Bollettino
Unione Matematica Italiana

Edoardo Ballico

Holomorphic vector bundles on certain holomorphically convex complex manifolds

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **9-B** (2006), n.2, p. 261–265.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2006_8_9B_2_261_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2006.

Bollettino U. M. I.
(8) **9**-B (2006), 261-265

Holomorphic Vector Bundles on Certain Holomorphically Convex Complex Manifolds.

EDOARDO BALLICO (*)

Sunto. – *Qui proviamo l'esistenza di fibrati vettoriali olomorfi non triviali su ogni varietà complessa* 0*-convessa ma non Stein e su certe classi di varietà complesse olomorficamente convesse.*

Summary. – *Here we prove the existence of non-trivial holomorphic vector bundles on every* 0*-convex but not Stein complex manifold and on certain classes of holomorphically convex complex manifolds.*

1. – Introduction.

A famous theorem of Grauert states that on a complex Stein space the holomorphic and the topological classification of vector bundles are the same. In particular every holomorphic vector bundle on a one-dimensional or a contractible Stein space is holomorphically trivial. A suitable extension of Grauert's theorem to 0-convex complex manifolds was proved by G. Henkin and J. Leiterer (see [6] and [4]). We just recall that a 0-convex space is a proper modification at finitely many points of a Stein space; these objects are called 1-convex spaces in [1]. In section 2 we will prove the following result.

THEOREM 1. – *Let X be a smooth n-dimensional* 0*-convex complex manifold which is not Stein. Then there exists a non-trivial holomorphic vector bundle E on X such that* $\mathrm{rank}(E) \leq n$.

J. Winkelmann proved that on any n-dimensional compact manifold there is a non-trivial holomorphic vector bundle of rank at most n ([10] and [11], Th. 7.13.1). In section 2 we will also prove the following result.

(*) The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

THEOREM 2. – *Let X be a connected and holomorphically convex complex manifold such that its Remmert reduction $f : X \to Z$ satisfies the inequalities* $1 \leq \dim(X) - \dim(Z) \leq 2$. *Then either its tangent bundle TX is not holomorphically trivial or there is a holomorphic line bundle on X which is topologically trivial but not holomorphically trivial. In particular there is a holomorphic vector bundle E on X such that* $\mathrm{rank}(E) \leq \dim(X)$ *and E is not holomorphically trivial.*

REMARK 1. – Let X be a connected 2-dimensional 0-convex manifold which is neither Stein nor compact. We claim the existence of a holomorphic line bundle on X which is not topologically trivial. Indeed, since X is neither Stein nor compact, its Remmert reduction $f : X \to Z$ is a modification and there is at least one irreducible compact curve $C \subset X$ contracted by f. Any such curve C has $C^2 < 0$, i.e. the normal bundle of C in X has degree $C^2 < 0$. Thus $\mathcal{O}_X(C)|C$ is not topologically trivial. Thus $\mathcal{O}_X(C)$ is not topologically trivial.

We want to thank very much the referee. His/her remarks completely changed the shape of this paper: the sequence of the sections, very often the exposition, often the proofs and sometimes even the statements.

2. – Proofs of Theorems 1 and 2.

PROOF OF THEOREM 1. We may assume that X is connected. By [11] we may assume that X is not compact, i.e. that its Remmert reduction $f : X \to Z$ is not constant. Let A be the union of all positive-dimensional compact complex subspaces of X. Since X is 0-convex but neither Stein nor compact, then $A \neq X$, f is a modification, $f|(X \setminus A)$ is an isomorphism onto $Z \setminus f(A)$ and f contracts each connected component of A to a different point of Z. If the tangent bundle TX is not trivial, then we take $E := TX$. Hence we may assume $TX \cong \mathcal{O}_X^{\oplus n}$. Thus $\Omega_X^1 \cong \mathcal{O}_X^{\oplus n}$. We distinguish two cases:

(i) Here we assume that each irreducible component of A has codimension one in X, i.e. we assume that A is a Cartier divisor of X. Let H be any irreducible component of A. If $\mathcal{O}_X(H)$ is not holomorphically trivial, then we are done. If $\mathcal{O}_X(H)$ is holomorphically trivial, then there is a holomorphic function u on X whose zero-locus is exactly H. Thus there is a holomorphic function v on Z which vanishes exactly at the point $f(H)$. Since the scheme-theoretic zero-locus of a holomorphic function is a Cartier divisor and $\dim(Z) = \dim(X) \geq 2$, we obtained a contradiction.

(ii) Here we assume the existence of an irreducible component of A with codimension at least two in X. Since A has no isolated point, this implies $n \geq 3$. Since X is normal, the universal property of the Remmert reduction implies that

Z is normal. Hence the tangent sheaf $TZ := (\Omega^1_Z)^*$ is the unique reflexive extension to Z of the tangent bundle TZ_{reg} of Z_{reg}. Since f induces an isomorphism between $X\backslash A$ and $Z\backslash f(A)$, we have $T(Z\backslash f(A)) \cong \mathcal{O}^{\oplus n}_{Z\backslash f(A)}$. Since $\mathcal{O}^{\oplus n}_Z$ is a reflexive extension of $\mathcal{O}^{\oplus n}_{Z\backslash f(A)}$, we obtain $TZ \cong \mathcal{O}^{\oplus n}_Z$. In particular TZ is locally free. Since $Z\backslash f(A)$ is smooth and $f(A)$ is the union of finitely many points, Sing(Z) has codimension at least $n \geq 3$ in X. Hence the local freeness of TZ implies that Z is smooth ([5], Corollary at p. 318). Hence Ω^1_Z is locally free. Thus the natural map $\psi : f^*(\Omega^1_Z) \to \Omega^1_Y$ is a map between holomorphic vector bundles with the same rank. Hence ψ is an isomorphism around $Q \in Y$ if and only if $\det(\psi)(Q) \neq 0$. Since $X\backslash A = \{Q \in X : \psi \text{ is an isomorphism at } Q\}$ and $\{\det(\psi) = 0\}$ is a Cartier divisor of X, we obtain that A is a Cartier divisor of X, contradiction.

Proof of Theorem 2. If $H^1(X, \mathcal{O}_X) \neq 0$, then the exponential sequence

$$0 \to \mathbf{Z} \to \mathcal{O}_X \to \mathcal{O}^*_X \to 0 \tag{1}$$

implies the existence of a holomorphic line bundle on X which is topologically trivial but not holomorphically trivial. Hence the goal is to obtain $H^1(X, \mathcal{O}_X) \neq 0$ under the assumption that TX is holomorphically trivial. Since f is proper, $R^jf_*(\mathcal{O}_X)$ is coherent for every $j \geq 0$. Hence the Leray spectral sequence of the map f gives $H^1(X, \mathcal{O}_X) \cong H^0(Z, R^1f_*(\mathcal{O}_X))$. Since Z is Stein, to conclude it is sufficient to prove that $R^1f_*(\mathcal{O}_X) \neq 0$ (Cartan's Theorem A). Thus it is sufficient to show that $R^1f_*(\mathcal{O}_X)$ has positive rank at a general point P of Z. By Sard's Lemma the smoothness of X implies that for a sufficiently general $P \in Z$ the fiber $f^{-1}(P)$ is smooth. Since f is proper, f is smooth in a neighborhood of such a fiber $f^{-1}(P)$. Since f is the Remmert reduction of X, $f^{-1}(P)$ is connected.

(i) Here we assume $\dim(Z) = \dim(X) - 1$. Thus $f^{-1}(P)$ is a connected smooth curve. Since f is smooth near $f^{-1}(P)$, the normal bundle of $f^{-1}(P)$ in X is trivial. Since TX is trivial, we obtain that $Tf^{-1}(P)$ is trivial, i.e. that $f^{-1}(P)$ is an elliptic curve. Thus the coherent sheaf $R^1f_*(\mathcal{O}_X)$ has rank one in a neighborhood of P and in particular it is non-zero.

(ii) Here we assume $\dim(Z) = \dim(X) - 2$. Since f is smooth near $f^{-1}(P)$, the normal bundle of $f^{-1}(P)$ in X is trivial. Since TX is trivial, we obtain that $Tf^{-1}(P)$ is trivial, i.e. that $f^{-1}(P)$ is a compact surface of the form G/Γ with G a two-dimensional simply connected complex Lie group and Γ a discrete cocompact subgroup of G. If $G \neq G'$, where G' is the commutator subgroup of G, then the Albanese torus Alb(G/Γ) of G/Γ is non-trivial and $H^1(G/\Gamma, \mathcal{O}_{G/\Gamma}) \neq 0$. Furthermore, the integer-valued function $h^1(f^{-1}(Q), \mathcal{O}_{f^{-1}(Q)})$ is a constant function of Q in a neighborhood of P. Hence $R^1f_*(\mathcal{O}_X)$ has rank $h^1(G/\Gamma, \mathcal{O}_{G/\Gamma}) > 0$ around P in this case. Now assume $G = G'$, i.e. assume G semi-simple. Hence the Lie algebra of G is semi-simple. Thus the Lie algebra of G is a product of simple Lie algebras. By the classification of all simple Lie algebras ([8],p. 74), every simple Lie algebra has dimension at least 3. Hence $\dim(G) \geq 3$, contradiction.

REMARK 2. – Let X be a connected 3-dimensional holomorphically convex manifold which is neither Stein nor compact. We claim the existence of a holomorphic vector bundle E on X such that $\mathrm{rank}(E) \leq 3$ and E is not holomorphically trivial. Indeed, let $f : X \to Z$ be the Remmert reduction of X and $B \subset X$ the union of all positive-dimensional compact subvarieties of X. Since X is not compact, $\dim(Z) > 0$. The cases $\dim(Z) = 1$ and $\dim(Z) = 2$ are just Theorem 2. Hence we may assume $\dim(Z) = 3$. Since X is not Stein, $f(B) \neq \emptyset$. If $f(B)$ is finite, then X is 0-convex and hence we may apply Theorem 1. Hence we may assume the existence of a one-dimensional irreducible component of $f(B)$. Since $\dim(f^{-1}(P)) > 0$ for every $P \in f(B)$, we obtain the existence of an irreducible hypersurface H of X such that $H \subseteq B$ and $f(H)$ is a curve. If $\mathcal{O}_X(H)$ is not holomorphically trivial, then we are done. Assume that $\mathcal{O}_X(H)$ is holomorphically trivial. Thus there exists a holomorphic function on X whose zero-locus is exactly H, i.e. a holomorphic function on Z whose zero-locus is exactly $f(H)$. Since $f(H)$ is a non-empty curve and $\dim(Z) = 3$, this is absurd.

PROPOSITION 1. – *Let X be a connected 2-dimensional complex manifold which is neither Stein nor compact. Then there is a holomorphic vector bundle E on X such that* $\mathrm{rank}(E) \leq 2$ *and E is not holomorphically trivial.*

PROOF. – We may assume that every holomorphic line bundle on X is trivial. Hence by the exponential sequence we may assume $H^1(X, \mathcal{O}_X) = 0$. Fix a discrete subset T of X with $T \neq \emptyset$. Consider all extensions

$$0 \to \mathcal{O}_X \to F \to \mathcal{I}_T \to 0 \tag{2}$$

of $\mathcal{I}_T$ by $\mathcal{O}_X$. Since T is a two-dimensional locally complete intersection in X, we have $\mathrm{Ext}^1(\mathcal{I}_T, \mathcal{O}_X) \cong \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_T$, $\mathrm{Ext}^0(\mathcal{I}_T, \mathcal{O}_X) \cong \mathrm{Ext}^0(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_X$ and $\mathrm{Ext}^i(\mathcal{I}_T, \mathcal{O}_X) = 0$ for every $i \geq 2$. Since $\dim(X) = 2$ and X is not compact, we have $H^2(X, G) = 0$ for every coherent analytic sheaf G on X and in particular $H^2(X, \mathcal{O}_X) = 0$ ([7], p. 236, Probleme 1, or [9]). Thus the local to global spectral sequence of the Ext-functors gives $\mathrm{Ext}^1(X; \mathcal{I}_T, \mathcal{O}_X) \cong \mathcal{O}_T$. In this isomorphism the middle term, F, of an extension (2) is locally free if and only if it corresponds to a nowhere vanishing section of $\mathcal{O}_T$. We stress that here we just use local duality on the two-dimensional regular local ring $\mathcal{O}_{X,P}$, i.e. essentially the adjunction formula $\omega_T \cong (\omega_X)|T \otimes \det(N_{T,X})$, where $N_{T,X}$ is the normal bundle of T in X. Hence for any $T \neq \emptyset$ we may find an extension (2) whose middle term is locally free (see e.g. [2], pp. 9–10). Assume $F \cong \mathcal{O}_X^{\oplus 2}$. Hence the injective map $\mathcal{O}_X \to F$ in (2) is induced by two holomorphic functions f_1, f_2 on X such that $T = \{f_1 = f_2 = 0\}$. We want to check that X is holomorphically convex; by Theorems 1 and 2 this would be sufficient to conclude. Set $D := \{f_1 = 0\}$. Since $T \neq \emptyset$, D is a non-empty effective Cartier divisor. By construction we have $\mathcal{O}_X(-D) \cong \mathcal{O}_X$. Thus

$H^1(X, \mathcal{O}_X(-D)) = 0$. From the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0 \tag{3}$$

we obtain the surjectivity of the restriction map $\rho : H^0(X, \mathcal{O}_X) \to H^0(D, \mathcal{O}_D)$. Let R be the union of all positive-dimensional compact irreducible components of D. Since X is connected, two-dimensional and non-compact, we have $H^2(X, \mathcal{F}) = 0$ for every coherent analytic sheaf $\mathcal{F}$ on X. Hence $H^2(X, \mathcal{O}_X(-D)) = 0$. Hence (3) gives $h^1(D, \mathcal{O}_D) = 0$. Since $T = \{f_1 = f_2 = 0\}$, T is scheme-theoretically given by an equation on D. Thus $\mathcal{O}_D(-T) \cong \mathcal{O}_D$. Hence $h^1(D, \mathcal{O}_D(-T)) = 0$. From the exact sequence

$$0 \to \mathcal{O}_D(-T) \to \mathcal{O}_D \to \mathcal{O}_T \to 0 \tag{4}$$

we obtain the surjectivity of the restriction map $\eta : H^0(D, \mathcal{O}_D) \to H^0(T, \mathcal{O}_T)$. Hence the restriction map $\eta \circ \rho : H^0(X, \mathcal{O}_X) \to H^0(T, \mathcal{O}_T)$ is surjective. Since this is true for every discrete subset T of X, X is holomorphically convex, as wanted. □

REFERENCES

[1] A. Andreotti and H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193-259.

[2] C. Banica and J. Le Potier, *Sur l'existence des fibrés holomorphes sur une surface non-algébrique*, J. Reine Angew. Math. **378** (1987), 1-31.

[3] J. Bingener, *Über formale komplexe Räume*, Manuscripta Math. **24** (1978), 253-293.

[4] M. Coltoiu, *On the Oka-Grauert principle for* 1*-convex manifolds*, Math. Ann. **310** (1998), 561-569.

[5] H. Flenner, *Extendability of differential forms on non-isolated singularities*, Invent. Math. **94** (1988), 317-326.

[6] G. Henkin and J. Leiterer, *The Oka-Grauert principle without induction over the base dimension*, Math. Ann. **311** (1998), 71-93.

[7] B. Malgrange, *Faisceaux sur les variétés analytique-reélles*, Bull. Soc. Math. France **85** (1957), 231-237.

[8] H. Samelson, *Notes on Lie Algebras*, Universitext, Springer, 1990.

[9] Y.-T. Siu, *Analytic sheaf cohomology groups of dimension n of n-dimensional complex spaces*, Trans. Amer. Math. Soc. **143** (1969), 77-94.

[10] J. Winkelmann, *Every compact complex manifold admits a holomorphic vector bundle*, Revue Roum. Math. Pures et Appl. **38** (1993), 743-744.

[11] J. Winkelmann, *Complex analytic geometry of complex parallelizable manifolds*, Mémoires Soc. Math. France 72-73, 1998.

Dept. of Mathematics, University of Trento
38050 Povo (TN), Italy
E-mail address: ballico@science.unitn.it

Pervenuta in Redazione
il 25 marzo 2003 e in forma rivista il 17 aprile 2004