Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **9-B** (2006), n.2, p. 311–317.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2006_8_9B_2_311_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2006.

Bollettino U. M. I.
(8) **9**-B (2006), 311-317

On the Projective Genus of Surfaces.

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Sunto. – *Sia $X \subset \mathbb{P}^N$ una superficie complessa liscia irriducibile e non degenere, $N \geq 4$. Definiamo genere proiettivo di X, denotato con $PG(X)$, come il genere geometrico della curva singolare della proiezione di X da un sottospazio lineare generico di codimensione quattro. Si denoti con $g(X)$ il genere sezionale di X. Nel presente lavoro congetturiamo che le uniche superfici per cui $PG(X) = g(X) - 1$ sono la superfice di del Pezzo in $\mathbb{P}^4$, in $\mathbb{P}^5$ e una fibrazione in coniche di grado 5 in $\mathbb{P}^4$. Dimostriamo che per $N \geq 5$ se $PG(X) = g(X) - 1 + \lambda$, λ un intero non negativo, allora $g(X) \leq \lambda + 1 + \alpha$ dove $\alpha = -2$ per uno scroll e $\alpha = 0$ altrimenti, e deduciamo la congettura per $N \geq 5$ da questo enunciato.*

Summary. – *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non degenerate surface over the complex numbers, $N \geq 4$. We define the projective genus of X, denoted by $PG(X)$, as the geometric genus of the singular curve of the projection of X from a general linear subspace of codimension four. Denote by $g(X)$ the sectional genus of X. In this paper we conjecture that the only surfaces for which $PG(X) = g(X) - 1$ are the del Pezzo surface in $\mathbb{P}^4$, in $\mathbb{P}^5$ and a conic bundle of degree 5 in $\mathbb{P}^4$. We prove that for $N \geq 5$ if $PG(X) = g(X) - 1 + \lambda$, λ a non negative integer, then $g(X) \leq \lambda + 1 + \alpha$ where $\alpha = -2$ for a scroll and $\alpha = 0$ otherwise, and deduce the conjecture for $N \geq 5$ from this statement.*

Denote by $X \subset \mathbb{P}^N$ a smooth irreducible non degenerate surface over the complex numbers, $N \geq 4$, and by Γ the singular scheme of the projection of X from a general linear subspace of codimension four. It is well known that Γ is a reduced curve and it can have only ordinary triple points as singularities. By Franchetta's Theorem (see [F1], [F2], [Se]) Γ is also irreducible except for the Veronese surface in $\mathbb{P}^5$, and its isomorphic projection in $\mathbb{P}^4$. Denote by $g(X)$ the sectional genus of X and by d its degree.

DEFINITION 1. – *We call projective genus of X, denoting it by $PG(X)$, the geometric genus of Γ, that is the arithmetic genus of a desingularization of Γ.*

In a joint paper Ciliberto, Mella and Russo ([CMR] p. 16 Theorem 4.10) deduce the classification of surfaces with one apparent double point from the following statement:

PROPOSITION 1. – *Let* $X \subset \mathbb{P}^r$, $r \geq 4$, *be a smooth irreducible non degenerate surface different from the Veronese surface in* $\mathbb{P}^4$ *and in* $\mathbb{P}^5$. $PG(X) = 0$ *if and only if X is one of the following*:

- *a rational normal scroll of degree* 3 *in* $\mathbb{P}^4$,
- *a rational normal scroll of degree* 4 *in* $\mathbb{P}^5$,
- *a del Pezzo surface of degree* $n = 4, 5$ *in* $\mathbb{P}^n$.

PROOF. – [CMR] p. 15 Proposition 4.8.

As they noted the ideas used in the proof of the above Proposition are useful to attack the problem of classifying surfaces with small projective genus, in the sense that will become clear in a moment.

First of all we recall a couple of statements.

PROPOSITION 2.

$$PG(X) = \frac{1}{2}(d^2 - 7d + 48) + (d - 12)\, g(X) + 9p_a(X) - 2(K_X^2 + 1)$$

PROOF. – [En], pp. 173-177, [P1].

PROPOSITION 3. – *If X is different from the Veronese surface in* $\mathbb{P}^5$ *and in* $\mathbb{P}^4$ *then*

$$g(X) \leq PG(X) + \mu$$

where $\mu = 0$ *if X is a scroll and* $\mu = 1$ *otherwise.*

PROOF. – [CMR] p. 15 Proposition 4.7.

Suppose now to fix ideas that $g(X) \leq 5$. By the classification in [Li1]-[Li4] and by Proposition 2 we are able to compute $PG(X)$ obtaining all the surfaces with $PG(X) \leq 5$ and $g(X) \leq 5$. This is the content of the following Proposition.

PROPOSITION 4. – *Let* $X \subset \mathbb{P}^N$, $N \geq 4$. *X is such that* $g(X) \leq 5$ *and* $PG(X) \leq 5$ *if and only if X is one of the following types.*

i) $PG(X) = -2$: *X is the Veronese surface in* $\mathbb{P}^5$ *or in* $\mathbb{P}^4$.

ii) $PG(X) = 0$: $g(X) = 0$ (*respectively* $g(X) = 1$) *and X is a rational normal scroll* (*respectively a del Pezzo surface*) *in* $\mathbb{P}^4$ (*respectively* $\mathbb{P}^4$, $\mathbb{P}^5$).

iii) $PG(X) = 1$: $g(X) = 0$ (*respectively* $g(X) = 1$) *and X is a rational normal scroll* (*respectively a del Pezzo surface*) *in* $\mathbb{P}^6$.

$g(X) = 1$, $d = 5$, *X is a scroll embedded in* $\mathbb{P}^4$ ([I1], p. 146).

$g(X) = 2$, *X is a conic bundle over* $\mathbb{P}^1$ *embedded in* $\mathbb{P}^4$ *of degree* 5.

iv) $PG(X) = 3$: $g(X) = 0$ (*respectively* $g(X) = 1$) *and* X *is a rational normal scroll* (*respectively a del Pezzo surface*) *in* $\mathbb{P}^7$.
$g(X) = 2$, $d = 6$, X *is a conic bundle over* $\mathbb{P}^1$ *embedded in* $\mathbb{P}^5$.
$g(X) = 3$, $d = 6$, X *is the blow-up of* $\mathbb{P}^2$ *in* 10 *points embedded by the linear system* $\mathcal{O}_{\mathbb{P}^4}(4)$ ([I1] *p.* 163 *Theorem* 4.1(iii)).

v) $PG(X) = 4$: $g(X) = 1$, $d = 6$, X *is a scroll.*
$g(X) = 4$, $d = 6$, X *is complete intersection of a quadric hypersurface and a cubic hypersurface of* $\mathbb{P}^4$ ([I2] *p.* 4 *Theorem* 1(i)).

vi) $PG(X) = 5$: $g(X) = 3$, $d = 8$, X *is a conic bundle over an elliptic curve* ([I1] *p.* 167 *Theorem* 4.1(v)).

See [BS] for a definition of the terms used in the Proposition. Note, though we will not dwell on this, that by [Li5] for example one can extend the above analysis to surfaces with $PG \leqslant 7$ and $g(x) \leqslant 7$. In fact in [Li5] one can find a list of all surfaces with sectional genus less or equal than 7. As before by Proposition 2 one can compute $PG(X)$, obtaining the desidered list.

In view of the above examples it seems natural to state the following:

CONJECTURE 1. – *Let* $X \subset \mathbb{P}^N$, $N \geqslant 4$, *such that* $PG(X) = g(X) - 1$ *then* X *is one of the following surfaces*:

- $g(X) = 1$ *and* X *is the del Pezzo surface in* $\mathbb{P}^4$, $\mathbb{P}^5$,
- $g(X) = 2$ *and* X *is a conic bundle of degree* 5 *in* $\mathbb{P}^4$.

Proving the Conjecture we could give a complete classification of surfaces $X \subset \mathbb{P}^N$, $N \geqslant 4$ such that $PG(X) \leqslant 7$. Indeed if X is such that $PG(X) \leqslant 7$ by Proposition 3 either $g(X) \leqslant 7$ or $g(X) = 8$ and $PG(X) = 7$. If $g(X) \leqslant 7$ than we already observed how to list all surfaces with $g(X) \leqslant 7$ and $PG(X) \leqslant 7$. If $g(X) = 8$ and $PG(X) = 7$ than $g(X) = PG(X) + 1$, and these surfaces are listed in the Conjecture.

We are able to prove the Conjecture only for $N \geqslant 5$. This is accomplished by the following Theorem.

THEOREM 1. – *Let* $X \subset \mathbb{P}^N$, $N \geqslant 5$, *suppose that* $PG(X) = g(X) - 1 + \lambda$, λ *a non negative integer, then*

$$g(X) \leqslant \lambda + 1 + \alpha$$

where $\alpha = -2$ *for a scroll and* $\alpha = 0$ *otherwise.*

Before starting the proof of the Theorem we give a definition.

DEFINITION 2. – *Let* $S \subset \mathbb{P}^4$ *a non degenerate surface. We say that* S *is a general surface of* $\mathbb{P}^4$ *if* S *has at most a finite number of improper double* *points, that is non normal double points whose tangent cone consists of two planes that intersect transversally in the point.*

We will need the following well known facts.

PROPOSITION 5. – *i) Let* $S \subset \mathbb{P}^N$, $N > 5$, *and* $\Lambda \subset \mathbb{P}^N$ *a general linear subspace of dimension* $N-6$. *The projection* $\pi_\Lambda : S \to \mathbb{P}^5$ *from* Λ *is an isomorphism over the image.*

ii) Let $S \subset \mathbb{P}^5$ *be a smooth surface, its projection from a general point is a general surface of* $\mathbb{P}^4$.

PROOF. – [GH] pp. 611-618, or [Mo] Appendix I.

PROOF. – [Proof of Theorem 1.] By Proposition 5 we may suppose that $X \subset \mathbb{P}^5$. We will make use of a slightly modified version of the method of *degeneration of projection* as described in [CMR] Section 3. Let $S \subset \mathbb{P}^4$ the general surface projection of X. Since X is not the Veronese surface, S is singular. Let $x \in Sing(S)$ (x is the image of a general secant line through the center of projection), let Θ_1, Θ_2 be the planes whose union is the tangent cone to S in x. Denote by L a general line passing through x, $L \cap \Theta_1 \cap \Theta_2 = \{x\}$, and $T_i = < L, \Theta_i >$, $i = 1, 2$.

For $t \in L$ a general point denote by $\delta_t : X \to \mathbb{P}^3$ the projection of S from t. Let U be a neighborhood of x in L such that δ_t is a morphism for $t \in U$, $t \neq x = 0$. Consider $\mathcal{S} = S \times U$; the maps δ_t for $t \in U$ give globally a rational map $\delta : \mathcal{S} \dashrightarrow \mathbb{P}^3_U$ defined everywhere except in $(x, 0) \in \mathcal{S}$. Let $\tilde{\mathcal{S}}$ be the blow-up of $\mathcal{S}$ in $(x, 0)$, $p : \tilde{\mathcal{S}} \to \mathcal{S}$ the associated projection. The exceptional divisor of this blow-up is a disjoint union of say $Z_1 \cong \mathbb{P}^2$ and $Z_2 \cong \mathbb{P}^2$.

REMARK 1. – *The morphism* $\phi : \tilde{\mathcal{S}} \to U$ *induced by the projection on the second factor is a flat family of varieties, moreover*:

1. S_t *fibre of* ϕ *over* $t \in U \setminus \{0\}$ *is isomorphic to* S, *whereas* S_0 *the fibre of* ϕ *over* $0 \in U$ *is equal to* $\tilde{S} \cup Z_1 \cup Z_2$ *where* $\tilde{S}$ *is the blow-up of* S *in* x. *Write the exceptional divisor of* $\tilde{S}$ *as disjoint union of* E_1, E_2 *then* $\tilde{S} \cap Z_i = E_i$ *where the intersection is transverse.*

2. *Denote by* $\tilde{\delta} : \tilde{\mathcal{S}} \to \mathbb{P}^3_U$ *the morphism induced by* δ, *the restriction to* S_t *of* $\tilde{\delta}$ *coincide with* δ_t *for* $t \neq 0$.

We introduce the following notations:

- δ_0 denotes the restriction of $\tilde{\delta}$ to X_0,
- $D(\tilde{\delta}) \subset \tilde{\mathcal{S}}$ denotes the double point locus (see [Fu], p. 166 or [L], p. 85 for the definition) of $\tilde{\delta}$ and $\varphi : D(\tilde{\delta}) \to U$ the map induced by the projection on the second factor $p_2 : \tilde{\mathcal{S}} \to U$,

- Λ_t the fibre of φ for $t \in U$ (from the definition of double point locus it follows immediately that Λ_t coincides with the double point locus of δ_t),
- S_{T_i} the curve intersection of T_i and S, and by $\tilde{S}_{T_i}$ its strict transform in $\tilde{S}$,
- Y_{T_i} the image of $\tilde{S}_{T_i}$ by the restriction of δ_0 to $\tilde{S}$, $Y_{T_i} \subset P_i := \delta_0(Z_i)$,
- Λ'_0 the double point locus of the restriction of δ_0 to $\tilde{S}$.

LEMMA 1. – *With the above notations*:

1. *the restriction of* δ_0 *to* Z_i *is an isomorphism on the image,* P_i, *which coincide with the plane of* $\mathbb{P}^3$ *projection of* T_i *from* x.

2. Λ_0 *is the union of* Λ'_0 *and* $\tilde{S}_{T_i}$, $i = 1, 2$, *on* $\tilde{S}$, *and* Y_{T_i} *on* Z_i.

PROOF. – The same argument of [CMR], p. 12 Lemma 3.1, [CMR] p. 13, Lemma 3.3.

Summarizing we have a flat family of curves, $\varphi : D(\tilde{\sigma}) \to U$, whose fibre for $t \neq 0$ is the double point locus of the projection of S from a general point. Moreover the central fibre is described by the above Lemma. Denote by $\mathcal{C} \subset \mathbb{P}^3_U$ the image of $D(\tilde{\delta})$ by $\tilde{\delta}$.

The projection on the second factor of $\mathbb{P}^3_U$ induces a map $\mathcal{C} \to U$ which again is a flat family, for $t \neq 0$ the fibre is the singular locus of the general projection of S from a point, and for $t = 0$ the fibre is union of Y_{T_1}, Y_{T_2} and the singular locus of the projection of S from x. Moreover Y_{T_i} is a reduced irreducible curve of geometric genus $g(X) - 1$ if X is not a scroll ([Ei] p. 65, Theorem 2.1(a), [CMR], p. 17, Proposition 5.2), and of genus $g(X)$ if it is a scroll.

Suppose that X is not a scroll. We follow the argument in [CMR] p. 14 proof of the Proposition 4.4. Let $\mathcal{C}'$ be the normalization of $\mathcal{C}$, $\mathcal{C}' \to U$ is again a flat family whose general curve is smooth irreducible of genus $g(X) - 1 + \lambda$. Denote by $\mathcal{C}'' \to U$ a semistable reduction of $\mathcal{C}' \to U$ such that every component of the central fibre, $\mathcal{C}''_0$, is smooth. $\mathcal{C}''_0$ is reduced connected ([Ha], p. 281 Ex. 11.4) and has arithmetic genus equal to $g(X) - 1 + \lambda$. But now $\mathcal{C}''_0$ has two components of genus $g(X) - 1$ which dominate Y_{T_1}, Y_{T_2}, then $g(X) \leq \lambda + 1$. The argument is similar for a scroll.

COROLLARY 1. – *Let* $X \subset \mathbb{P}^N$, $N \geq 5$, *be such that* $PG(X) = g(X) - 1$ *then* $g(X) = 1$ *and* X *is the del Pezzo surface of degree* 5 *in* $\mathbb{P}^5$.

PROOF. – X can not be the Veronese surface nor a scroll (by Proposition 2). Then by Theorem 1 it follows that $g(X) = 1$.

A similar argument and Proposition 4 prove the following:

COROLLARY 2. – *There is no $X \subset \mathbb{P}^N$, $N \geqslant 5$, such that $PG(X) = g(X)$.*

The author give warm thanks to Ciro Ciliberto and Antonio Lanteri for helpful conversations.

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Pervenuta in Redazione
il 22 novembre 2003