t)} \right)_x =$$

$$-\frac{D_2}{\sqrt{\pi t}} \frac{\exp(-Y^2(x,\,t))}{\Theta^2(x,\,t)} \left(Y(x,\,t) + \frac{D_2}{\sqrt{\pi}} \frac{\exp(-Y^2(x,\,t))}{\Theta(x,\,t)} \right);$$

b)

$$\Theta_t(x,\,t) = \frac{2D_2}{\sqrt{\pi}} \exp(-Y^2(x,\,t))\, Y_t(x,\,t) =$$

$$-\frac{D_2}{\sqrt{\pi t}} \exp(-Y^2(x,\,t)) \left(Y(x,\,t) + \frac{D_2}{\sqrt{\pi}} \frac{\exp(-Y^2(x,\,t))}{\Theta(x,\,t)} \right)$$

that is Eq. (12);

c)

$$\Theta(0,\,t) = C_2 - D_2 \operatorname{erf}(r) = \frac{1}{a + b\theta_f} - \frac{b(\theta_0 - \theta_f)}{(a + b\theta_f)(a + b\theta_0)} = \frac{1}{a + b\theta_0};$$

d)

$$\Theta_x(0,\,t) = \frac{D_2}{\sqrt{\pi t}} \frac{\exp(-Y^2(0,\,t))}{\Theta(0,\,t)} = \frac{r}{\sqrt{t}}, \quad \text{that is (13);}$$

e)

$$\Theta(s(t),\,t) = C_2 + D_2 \operatorname{erf}(\Lambda_2 - r) = \frac{1}{a + b\theta_f}, \quad \text{that is (15);}$$

f)

$$\Theta_x(s(t),\,t) = \frac{D_2}{\sqrt{\pi t}} \frac{\exp(-Y^2(s(t),\,t))}{\Theta(s(t),\,t)} = \frac{(a + b\theta_f)\, D_2}{\sqrt{\pi t}} \exp(-(\Lambda_2 - r)^2) =$$

$$\frac{a + b\theta_f}{a + b\theta_0} \frac{1}{\sqrt{t}} r \exp(r^2) \exp(-(\Lambda_2 - r)^2) = \frac{a + b\theta_f}{a + b\theta_0} \frac{1}{\sqrt{t}} r \exp(r^2) \frac{\operatorname{erf}^{-1}(F(r))}{W_3(r)} =$$

$$\frac{a + b\theta_f}{a + b\theta_0} \frac{1}{\sqrt{t}} r \exp(r^2) \frac{\operatorname{erf}^{-1}(F(r))}{W_4(r)} =$$

$$\frac{1}{\sqrt{t}} \frac{bl}{c(a + b\theta_f) - bl} (\Lambda_2 - r) = \frac{bl\lambda_2}{c\sqrt{t}} = \frac{bl}{c} \dot{s}(t), \quad \text{that is (14).}$$

(iii) Now in order to complete the proof, we just have to proof the existence of a solution of the integral equation (76). If we define $Y(x, t)$ by (77) then, Eq. (76) is equivalent to the following Cauchy differential problem

$$(83)\qquad \begin{cases} \dfrac{\partial Y}{\partial x}(x, t) = \dfrac{1}{2\sqrt{t}} \dfrac{1}{(C_2 + D_2 \operatorname{erf}(Y(x, t)))} \equiv G_2(x, t, Y(x, t)), \\ 0 < x < s(t), \quad t > 0, \quad Y(0, t) = -r, \end{cases}$$

with a positive parameter $t > 0$. We have $\left| \dfrac{\partial G_2}{\partial Y} \right| \leqslant \dfrac{D_2}{C_2^2 \sqrt{\pi t}}$ which is bounded for all $t \geqslant t_0 > 0$, $0 \leqslant x \leqslant s(t)$, for an arbitrary positive time t_0. Then, problem (83) (i.e. the integral Eq. (76)) has a unique solution for $t \geqslant t_0 > 0$, for an arbitrary positive time t_0.

(iv) It follows from elementary but tedious computation. ■

REMARK 2. – If Θ is the solution of the integral equation (76) then Θ is strictly monotone in variable x. We obtain that $\theta(x, t) = (1/\Theta(x, t) - a)/b$ does not have a limit when $(x, t) \to (0, 0)$ but $\theta(x, t)$ is bounded in a neighborhood of $(0, 0)$ checking that

$$\theta_f = \lim_{(\eta, \tau) \mapsto (0, 0)} \inf \theta(\eta, \tau) \leqslant \theta(x, t) \leqslant \theta_0 = \lim_{(\eta, \tau) \mapsto (0, 0)} \sup \theta(\eta, \tau),$$

$$\text{for } 0 \leqslant x \leqslant s(t),\ t > 0 .$$

The result of the Theorem 3 is also true if we replace condition (10) by $a + b\theta_f = bl/c$.

THEOREM 4. – If condition $a + b\theta_f = bl/c$ is satisfied then the solution of the problem (2), (4)-(6) and (52) is given by

$$(84)\qquad \theta(x, t) = \frac{1}{b}\left[\frac{1}{\Theta(x, t)} - a\right] \qquad s(t) = \frac{2\Lambda_3 \exp(\Lambda_3^2)}{a + b\theta_0}\sqrt{t}$$

where Θ is the unique solution in variable x of the following integral equation

$$(85)\qquad \begin{cases} \Theta(x, t) = \dfrac{1}{a + b\theta_f}\left(1 + \dfrac{b(\theta_0 - \theta_f)}{(a + b\theta_0)} \dfrac{\operatorname{erf}\left(\dfrac{\int_0^x \dfrac{d\eta}{\Theta(\eta, t)}}{2\sqrt{t}} - \Lambda_3\right)}{\operatorname{erf}(\Lambda_3)}\right), \\ 0 \leqslant x \leqslant s(t) \end{cases}$$

for $t \geqslant t_0 > 0$ with t_0 an arbitrary positive time and Λ_3 is the unique solution of the equation

$$x \exp(x^2) \operatorname{erf}(x) = \frac{b(\theta_0 - \theta_f)}{\sqrt{\pi}(a + b\theta_f)}, \quad x > 0. \tag{86}$$

PROOF. – If we define (11) then, problem (2), (4)-(6) and (52) is transformed in (12), (14)-(16) and (53). We also define (18) and (19) and we obtain (23), (24), (54), (55) and

$$\frac{\partial \Psi}{\partial \chi}(S(t), t) = \dot{s}(t). \tag{87}$$

Now, introducing (26) and (27) we get (28), (31), (58) and (59). Taking into account (14), (15) and (19) we have

$$\frac{\partial \Theta}{\partial x}(s(t), t) = \frac{lb}{c}\dot{s}(t) \Leftrightarrow \varphi'\left(\frac{\varphi'(0)(a + b\theta_0)}{2}\right)\frac{1}{2\sqrt{t}}(a + b\theta_f) = \frac{lb}{c}\dot{s}(t)$$

then $s(t) = \lambda_3\sqrt{t}$ where $\lambda_3 = \varphi'\left(\frac{\varphi'(0)(a + b\theta_0)}{2}\right)$. From (19) we have

$$\begin{aligned} S(t) &= \frac{\partial \chi}{\partial x}(s(t), t)\,\dot{s}(t) + \frac{\partial \chi}{\partial t}(s(t), t) = \\ &= \frac{1}{\Theta(s(t), t)}\dot{s}(t) + \frac{\partial \Theta}{\partial x}(0, t) - \frac{\partial \Theta}{\partial x}(s(t), t) = \\ &= (a + b\theta_f)\,\dot{s}(t) + \frac{\partial \Theta}{\partial x}(0, t) - \frac{lb}{c}\dot{s}(t) = \frac{\partial \Theta}{\partial x}(0, t) \end{aligned} \tag{88}$$

then

$$\dot{S}(t) = \varphi'(0)\frac{1}{2\sqrt{t}}\frac{1}{\Theta(0, t)} = \varphi'(0)\frac{1}{2\sqrt{t}}(a + b\theta_0) \tag{89}$$

that is (28) when

$$\Lambda_3 = \frac{\varphi'(0)(a + b\theta_0)}{2} \quad \text{(i.e. } \lambda_3 = \varphi'(\Lambda_3)\text{)}. \tag{90}$$

If we integrate (58) we obtain

$$\varphi(\xi) = K_3 \exp(\Lambda^2)\frac{\sqrt{\pi}}{2}\operatorname{erf}(\xi - \Lambda_3) + C_3 \tag{91}$$

where

$$K_3 = \frac{2b(\theta_0 - \theta_f)\exp(-\Lambda_3^2)}{\sqrt{\pi}(a+b\theta_0)(a+b\theta_f)\operatorname{erf}(\Lambda_3)}, \qquad C_3 = \frac{1}{a+b\theta_f}. \tag{92}$$

We know that $\varphi'(0) = K_3$ then Λ_3 must satisfy equation (86) which has a unique solution $\Lambda_3 > 0$ for all data. Taking into account (91) we get

$$\lambda_3 = \frac{2\Lambda_3 \exp(\Lambda_3^2)}{(a+b\theta_0)} \tag{93}$$

and then the free boundary $s(t)$ is given by (84).

Furthermore θ and s are the solution of problem (2), (4)-(6) and (52) with condition $a + b\theta_f = \frac{bl}{c}$ if and only if Θ (defined by (11)) and s are the solution of (12), (14)-(16) and (53). Then, Θ must satisfy the integral equation (85). This integral equation is of the same type of (76), then it has a unique solution for all $t \geqslant t_0 > 0$ with t_0 an arbitrary positive time. We reach the thesis following an argument similar to the one developed in Theorem 3. ■

Finally we study the last case $a + b\theta_f < \frac{bl}{c}$. Doing the same transformations that in the case $a + b\theta_f > \frac{bl}{c}$ we obtain (57)-(63) with coefficients λ_4, Λ_4 and p instead of λ_2, Λ_2 and r being

$$\Lambda_4 = p + \operatorname{erf}^{-1}[\mathrm{F(p)}], \qquad \text{with } p > r_1 = F^{-1}(1) \tag{94}$$

and F was defined by (69). Furthermore $\Lambda_4 < p$, with $p > r_0 = F^{-1}(0)$ $(> r_1)$. We have

$$\varphi(\xi) = K_4 \exp(p^2)\frac{\sqrt{\pi}}{2}\operatorname{erf}(\xi - p) + C_4$$

where

$$K_4 = \frac{2}{\sqrt{\pi}} \frac{\exp(-p^2)\, b(\theta_0 - \theta_f)}{(a+b\theta_f)(a+b\theta_0)(\operatorname{erf}(p) - \operatorname{erf}(p - \Lambda_4))} > 0, \tag{95}$$

$$C_4 = \frac{1}{a+b\theta_f}\left(1 + \frac{b(\theta_0 - \theta_f)\operatorname{erf}(p - \Lambda_4)}{(a+b\theta_0)(\operatorname{erf}(p) - \operatorname{erf}(p-\Lambda_4))}\right), \tag{96}$$

and p must verify the following equation

$$W_3(x) = W_4(x), \qquad x > r_0 \tag{97}$$

Let $\eta = \left(1 - \frac{c}{bl}(a + b\theta_f)\right)\frac{(a+b\theta_f)}{(a+b\theta_0)}$ and $Z(x) = x\exp(x^2)$, $x > 0$. It's easy to see that $\eta \in (0, 1)$ if and only if $a + b\theta_f < \frac{bl}{c}$ i.e. our hypothesis. Then, equation (97) is equivalent to

$$U(x) = \eta\,, \qquad x > r_0 \ \text{ for } \ 0 < \eta < 1 \tag{98}$$

where function U is defined by

$$U(x) = -\frac{Z(\mathrm{erf}^{-1}(F(x)))}{Z(x)} > 0\,, \quad x > r_0 = F^{-1}(0)\,. \tag{99}$$

Function U has the following properties: $\lim_{x \mapsto r_0} U(x) = \frac{a + b\theta_f}{a + b\theta_0} > \eta$.

Then we have at least one solution of the equation (97). ■

Then, we have the following result whose proof is parallel to the one of Theorem 3.

THEOREM 5. – If the condition $a + b\theta_f < \frac{bl}{c}$ is satisfied then the free boundary problem (2), (4)-(6) and (52) has at least one solution (θ, s) for $t \geq t_0 > 0$ (with t_0 an arbitrary positive time) which is given by

$$\theta(x, t) = \frac{1}{b}\left[\frac{1}{\Theta(x, t)} - a\right] \qquad s(t) = \frac{2(p - \Lambda_4)\sqrt{t}}{\frac{bl}{c} - (a + b\theta_f)} \tag{102}$$

with Λ_4 is given by (94) and p is a solution of Eq. (97), and $\Theta(x, t)$ is the corresponding solution of the equivalent integral equation

$$\begin{cases} \Theta(x, t) = \\ \dfrac{1}{a + b\theta_f}\left[1 + \dfrac{b(\theta_0 - \theta_f)}{(a + b\theta_0)} \dfrac{\left(\mathrm{erf}(p - \Lambda_4) + \mathrm{erf}\left(\dfrac{\int_0^x \frac{d\eta}{\Theta(\eta, t)}}{2\sqrt{t}} - p\right)\right)}{\mathrm{erf}(p) - \mathrm{erf}(p - \Lambda_4)}\right], \\ 0 \leq x \leq s(t)\,; \end{cases} \tag{103}$$

with $t \geq t_0 > 0$ is a parameter.

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REFERENCES

[1] V. ALEXIADES - A. D. SOLOMON, *Mathematical modeling of melting and freezing processes*, Hemisphere - Taylor & Francis, Washington (1983).

[2] I. ATHANASOPOULOS - G. MAKRAKIS - J. F. RODRIGUES (EDS.), *Free Boundary Problems: Theory and Applications*, CRC Press, Boca Raton (1999).

[3] J. R. BARBER, *An asymptotic solution for short-time transient heat conduction between two similar contacting bodies*, Int. J. Heat Mass Transfer, **32**, No 5 (1989), 943-949.

[4] D. A. BARRY - G. C. SANDER, *Exact solutions for water infiltration with an arbitrary surface flux or nonlinear solute adsorption*, Water Resources Research, **27**, No 10 (1991), 2667-2680.

[5] G. BLUMAN - S. KUMEI, *On the remarkable nonlinear diffusion equation*, J. Math Phys., **21** (1980), 1019-1023.

[6] A. C. BRIOZZO - M. F. NATALE - D. A. TARZIA, *Determination of unknown thermal coefficients for Storm's-type materials through a phase-change process*, Int. J. Non-Linear Mech., **34** (1999), 324-340.

[7] P. BROADBRIDGE, *Non-integrability of non-linear diffusion-convection equations in two spatial dimensions*, J. Phys. A: Math. Gen., **19** (1986), 1245-1257.

[8] P. BROADBRIDGE, *Integrable forms of the one-dimensional flow equation for unsaturated heterogeneous porous media*, J. Math. Phys., **29** (1988), 622-627.

[9] J. R. CANNON, *The one-dimensional heat equation*, Addison - Wesley, Menlo Park (1984).

[10] H. S. CARSLAW - J. C. JAEGER, *Conduction of heat in solids*, Oxford University Press, London (1959).

[11] J. M. CHADAM - H. RASMUSSEN H. (EDS.), *Free boundary problems involving solids*, Pitman Research Notes in Mathematics Series **281**, Longman, Essex (1993).

[12] M. N. COELHO PINHEIRO, *Liquid phase mass transfer coefficients for bubbles growing in a pressure field: a simplified analysis*, Int. Comm. Heat Mass Transfer, **27**, No 1 (2000), 99-108.

[13] J. CRANK, *Free and moving boundary problems*, Clarendon Press, Oxford (1984).

[14] J. I. DIAZ - M. A. HERRERO - A. LIÑAN - J. L. VAZQUEZ (Eds.), *Free boundary problems: theory and applications*, Pitman Research Notes in Mathematics Series **323**, Longman, Essex (1995).

[15] A. FASANO - M. PRIMICERIO (Eds.), *Nonlinear diffusion problems*, Lecture Notes in Math., N. 1224, Springer Verlag, Berlin (1986).

[16] A. S. FOKAS - Y. C. YORTSOS, *On the exactly solvable equation* $S_t = [(\beta S + \gamma)^{-2} S_x]_x + \alpha(\beta S + \gamma)^{-2} S_x$ *occurring in two-phase flow in porous media*, SIAM J. Appl. Math., **42**, No 2 (1982), 318-331.

[17] N. KENMOCHI (Ed.), *Free Boundary Problems: Theory and Applications, I,II*, Gakuto International Series: Mathematical Sciences and Applications, Vol. **13**, **14**, Gakkotosho, Tokyo (2000).

[18] J. H. KNIGHT - J. R. PHILIP, *Exact solutions in nonlinear diffusion*, J. Engrg. Math., 8 (1974), 219-227.
[19] G. LAMÉ - B. P. CLAPEYRON, *Memoire sur la solidification par refroidissement d'un globe liquide*, Annales Chimie Physique, **47** (1831), 250-256.
[20] V. J. LUNARDINI, *Heat transfer with freezing and thawing*, Elsevier, Amsterdam (1991).
[21] A. MUNIER - J. R. BURGAN - J. GUTIERREZ - E. FIJALKOW - M. R. FEIX, *Group transformations and the nonlinear heat diffusion equation*, SIAM J. Appl. Math., **40**, No 2 (1981), 191-207.
[22] M. F. NATALE - D. A. TARZIA, *Explicit solutions to the two-phase Stefan problem for Storm-type materials*, J. Phys. A: Math. Gen., **33** (2000), 395-404.
[23] M. F. NATALE - D. A. TARZIA, *Explicit solutions to the one-phase Stefan problem with temperature-dependent thermal conductivity and a convective term*, Int. J. Engng. Sci., **41** (2003), 1685-1698.
[24] R. PHILIP, *General method of exact solution of the concentration-dependent diffusion equation*, Australian J. Physics, **13** (1960), 1-12.
[25] A. D. POLYANIN - V. V. DIL'MAN, *The method of the «carry over» of integral transforms in non-linear mass and heat transfer problems*, Int. J. Heat Mass Transfer, **33**, No 1 (1990), 175-181
[26] C. ROGERS, *Application of a reciprocal transformation to a two-phase Stefan problem*, J. Phys. A: Math. Gen., **18** (1985), 105-109.
[27] C. ROGERS, *On a class of moving boundary problems in non-linear heat condition: Application of a Bäcklund transformation*, Int. J. Non-Linear Mech., **21** (1986), 249-256.
[28] C. ROGERS - P. BROADBRIDGE, *On a nonlinear moving boundary problem with heterogeneity: application of reciprocal transformation*, Journal of Applied Mathematics and Physics (ZAMP), **39** (1988), 122-129.
[29] G. C. SANDER - I. F. CUNNING - W. L. HOGARTH - J. Y. PARLANGE, *Exact solution for nonlinear nonhysteretic redistribution in vertical soli of finite depth*, Water Resources Research, **27** (1991), 1529-1536.
[30] D. A. TARZIA, *An inequality for the coefficient σ of the free boundary $s(t) = 2\sigma\sqrt{t}$ of the Neumann solution for the two-phase Stefan problem*, Quart. Appl. Math., **39** (1981), 491-497.
[31] D. A. TARZIA, *A bibliography on moving - free boundary problems for the heat-diffusion equation. The Stefan and related problems*, MAT-Serie A #**2** (2000) (with 5869 titles on the subject, 300 pages). See www.austral.edu.ar/MAT-SerieA/2(2000)/
[32] P. TRITSCHER - P. BROADBRIDGE, *A similarity solution of a multiphase Stefan problem incorporating general non-linear heat conduction*, Int. J. Heat Mass Transfer, **37**, No 14 (1994), 2113-2121.

Depto. Matemática, FCE, Universidad Austral
Paraguay 1950, S2000FZF Rosario, Argentina
E-mail: Maria.Natale@fce.austral.edu.ar
Domingo.Tarzia@fce.austral.edu.ar

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