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Lucia Marino

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Conductor and Separating Degrees for Sets of Points in $\mathbb{P}^r$ and in $\mathbb{P}^1 \times \mathbb{P}^1$.

LUCIA MARINO

Sunto. – *In questo lavoro generalizziamo alcuni risultati sui gradi del conduttore, noti in $\mathbb{P}^2$, al caso di schemi 0-dimensionali di $\mathbb{P}^r$. Nella prima parte consideriamo il problema di caratterizzare la sequenza dei gradi dei generatori del conduttore in accordo con una fissata funzione di Hilbert per un insieme di punti in $\mathbb{P}^r$ e determiniamo il grado del conduttore di ogni punto in una r-parziale intersezione. Inoltre diamo la definizione di separating degree di un punto per uno schema 0-dimensionale su una quadrica liscia $Q = \mathbb{P}^1 \times \mathbb{P}^1$ e proviamo dei risultati per esso nel caso di sottoschemi speciali.*

Summary. – *We attempt to generalize conductor degree's results, known in $\mathbb{P}^2$, to the case of 0-dimensional schemes of $\mathbb{P}^r$. In the first part of this paper, we consider the problem of characterizing the sequences generators's degrees of the conductor which are compatible with a fixed postulation (or Hilbert function) for a set of points in $\mathbb{P}^r$ and we determine the conductor degree of every point in a r-partial intersection. In addition, we define the separating degree of a point for a 0-dimensional subscheme of a smooth quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and we give some results in case of special subschemes.*

1. – Introduction.

Given a set of s distinct points, $\mathbb{X} = \{P_1, ..., P_s\}$ in the projective space $\mathbb{P}^r$, we have the ideal of $\mathbb{X}$, $I_{\mathbb{X}} \subseteq S = k[x_0, ..., x_r]$, which is generated by the forms vanishing on the points. The reduced 1-dimensional coordinate ring of $\mathbb{X}$, $A = S/I_{\mathbb{X}}$, has as its integral closure (in its total ring of fractions) the ring $B = \oplus_{i=1}^{s} k[T_i]$, and the inclusion map $A \to B$ is made up of the various projections $A \to A/p_i$, p_i the minimal prime ideal of A corresponding to the point P_i of $\mathbb{X}$. The conductor associated to the inclusion (considered as an ideal of B) has the simple form $\mathcal{C} = (T_1^{n_1}, ..., T_s^{n_s})$ and the first question one might ask is an interpretation (geometrically) of the positive integers n_i. This was done many years ago by F. Orecchia [9] who showed that n_i is nothing more than the least degree of a hypersurface in $\mathbb{P}^r$ which vanishes at every point of $\{X \setminus P_i\}$. This n_i is denominated as *conductor degree of the point* P_i on the scheme $\mathbb{X}$. Later, Geramita, Maroscia and Roberts, see [4], algebraicized the problem and considered the problem of finding out what possible sequences $(n_1, ..., n_s)$ could exist for a set of points in $\mathbb{P}^r$. The Hilbert function of a finite set of points in projective r-space is an algebraic tool used to describe information about the way the points are geometrically situated. Mainly, the conductor degree of a point $P_i \in \mathbb{X}$, is strongly related to the Hilbert function of $\mathbb{X}$; precisely,

$$HF(\mathbb{X} \setminus \{P_i\}, j) = \begin{cases} H(i) & j < n_i \\ H(i) - 1 & j \geq n_i \end{cases}$$

Hence, for every zero-dimensional differentiable 0-sequence H it is natural to say that an integer d such that

$$H_d(j) = \begin{cases} H(j) & j < d \\ H(j) - 1 & j \geq d \end{cases}$$

is again a zero-dimensional differentiable 0-sequence, is a *permissible values for the conductor degrees with respect to* H. Moreover, in this paper we will also need the notion of r-partial intersection introduced by Ragusa, Zappalà in [11]. A review of basic facts on these schemes occupies all of Section 2.

In Section 3, it is shown how the conductor degree d of $P \in \mathbb{X} \subseteq \mathbb{P}^r$ is connected with the last graded Betti numbers of the ideal $I_{\mathbb{X}}$ generalizing the Theorem 3.9 of [1] to 0-dimensional subschemes of $\mathbb{P}^r$. Thus we can build the set of all permissible values for the conductor degree with respect to graded Betti numbers B of $I_{\mathbb{X}}$, which will be indicated by S_B.

On the other hand, we consider the problem of characterizing the sequences $(n_1, ..., n_s)$ which are compatible with a fixed Hilbert function for a set of points in $\mathbb{P}^r$. The following question arises

Question: Given a zero-dimensional differentiable 0-sequence $H = \{h_i\}$ with $h_1 \leq r + 1$, i.e. a possible Hilbert function for a set of distinct points $\mathbb{X}$ of $\mathbb{P}^r$,
is it possible to find a configuration $\mathbb{Y} \subseteq \mathbb{P}^r$ *with* $HF(\mathbb{Y}) = H$ *such that for every* $d \in S_H$ $\exists$ $P \in \mathbb{Y}$ *with* $d = c.deg_{\mathbb{Y}}$ P ?

In Section 4 we will give an affirmative answer to the above question. Here, we will assume that the reader is familiar with the notation of r-type vectors as introduced in [6]. In that paper, an algorithm for switching between a Hilbert function and its corresponding r-type vector was found. We will also need the notion of k-configuration. In particular, given an r-type vector with $r \geq 1$, then we can associate to it a k-configuration, see Definition 4.1 of [6].

In Section 5, since a k-configuration corresponding to a r-type vector with $(n \geq 1)$ is a particular r-partial intersection of $\mathbb{P}^r$ we generalize the Theorem 5.11

of [12] determining the conductor degree of any point in a r-partial intersection in $\mathbb{P}^r$.

In the rest of paper, Section 6 – 7, we study 0-dimensional subschemes on $\mathbb{P}^1 \times \mathbb{P}^1$ with particular attention to the *separating degrees* for the points of these subschemes. This new investigation justifies the use of the name *separating degree* of a point P on the 0-dimensional schemes $\mathbb{X}$ on a smooth quadric Q. We make first steps in trying to understand the situation in this environment and so we give new results on the separating degree of a point P of $\mathbb{X} \subseteq Q$.

2. – Preliminaries.

Denote $\mathbb{P}^r$ the r-dimensional projective space over an infinitive field k and let $S = k[x_0, ..., x_r]$ be the usual coordinate ring of $\mathbb{P}^r$.

If $I_{\mathbb{X}} \subseteq S$ is the (saturated) ideal associated to a projective scheme $\mathbb{X} \subseteq \mathbb{P}^r$, we will use the notation $H_{\mathbb{X}}$ to indicate the Hilbert function of $S/I_{\mathbb{X}}$.

We recall now some known results from ([4]).

DEFINITION 2.1. – *Let $\mathbb{X}$ be a set of distinct points in $\mathbb{P}^n$. We say that $f \in R$ is a* separator *for $P \in \mathbb{X}$ if $f(P) \neq 0$ and $f(Q) = 0\ \forall\ Q \in \mathbb{X} \setminus \{P\}$. The minimal degree of a separator for P is called the* conductor degree of P in $\mathbb{X}$*; we denote it as $c.deg_{\mathbb{X}} P$.*

Thus

$$c.deg_{\mathbb{X}}\ P = min\{d \mid \exists\, f \in R_d :\ f(P) \neq 0, f(Q) = 0\ \forall\ Q \in \mathbb{X} \setminus \{P\}\}.$$

Let $\mathbb{X}' = \mathbb{X} \setminus \{P\}$. If $a = c.deg_{\mathbb{X}}\ P$ then

$$f \in (I_{\mathbb{X}'})_a \setminus (I_{\mathbb{X}})_a.$$

So we get ([4], Lemma 2.3)

$$\dim (I_{\mathbb{X}'})_t = \begin{cases} \dim (I_{\mathbb{X}})_t & t < a \\ \dim (I_{\mathbb{X}})_t + 1 & t \geq a \end{cases}$$

and the Hilbert function of $\mathbb{X}'$ is the following

$$H_{\mathbb{X}'}(t) = \begin{cases} H_{\mathbb{X}}(t) & t < a \\ H_{\mathbb{X}}(t) - 1 & t \geq a \end{cases}$$

i.e.

$$\triangle H_{\mathbb{X}'}(t) = \begin{cases} \triangle H_{\mathbb{X}}(t) & t \neq a \\ \triangle H_{\mathbb{X}}(t) - 1 & t = a \end{cases}$$

This motivates the following definition, see [[4], Definition 4.1].

DEFINITION 2.2. – *Let* $H = \{b_i\}$, $i \geq 0$, *be a zero-dimensional differentiable* 0-*sequence. We say that d is a permissible value for the conductor degree with respect to H if the sequence* H_d

$$H_d(i) = \begin{cases} b_i & i < d \\ b_i - 1 & i \geq d \end{cases}$$

is again a zero-dimensional differentiable 0-*sequence.*

The set of all permissible values for the conductor degree with respect to H will be denoted S_H.

By the above observation

REMARK 2.3. – The conductor degree of P in $\mathbb{X}$ is a permissible value for the Hilbert function of $A = R/I_{\mathbb{X}}$.

First of all we recall some results on the conductor degree in $\mathbb{P}^r$. We know that the possible values of the conductor degrees of a set of points $\mathbb{X}$, with Hilbert function H, can be read from the first difference function $\triangle H_{\mathbb{X}}$.

Now, let H be the Hilbert function of a set of distinct points $\mathbb{X}$ in $\mathbb{P}^r$, define

$$\sigma(H_{\mathbb{X}}) = min\{n \mid H_{\mathbb{X}}(n) = H_{\mathbb{X}}(n+1)\}.$$

It is well known, see Geramita, Kreuzer and Robbiano ([3], Proposition 1.14), that $\sigma(H_{\mathbb{X}})$ is an element of the set of the permissible values of the conductor degree with respect to H and moreover for every set of points $\mathbb{X}$, with Hilbert function H,

$$\exists\ P \in \mathbb{X}\ :\ c.deg_{\mathbb{X}} P = \sigma(H_{\mathbb{X}}).$$

For the reader 's convenience now we define the *partial intersection* subschemes of $\mathbb{P}^r$ and we state the main properties of these schemes that will be frequently used in the sequel. For this reason, in the present section, we state some results that have already been proved by Ragusa- Zappalà [11].

Let $(\mathcal{P}, \leq)$ be a poset. We denote, for every $H \in \mathcal{P}$,

$$\mathcal{S}_H = \{K \in \mathcal{P} \mid K\ < H\},\ \bar{\mathcal{S}}_H = \{K \in \mathcal{P} \mid K \leq H\}.$$

DEFINITION 2.4. – *A subset* $\mathcal{A}$ *of the poset* $\mathcal{P}$ *is said a left segment if for every* $H \in \mathcal{A}$, $\mathcal{S}_H \subseteq \mathcal{A}$. *In particular, when* $\mathcal{P} = \mathbb{N}^r$ *with the ordering induced by the natural ordering in* $\mathbb{N}$, *a finite left segment will be mentioned as a r-left segment.*

Note that every r-left segment $\mathcal{A}$ has sets of generators and among those there is a unique minimal set of generators consisting of the maximal elements of $\mathcal{A}$, which will be denoted by $\mathcal{G}(\mathcal{A})$.

In the sequel, if

$$a_i : \mathbb{N}^r \to \mathbb{N}$$

for a r-left segment $\mathcal{A}$, we set

$$a_i = max\{a_i(H)|\ H \in \mathcal{A}\},\ for\ 1 \leq i \leq r.$$

The r-tuple

$$T = (a_1, ..., a_r)$$

will be called the *size* of $\mathcal{A}$.

If $\mathcal{A}$ is a r-left segment, $\mathcal{F}(\mathcal{A})$ will denote the set of minimal elements of $\mathbb{N}^r \setminus \mathcal{A}$, i.e.

$$\mathcal{F}(\mathcal{A}) = \{H \in \mathbb{N}^r \setminus \mathcal{A} \mid S_H \subseteq \mathcal{A}\}.$$

If $\mathcal{A}$ is a r-left segment, consider r families of hyperplanes of $\mathbb{P}^r$

$$\{r_{1,j}\}_{1 \leq j \leq a_1}, \{r_{2,j}\}_{1 \leq j \leq a_2}, ..., \{r_{r,j}\}_{1 \leq j \leq a_r}$$

sufficiently generic, in the sense that $\{r_{1,j_1} \cap r_{2,j_2} \cap ... \cap r_{r,j_r}\}$ are $\prod_{i=1}^{r} a_i$ distinct points of $\mathbb{P}^r$. For every $H = (j_1, ..., j_r) \in \mathcal{A}$, we denote

$$P_H = \cap_{h=1}^{r} r_{h,j_h}.$$

With this notation the subscheme of $\mathbb{P}^r$

$$\mathbb{V} = \bigcup_{H \in \mathcal{A}} P_H$$

is called a *r-partial intersection* with support on the r-left segment $\mathcal{A}$.

In the sequel, if $H = (m_1, ..., m_r) \in \mathbb{N}^r$, then we will write

$$\upsilon(H) = m_1 + ... + m_r.$$

We recall how to compute the Hilbert function of a partial intersection $\mathbb{V}$ of codimension r in terms of its support $\mathcal{A}$.

THEOREM 2.5. – *If $\mathbb{V} \subset \mathbb{P}^r$ is a partial intersection of codimension r with support on $\mathcal{A}$, then the first difference of its Hilbert function is*

$$\triangle H_{\mathbb{V}}(n) = |\{H \in \mathcal{A} \mid \upsilon(H) = n + r\}|.$$

We recall how to compute a minimal set of generators of a partial intersection scheme of codimension r of $\mathbb{P}^r$. Now, set

$$I_{r_{ij}} = (f_{ij}),$$

where

$$f_{ij} \in R_1$$

for all i,j. To every

$$H = (m_1, ..., m_r) \in \bar{S}_T$$

we associate the following form

$$P_H = \prod_{i=1}^{r} \prod_{j=1}^{m_i-1} f_{ij}.$$

THEOREM 2.6. – *If $\mathbb{V} \subset \mathbb{P}^r$ is a partial intersection of codimension r with support on $\mathcal{A}$.*

Then a minimal set of generators for $I_{\mathbb{V}}$ is

$$\{P_H \mid H \in \mathcal{F}(\mathcal{A})\ \}.$$

The previous theorem, in particular, shows that the the first graded Betti numbers of partial intersections are determined by its support.

COROLLARY 2.7. – *Let $\mathbb{V}$ be as above then its first graded Betti numbers depend only on $\mathcal{A}$ and they are the following integers*

$$d_H = \upsilon(H) - r\ \forall\ H \in \mathcal{F}(\mathcal{A}).$$

Now we recall how to compute the last graded Betti numbers of a r-codimensional partial intersection in terms of its support.

THEOREM 2.8. – *Let $\mathbb{V} \subset \mathbb{P}^r$ be a partial intersection of codimension r with support $\mathcal{A}$.*

Then the last graded Betti numbers of $\mathbb{V}$ are

$$s_H = \upsilon(H)\ \forall\ H \in \mathcal{G}(\mathcal{A}).$$

3. – Permissible values S_B in $\mathbb{P}^r$.

THEOREM 3.1. – *Let $\mathbb{X}$ a finite set of distinct points in $\mathbb{P}^2$ with minimal free resolution*

$$0 \to \oplus_{i=1}^{n-1} S(-s_i) \to \oplus_{i=1}^{n} S(-g_i) \to I_{\mathbb{X}} \to 0 \ (1)$$

then for every $P \in \mathbb{X}$ the conductor degree of P in $\mathbb{X}$ is a permissible value for the conductor degrees with respect to graded Betti numbers, i.e.

$$c.deg_{\mathbb{X}}\ P = s_i - 2$$

for some i.

PROOF. – (see [1], Thorem 3.9).

In this section we can generalize Theorem 3.1 for distinct points of $\mathbb{P}^r$.

If $V \subset \mathbb{P}^r$ is a r-codimensional aCM scheme with minimal free resolution

$$0 \to \bigoplus_{j \in B_r} S(-j)^{a_{rj}} \to \ldots \to \bigoplus_{j \in B_1} S(-j)^{a_{1j}} \to I_V \to 0 \; (**)$$

where $B_h = \{j | \; a_{hj} \neq 0\}$ for every $h = 1, \ldots, r$ and the integers $\{a_{ij}\}_j$ will denote the i - th graded Betti numbers.

PROPOSITION 3.2. – *If $V \subset \mathbb{P}^r$ is a set of points with minimal free resolution $(**)$ for every P of V*

$$\exists \; j \in B_r \; | \quad c.deg_V P = j - r$$

PROOF. – Here we repeat the same argument as in Proposition 3.1. Let C be a separator of minimal degree of the point $P \in V$, say

$$deg \; C = a;$$

when no confusion can arise we will not distinguish between the curve C and its defining form.

We recall that

$$(I_V : C) = \mathcal{L}$$

is saturated thus this ideal is the ideal of the separated point P.

We will see that (I_V, C) is the saturated ideal of the remaining elements $V' = V \setminus \{P\}$. In fact consider the following exact sequence:

$$0 \to S/(I_V : C)(-a) \stackrel{\cdot C}{\to} S/I_V \to S/(I_V, C) \to 0$$

where the first map is multiplication by C and the second is the canonical map .

We know that $(I_V, C) \subseteq \text{sat}(I_V, C)$. In order to prove the other inclusion, it is enough to show that both ideals have the same Hilbert function.

By Lemma 2.3 of [4], we have

$$H_{S/sat(I_V,\, C)}(t) = H_{V'}(t) = \begin{cases} H_V(t) & t < a \\ H_V(t) - 1 & t \geq a \end{cases}$$

On the other hand, from the exact sequence (1) we get that

$$H_{S/(I_V,\, C)}(t) = H_V(t) - H_{\mathcal{L}}(t-a) = \begin{cases} H_V(t) & t < a \\ H_V(t) - 1 & t \geq a \end{cases}$$

Hence the Hilbert functions agree, so the two ideals have to coincide. Then the ideal $(I_V, \mathcal{L})$ is the defining ideal of the remaining points V'.

Take minimal free resolutions for $S/\mathcal{L}(-a)$, S/I_V :

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
 & 0 & 0 & & S(-a-r) \\
 & \downarrow & \downarrow & & \downarrow \\
 & S(-a-r) & \bigoplus_{j\in B_r} S(-j)^{a_{rj}} & & \cdots \\
 & \downarrow & \downarrow & & \downarrow \\
 & S(-a-(r-1))^{\binom{r}{r-1}} & \bigoplus_{j\in B_{r-1}} S(-j)^{a_{r-1j}} & & \cdots \\
 & \downarrow & \downarrow & & \downarrow \\
 & \cdots & \cdots & & \cdots \\
 & \downarrow & \downarrow & & \downarrow \\
 & S(-a-2)^{\binom{r}{2}} & \bigoplus_{j\in B_2} R(-j)^{a_{2j}} & & S(-a-1)^r \oplus \bigoplus_{j\in B_2} S(-j)^{a_{2j}} \\
 & \downarrow & \downarrow & & \downarrow \\
 & S(-a-1)^r & \bigoplus_{j\in B_2} S(-j)^{a_{1j}} & & S(-a) \oplus \bigoplus_{j\in B_1} R(-j)^{a_{1j}} \\
 & \downarrow & \downarrow & & \downarrow \\
 & S(-a) & S & & S \\
 & \downarrow & \downarrow & & \downarrow \\
0 & \rightarrow S/\mathcal{L}(-a) & \stackrel{\cdot C}{\rightarrow} S/I_V & & \rightarrow S/(I_V, C) \rightarrow 0
\end{array}$$

Applying the *mapping cone* construction ([8]) to the resolutions of S/I_V and $S/(I_V : C) = S/\mathcal{L}$, from the sequence (1), we get a resolution for the ideal $S/(I_V,\ C)$.

The following construction gives a resolution that, generally, is not minimal, but since $(I_V, \mathcal{L})$ is saturated, the resolution of $(I_V, \mathcal{L})$ must have length r. As it is been observed in [2], using a result of [10], the term $S(-a-r)$ must cancel with something in

$$S(-a-(r-1))^{\binom{r}{r-1}} \oplus_{j\in B_r} S(-j)^{a_{rj}},$$

that is

$$-a-r = -j$$

for some j, i.e.

$$c.deg_V P = j - r$$

for some $j \in B_r$. □

According to this result it makes sense to give the following definition in $\mathbb{P}^r$.

DEFINITION 3.3. – *We define the set of permissible values for the conductor degrees with respect to given graded Betti numbers B and will be denote by S_B, the following set*

$$S_B = \{j - r \mid j \in B_r\}.$$

Following [4], we can see that if B is the set of graded Betti numbers whose corresponding Hilbert function is H, then the set of permissible values for the conductor degree with respect to graded Betti numbers S_B is contained in S_H, i.e.

$$S_B \subseteq S_H.$$

4. – Permissible conductor values for a particular set of distinct points in $\mathbb{P}^r$.

Question: Given a zero-dimensional differentiable 0-sequence $H = \{h_i\}$ with $h_1 \leq r+1$, i.e. a possible Hilbert function for a set of distinct points $\mathbb{X}$ of $\mathbb{P}^r$,

is it possible to find a configuration $\mathbb{Y} \subseteq \mathbb{P}^r$ *with* $HF(\mathbb{Y}) = H$ *such that for every* $d \in S_H$ $\exists$ $P \in \mathbb{Y}$ *with* $d = c.deg_{\mathbb{Y}}\ P$?

In this section we will give an affirmative answer to this question. Let S_r denote the set of all Hilbert functions of finite sets of points in $\mathbb{P}^r$. It is introduced a new character (the r-type vector), which is an alternative to the Hilbert function for the set of points $\mathbb{X}$. This was done by Geramita, Harima, Su Shin ([6]) which showed that this new character is equivalent to the Hilbert function as a tool to describe finite sets of points in $\mathbb{P}^r$. Moreover, they showed how to associate, to any Hilbert function $H \in S_r$, a special point set in $\mathbb{P}^r$ said *k-configuration* which, naturally, has Hilbert function H and is *extremal* with respect to the graded Betti numbers (see [6], Theorem 3.7).

In [12] Theorem 5.11, the author computes the conductor degree of each point of a k-configuration; by this result, we can observe that if $d \in S_H$, considered the truncation to d of the first difference of Hilbert function H, and the k-configuration $\mathbb{X}''$ corresponding to such a truncation one gets

$$\exists P \in \mathbb{X}'' \subseteq \mathbb{X} : c.deg_{\mathbb{X}''}P = d \text{ and } c.deg_{\mathbb{X}''}P = c.deg_{\mathbb{X}}P.$$

This shows that a k-configuration of points of $\mathbb{P}^r$ associated to the given Hilbert function H furnished a positive answer to the question posed at the beginning of this section.

EXAMPLE 4.1. – Let H be the following 0-sequence differentiable 0-dimensional, i.e. a possible Hilbert function of 37 distinct points of $\mathbb{P}^3$

$$H:\ 1\ 4\ 10\ 19\ 26\ 33\ 37\ 37\ 37\ \ldots$$

$$\triangle H:\ 1\ 3\ 6\ 9\ 7\ 7\ 4\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots$$

It could be observed that the set S_H is

$$S_H = \{3, 4, 5, 6\}$$

Now we build a k-configuration of points of $\mathbb{P}^3$, $\mathbb{X}$ with the given Hilbert function H.

For this reason we begin writing the maximal 0-sequence in codimension 2 less than o equal to $\triangle H$ and there exists a scheme of distinct points with the given maximal 0-sequence which will admit all elements in S_H as conductor degrees:

$$\triangle H_1 : \ 1\ 2\ 3\ 4\ 5\ 6\ 4\ 0\ 0\ ...$$

after subtraction we have:

$$D_1 : 1\ 3\ 5\ 2\ 1\ 0\ 0\ 0\ 0...$$

Now, we repeat the argument on D_1 and we get $\triangle H_2$:

$$\triangle H_2 : \ 1\ 2\ 3\ 2\ 1\ 0\ 0\ 0...$$

after subtraction we have:

$$D_2 : 1\ 2\ 0\ 0\ 0\ 0...$$

And again maximal $\triangle H_3$ from D_2 is :

$$\triangle H_3 : \ 1\ 2\ 0\ 0\ ...$$

after subtraction we have:

$$D_3 : 0\ 0\ 0\ ...$$

Consider a partial intersection whose support $\mathcal{A}$ is given by $\mathcal{A}' \cup \mathcal{A}'' \cup \mathcal{A}'''$ where

$$\mathcal{A}' =< (1,1,7),(1,2,6),(1,3,5),(1,4,4),(1,5,2),(1,6,1) >$$

$$\mathcal{A}'' =< (2,1,5),(2,2,3),(2,3,1) >$$

$$\mathcal{A}''' =< (3,1,2),(3,2,1) >$$

It could be observed such a partial intersection can be seen as a k-configuration associated to our Hilbert function.

On the other hand,by [11], the point P corresponding to the element (a_1, a_2, a_3) has conductor degree

$$a_1 + a_2 + a_3 - 3.$$

Then the points corresponding to the generators have conductor degrees in $\mathbb{X}$, respectively

$$6, 6, 6, 6, 5, 5, 5, 4, 3, 3, 3.$$

5. – Conductor sequence of partial intersections of $\mathbb{P}^r$.

In this section, we generalize a result of the work of [12] (Theorem 5.11). Precisely, we determine the degree of each point of any r-partial intersection in $\mathbb{P}^r$, observing that a k-configuration is a particular r-partial intersection.

Let

$$G(\mathcal{A}) = \{H_1, H_2, ..., H_s\}$$

be the minimal set of generators for $\mathcal{A}$ where we denote

$$H_i = (a_{i,1}, ..., a_{i,r})$$

for $1 \leq i \leq s$. In the sequel we will use the above notation and will denote the r-tuple $(1, ..., 1)$ by I.

LEMMA 5.1. – *Let $V \subset \mathbb{P}^r$ be a r-partial intersection with support on the r-left segment $\mathcal{A}$. If*

$$H = (j_1, ..., j_r) \in \mathcal{A}$$

and

$$P_H = \bigcap_{l=1}^{r} r_{hj_h}$$

then

$$c.deg_V\ P_H = max\{\upsilon(K) - r \ \ \forall\ K \in G(\mathcal{A}), K \geq\ H\}$$

PROOF. – Denote

$$T_H = \{H' \in \mathcal{A} |\ H \leq H'\}.$$

and

$$\mathcal{A}' = \{K + I - H \mid \forall\ K \in T_H\}.$$

We show that if $J \in \mathcal{A}'$ and $J' \leq J$ then

$$J' \in \mathcal{A}',$$

thus $\mathcal{A}'$ is a r-left segment with the order induced by the natural order on $\mathbb{N}$. In fact if $J \in \mathcal{A}'$ we can write

$$J = K + I - H$$

where $K \in \mathcal{A}$ and $K \geq H$. Moreover $J' \leq J$, thus exists a r-tuple $(d_1, ..., d_r)$ such that

$$J = J' + (d_1, ..., d_r).$$

So that

$$K + I - H = J' + (d_1, ..., d_r),$$

$$K - (d_1, ..., d_r) + I - H = J'.$$

To show that $J' \in \mathcal{A}'$ it is sufficient to prove that

$$K-(d_1,...,d_r)\geq H,$$

i.e.

$$K-(d_1,...,d_r)+I-H\geq I.$$

This is obvious since it is always true that $J'\geq I$.

Consider the r-partial intersection with respect to r families of hyperplanes of $\mathbb{P}^r$

$$\{b_{1,s}\}_{1\leq s\leq r},\{b_{2,s}\}_{1\leq s\leq r},...,\{b_{r,s}\}_{1\leq s\leq r}$$

where

$$b_{h,k}=r_{h\ i_h+k-1}.$$

We denote V' be the partial intersection of $\mathbb{P}^r$ with support on $\mathcal{A}'$.

Note that

$$P_{\bar{H}}\in V\Leftrightarrow P_{\bar{H}'}\in V',$$

where

$$\bar{H}'=\bar{H}+I-H.$$

Thus our point P_H of V becames

$$P_I\in V'.$$

By Cayley- Bacharach theorem we know that the point P_I has conductor degree (as point of V') such that

$$c.deg_{V'}\ P_I\geq \upsilon(K)-r\ \ \forall\ K\in G(\mathcal{A}').$$

Then we have that

$$c.deg_{V'}\ P_I\geq max\{\upsilon(K)-r\ \ \forall\ K\in G(\mathcal{A}')\}$$

i.e.

$$c.deg_{V'}\ P_I= max\{\upsilon(K)-r\ \ \forall\ K\in G(\mathcal{A}')\}$$

in the partial intersection V'.

Let $\mathcal{M}'$ be such a maximum. This means that exists a separator $\tilde{\mathcal{C}}$ of P_I of minimal degree $\mathcal{M}'$ in our partial intersection V'.

Let

$$M=max\{\upsilon(K)-r\ \forall\ K\in G(\mathcal{A}),K\geq H\}.$$

Our aim is to show that

$$c.deg_V\ P_H=M.$$

To do this, it could be observed that

$$V\supset CI((a_{j_k1},...,a_{j_kr})\ \forall\ k=1,...,t$$

where $CI((a_{j_k1},...,a_{j_kr})$ is the left segment generated by $\{(a_{j_k1},...,a_{j_kr}\}$. This means that for Cayley- Bacharach,

$$c.deg_V\ P_H \geq M.$$

To show that

$$c.deg_V\ P_H = M$$

it is sufficient to find a separator of P_H of degree M.

Now such a separator is

$$C = \tilde{\mathcal{C}} \cdot R_{1,1} \cdot ... \cdot R_{1,i_1-1} \cdot R_{2,1} \cdot ... \cdot R_{2,i_2-1} \cdot R_{r,1} \cdot ... \cdot R_{r,i_r-1}$$

Note that

$$deg\ C = \mathcal{M}' + \upsilon(H) - r = M,$$

as we required. □

6. – Generalities for subschemes of smooth quadrics.

Here we first collect some terminology (see [5] for details). Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be a quadric and let $\mathcal{O}_Q$ be its structure sheaf.

If $D \subset Q$ is any divisor of type (a, b) we denote by

$$\mathcal{O}_Q(a, b)$$

the associated sheaf.

The k-algebra

$$S = \bigoplus_{a,b} H^0(Q, \mathcal{O}_Q(a, b))$$

is the coordinate ring of Q.

For any sheaf $\mathcal{F}$ on Q, we set

$$\mathcal{F}(a, b) = \mathcal{F} \otimes \mathcal{O}_Q(a, b).$$

Note that

$$S = H^0_*(a, b) = \bigoplus_{a,b\geq 0} H^0(a, b)$$

where $H^0_*(a, b) = H^0(Q, \mathcal{O}_Q(a, b))$. S is, in a natural way, a k-algebra using product of sections. It is easy to check that S is generated, as a k-algebra, by $H^0(1, 0)$ and $H^0(0, 1)$ (both vector spaces of dimension 2) since for every $a, b \geq 0$ the map

$$H^0(a, b) \otimes H^0(1, 0) \otimes H^0(0, 1) \to H^0(a + 1, b + 1)$$

given by the product, is surjective.

S is a bi-graded k-algebra taking $H^0(a,b) = S_{(a,b)}$ as the homogeneous component of degree (a,b).

When

$$s \in H^0(a,b)$$

its zero locus $(s)_0$ will be called a curve of type (a,b).

In particular $L = (l)_0$ and $L' = (l')_0$, with $l \in H^0(1,0)$ and $l' \in H^0(0,1)$ will be mentioned as lines of type $(1,0)$ or $(1,0)$-lines, and lines of type $(0,1)$ or $(0,1)$-lines respectively.

When no confusion can arise we will not distinguish between curves and their defining forms.

Let u, u' and v, v' be bases for $H^0(1,0)$ and $H^0(0,1)$; then we have a bi-graded ring isomorphism

$$S \cong k[u,u'] \otimes k[v,v'].$$

We use the above isomorphism to identify elements of S and elements of $k[u,u'] \otimes k[v,v']$. We deal only with bi-homogeneous ideals of S, i.e. ideals generated by elements which are homogeneous both with respect to u, u' and v, v'.

Let $\mathbb{X} \subset Q$ be a 0-dimensional subscheme, i.e. a subscheme associated to a saturated ideal in S of height 2.

In this paper we shall, for simplicity, concentrate on the case when $\mathbb{X}$ consists of distinct points.

We can associate to any 0-dimensional subscheme $\mathbb{X}$ of Q the bi-graded S-algebra

$$S_{\mathbb{X}} = S/I_{\mathbb{X}},$$

where $I_{\mathbb{X}}$ is the homogeneous saturated ideal of $\mathbb{X}$ in S.

By analogy with the definition of Hilbert functions for graded modules, we can define the function

$$M_{\mathbb{X}} : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{N}$$

by

$$M_{\mathbb{X}}(i,j) = dim_k(S_{\mathbb{X}})_{(i,j)} - dim_k(I_{\mathbb{X}})_{(i,j)}$$

where for every bi-graded S-module N we denote by $(N)_{(i,j)}$ the component of N of degree (i,j).

DEFINITION 6.1. – *The function $M_{\mathbb{X}}$ produces a matrix with integer entries,*

$$M_{\mathbb{X}} = (M_{\mathbb{X}}(i,j))$$

which will be called the Hilbert matrix of $\mathbb{X}$.

REMARK 6.2. – Note that

$$M_{\mathbb{X}}(i,j) = 0 \textit{ for } i < 0 \textit{ or } j < 0.$$

So, from now on we restrict ourselves to the range $i \geq 0, j \geq 0$.

When no confusion can arise we will use the notation

$$M_{\mathbb{X}} = (m_{ij}).$$

From the defining exact sequence

$$0 \to \mathcal{I}_{\mathbb{X}} \to \mathcal{O}_Q \to \mathcal{O}_{\mathbb{X}} \to 0$$

taking cohomology we have for $i,j \geq 0$:

$$h^1(\mathcal{I}_{\mathbb{X}}(i,j)) = h^0(\mathcal{I}_{\mathbb{X}}(i,j)) - h^0(i,j) + h^0(\mathcal{O}_{\mathbb{X}}(i,j)) = deg\mathbb{X} - m_{ij}$$

$$h^2(\mathcal{I}_{\mathbb{X}}(i,j)) = 0$$

since

$$h^1(\mathcal{O}_{\mathbb{X}}(i,j)) = 0$$

and in that range $H^1(i,j) = H^2(i,j) = 0$.

It will be useful in the sequel to consider in $\mathbb{Z} \times \mathbb{Z}$ the partial ordering induced by the usual one in $\mathbb{Z}$; we will denote it by $\leq$.

REMARK 6.3. – Thinking of Q as a subvariety of $\mathbb{P}^3$ by the Segre embedding, $\mathbb{X}$ becomes a subscheme of $\mathbb{P}^3$. In this case, if $HF(\mathbb{X}, -)$ is the Hilbert function of $\mathbb{X}$ in $\mathbb{P}^3$, one has

$$HF(\mathbb{X}, i) = m_{ii} \text{ for } i \geq 0.$$

This easily follows taking cohomology of the defining exact sequence of Q in $\mathbb{P}^3$ and of the exact sequence

$$0 \to \mathcal{I}_Q \to \mathcal{I}'_{\mathbb{X}} \to \mathcal{I}_{\mathbb{X}} \to 0$$

where $\mathcal{I}_Q$ and $\mathcal{I}'_{\mathbb{X}}$ are the ideal sheaves of Q and $\mathbb{X}$ in $\mathbb{P}^3$.

Let $M = (m_{ij})$ be a matrix, with $i, j \in \mathbb{Z}$; we will use the following notation: we set

$$\triangle^R M = (a_{ij}), \ \triangle^C M = (b_{ij})$$

for the *matrices of differences by rows and by columns of* M, respectively. Thus we have

$$a_{ij} = m_{i,j} - m_{i,j-1}$$

$$b_{ij} = m_{i,j} - m_{i-1,j}.$$

It is easy to check that

$$\triangle^R(\triangle^C M) = \triangle^C(\triangle^R M).$$

This matrix will be denoted by

$$\triangle M = (c_{ij})$$

and referred to as the first difference matrix of M.

DEFINITION 6.4. – *Let $M' = (m'_{ij})$ be a matrix such that $m'_{ij} = 0$ for $i < 0$ or $j < 0$. We say that M' is admissible when its difference $\triangle M' = (c'_{ij})$ satisfies the following conditions:*

1) $c'_{ij} \leq 1$ *and* $c'_{ij} = 0$ *for* $i \gg 0$ *or* $j \gg 0$*;*

2) *if* $c'_{ij} \leq 0$ *then* $c'_{rs} \leq 0$ *for any* $(r,s) \geq (i,j)$*;*

3) *for every* (i,j)

$$0 \leq \sum_{t=0}^{j} c'_{it} \leq \sum_{t=0}^{j} c'_{i-1,t},$$

and

$$0 \leq \sum_{t=0}^{j} c'_{tj} \leq \sum_{t=0}^{j} c'_{t,j-1}.$$

When M' is an admissible matrix the non-zero part of $\triangle M'$ is contained in a rectangle with opposite vertices $(0,0)$, (a,b) and the elements of the first row (resp. of the first column) are $c'_{0j} = 1$ if $j \leq b$ and $c'_{0j} = 0$ if $j > b$ (resp. $c'_{i0} = 1$ if $i \leq a$, and $c'_{i0} = 0$ if $i > a$). In this case we say M', or $\triangle M'$, has *size (a,b).*

Not every admissible matrix is the Hilbert matrix of some 0-dimensional subscheme of Q.

Let $\mathbb{X} \subset Q$ be a 0-dimensional subscheme and $I_{\mathbb{X}} \subset S$ the saturated ideal of $\mathbb{X}$. Note that

$$1 \leq depth S_{\mathbb{X}} \leq 2:$$

in fact $I_{\mathbb{X}}$ contains an S- sequence of length 2, and in $S_{\mathbb{X}}$ there is a regular element (it is enough to take an element of S which does not vanish at any point of $\mathbb{X}$). Therefore $I_{\mathbb{X}}$ has an S-free minimal resolution of length ≤ 3 with morphisms of degree $(0,0)$.

If this resolution has length 2, $\mathbb{X}$ is called arithmetically Cohen-Macaulay (ACM for short).

PROPOSITION 6.5. – *Let $\mathbb{X} \subset Q$ be a 0-dimensional subscheme and let*

$$0 \to \bigoplus_{i=1}^{p} S(-a_{3i}, -a'_{3i}) \to \bigoplus_{i=1}^{n} S(-a_{2i}, -a'_{2i}) \to \bigoplus_{i=1}^{m} S(-a_{1i}, -a'_{1i}) \to I(\mathbb{X}) \to 0 \quad (1)$$

be the minimal free resolution of $I_{\mathbb{X}}$. *Then we have:*

1) $n+1=m+p$;

2) $\sum_{i=1}^{m} a_{1i} - \sum_{i=1}^{n} a_{2i} + \sum_{i=1}^{p} a_{3i} = \sum_{i=1}^{m} a'_{1i} - \sum_{i=1}^{n} a'_{2i} + \sum_{i=1}^{p} a'_{3i} = 0$;

3) $deg\,\mathbb{X} = -\sum_{i=1}^{m} a_{1i}a'_{1i} + \sum_{i=1}^{n} a_{2i}a'_{2i} - \sum_{i=1}^{p} a_{3i}a'_{3i}$;

4) *for every* $i=1,2,...,m$ *there exists* j $(1\le j\le n)$ *such that*

$$(a_{2j}, a'_{2j}) > (a_{1i}, a'_{1i});$$

5) *The following relations between the given resolution of* $I_{\mathbb{X}}$ *and the matrices* $M_{\mathbb{X}}=(m_{ij})$, $\triangle M_{\mathbb{X}}=(c_{ij})$, $\triangle^2 M_{\mathbb{X}}=(d_{ij})$ *hold:*

$$m_{rs} = (r+1)(s+1) - \sum_{(h,k)\le(r,s)} (r+1-h)(s+1-k)(a_{hk}-\beta_{hk}+\gamma_{hk})$$

$$c_{rs} = 1 - \sum_{(h,k)\le(r,s)} (a_{hk}-\beta_{hk}+\gamma_{hk})$$

$$d_{00}=1,$$

and for every $(r,s)>(0,0)$

$$d_{rs} = -a_{rs}+\beta_{rs}-\gamma_{rs};$$

6) *if* $\triangle M_{\mathbb{X}}$ *is of size* (a,b) *then for every*

$$(i,j)\ge(a+2,b+2)$$

one has $a_{ij}=\beta_{ij}=\gamma_{ij}=0$.

PROOF. – See [5], Proposition 3.3. □

With the notation of resolution (1), we set the following:

$$a_{hk} = \sharp\{(a_{1i}, a'_{1i}) = (h,k)\}$$

$$\beta_{hk} = \sharp\{(a_{2i}, a'_{2i}) = (h,k)\}$$

$$\gamma_{hk} = \sharp\{(a_{3i}, a'_{3i}) = (h,k)\}$$

As we know not every 0-dimensional subscheme $\mathbb{X}\subset Q$ is ACM. There is a very simple characterization of the *ACM* 0-dimensional subschemes of Q in terms of their Hilbert Matrix.

An admissible matrix M' will be called an *ACM matrix* if $\triangle M'$ has only nonnegative entries.

If an ACM matrix M' of size (a,b) is such that $\triangle M'$ has entries

$$c'_{ij}=1$$

for every $(i, j) \le (a, b)$, it is trivial to verify that M' is the Hilbert matrix of a complete intersection of type

$$(a+1, 0), \ (0, b+1).$$

Let M' be an ACM matrix of size (a, b).

We say that (i, j) is a *corner* for $\triangle M'$ if

$$(i, j) = (0, b+1)$$

or

$$(i, j) = (a+1, 0)$$

or even if $c'_{ij} = 0$ and $c'_{i-1,j} = c'_{i,j-1} = 1$. We say that (i, j) is a *vertex* for $\triangle M'$ if

$$c'_{i-1,j} = c'_{i,j-1} = 0$$

and

$$c'_{i-1,j-1} = 1;$$

in this case, of course, $c'_{ij} = 0$.

One can check for an ACM matrix M' that the entries of $\triangle^2 M' = (d'_{ij})$ are:

$$d'_{ij} = \begin{cases} 1 & \textit{if } (i,j) = (0,0) \textit{ or } (i,j) \textit{ is a vertex} \\ -1 & \textit{if } (i,j) \textit{ is a corner} \\ 0 & \textit{otherwise} \end{cases}$$

Recall that $\mathbb{X} \subset Q$ is an ACM 0-dimensional subscheme if and only if the minimal free resolution of $I_{\mathbb{X}}$ is of type (1) with $\gamma_{ij} = 0$ for all (i, j).

THEOREM 6.6. – *Let $\mathbb{X} \subset Q$ be a 0-dimensional subscheme, and let $M_{\mathbb{X}}$ be its Hilbert matrix. $\mathbb{X}$ is an ACM scheme if and only if $M_{\mathbb{X}}$ is an ACM matrix. Furthermore, in this case, the minimal free resolution of $I_{\mathbb{X}}$ looks like*

$$0 \to \bigoplus_{i=1}^{m-1} S(-a_{2i}, -a'_{2i}) \to \bigoplus_{i=1}^{m} S(-a_{1i}, -a'_{1i}) \to I_{\mathbb{X}} \to 0$$

where (a_{2i}, a'_{2i}) runs over all the vertices and (a_{1i}, a'_{1i}) runs over all the corners of $\triangle M_{\mathbb{X}}$.

PROOF. – See [5], Proposition 4.1. □

Note that the Hilbert matrix of an ACM 0-dimensional subscheme of Q completely determines the graded Betti numbers of this ideal sheaf, although this is not true for 0-dimensional subscheme of $\mathbb{P}^n$.

As it is known (see [5] Example 2.14), not every admssible matrix is the Hilbert matrix of some 0-dimensional subschemes of Q, but every ACM matrix is the Hilbert matrix of an ACM subscheme of Q.

7. – Separating degrees for 0-dimensional subschemes of Q: the ACM case.

We are ready to generalize the definitions given for points in a projective space to the case of points on a smooth quadric. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric, and $S = k[u, u'] \times k[v, v']$ its coordinate ring.

DEFINITION 7.1. – *Let $\mathbb{X} \subset Q$ be a 0-dimensional subscheme.*
We say that a form $f \in S$ is a separator for $P \in \mathbb{X}$ if $f(P) \neq 0$ and $f(Q) = 0 \ \forall \ Q \in \mathbb{X} \setminus \{P\}$.

The set of minimal bi-degrees of separators for P, *with respect to the partial ordering induced by the usual one in $\mathbb{Z}$, is called the* separating degrees of P in $\mathbb{X}$*; we denote it by*

$$s.deg_{\mathbb{X}}P.$$

We should be observe that the cardinality of this set is not always one: this is a very big difference with the conductor degree of P in a scheme of distinct points of $\mathbb{P}^r$.

THEOREM 7.2. – *(Cayley-Bacharach on Q). Let*

$$\mathbb{Y} = C.I.((a, 0), (0, b)) \subset Q$$

be a complete intersection on Q and let P be a point of $\mathbb{Y}$ and $\mathbb{Y}' = \mathbb{Y} \setminus \{P\}$ for $P \in \mathbb{Y}$.

Then

$$\triangle M_{\mathbb{Y}'} = \begin{cases} \triangle M_{\mathbb{Y}}(i, j) - 1 & \text{if } (i, j) = (a - 1, b - 1) \\ \triangle M_{\mathbb{Y}} & \text{otherwise} \end{cases}$$

PROOF. – The proof is trivial. □

Before stating the following result we need some terminology. Let $\mathcal{A}$ be a 2-left segment

$$\mathcal{A} =< (a_1, b_1), ..., (a_{n-1}, b_{n-1}) >$$

with

$$a_1 < a_2 < ... < a_{n-1}$$

and

$$b_1 > b_2 > ... > b_{n-1}.$$

Consider

$$\{L_i\}_{1 \leq i \leq a_{n-1}}$$

which are a_{n-1} lines of type $(1, 0)$ and

$$\{R_j\}_{1 \leq j \leq b_1}$$

b_1 $(0,1)$-lines. For every $H = (i, j) \in \mathcal{A}$ we denote

$$P_{ij} = L_i \cap R_j.$$

With this notation

$$\mathbb{X} = \bigcup_{H \in \mathcal{A}} P_{ij} \subset Q$$

is an *ACM 0-dimensional subscheme with support on* $\mathcal{A}$. (Indeed every 0-dimensional ACM subscheme of Q can be obtained in this way).

Then every pair

$$H = (i, j) \in \mathcal{A}$$

will determine two positive integers h and k such that

$$a_1 < ... < a_{h-1} < i \leq a_h < ... < a_{n-1}$$

and

$$b_1 > ... > b_{k-1} > j \geq b_k > ... > b_{n-1}.$$

REMARK 7.3. – With the above notation, note that $h \leq k$. Namely, since $i \leq a_h$ thus $j \leq b_h$; so by $j \geq b_k$ we get

$$b_k \leq j \leq b_h \ \Rightarrow h \leq k.$$

PROPOSITION 7.4. – *Let* $\mathbb{X} \subset Q$ *be an ACM 0-dimensional subscheme with support on*

$$\mathcal{A} =< (a_1, b_1), ..., (a_{n-1}, b_{n-1}) >$$

then the separating degrees of its points is given by

$$s.deg_{\mathbb{X}}\ P_{ij} = \{(a_k - 1, b_h - 1)\}$$

$\forall (i,\ j) \in \mathcal{A}$.

PROOF. – Take P_{ij} in the ACM 0-dimensional subscheme $\mathbb{X} \subset Q$. Now let h and k be the two integers described above and let

$\mathbb{Y}_h = C.I.((a_h, 0), (0, b_h))$

$\mathbb{Y}_{h+1} = C.I.((a_{h+1}, 0), (0, b_{h+1}))$

...

$\mathbb{Y}_k = C.I.((a_k, 0), (0, b_k))$.

Thus, in particular the point P_{ij} of the ACM 0-dimensional subscheme $\mathbb{X}$ is a point of the complete intersections $\mathbb{Y}_h, \mathbb{Y}_{h+1}, ..., \mathbb{Y}_k$.

By the Cayley- Bacharach theorem 7.2 we know that

$$s.deg_{\mathbb{Y}_h}\ P_{ij} = \{(a_h - 1, b_h - 1)\}$$

$$s.deg_{\mathbb{Y}_{h+1}}\ P_{ij} = \{(a_{h+1} - 1, b_{h+1} - 1)\}$$

$$\dots$$

$$s.deg_{\mathbb{Y}_k}\ P_{ij} = \{(a_k - 1, b_k - 1)\}.$$

Consider

$$(a_k - 1, b_h - 1),$$

from the previous equalities we have that any element of $s.deg_{\mathbb{X}}\ P_{ij}$ is $\geq (a_k - 1, b_h - 1)$. Now to show that

$$s.deg_{\mathbb{X}}\ P_{ij} = \{(a_k - 1, b_h - 1)\},$$

it is sufficient to find a separator for P_{ij} of bi-degree $(a_k - 1, b_h - 1)$. Now such a separator is given by

$$C = R_{1,0} \cdot \ldots \cdot R_{i-1,0} \cdot \widehat{R_{i,0}} \cdot R_{i+1,0} \cdot \ldots \cdot R_{k,0} \cdot R_{0,1} R_{0,2} \cdot \ldots \cdot R_{0,j-1} \cdot \widehat{R_{0,j}} \cdot R_{0,j+1} \cdot \ldots \cdot R_{0,h}$$

one notes that

$$deg\ C = (a_k - 1, b_h - 1)$$

as required.

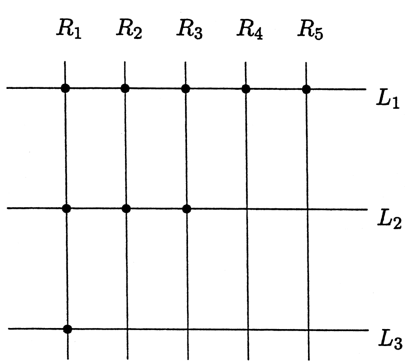
COROLLARY 7.5. – *Let $\mathbb{X} \subset Q$ be an ACM 0-dimensional subscheme then*

$$|s.deg_{\mathbb{X}}\ P| = 1$$

for each point P of $\mathbb{X}$.

Then Corollary 7.5 states that the definition of separating degrees of a point of an ACM 0-dimensional subscheme of Q looks like the definition given for 0-dimensional subschemes of $\mathbb{P}^r$.

EXAMPLE 7.6. – Let us consider the following diagram, when L_i are lines of type $(1,0)$ on Q and R_j lines of type (0,1).



The 0-dimensional ACM scheme $\mathbb{X}$ represented by the marked points has Hilbert matrix

M=

	0	1	2	3	4	5	...
0	1	2	3	4	5	5	
1	2	4	6	7	8	8	
2	3	5	7	8	9	9	
3	3	5	7	8	9	9	
4	3	5	7	8	9	9	
5	...	...	...	...	...	...	

Let

$$P_{ij} = L_i \cap R_j$$

be the points of $\mathbb{X}$.

If, for example we consider the scheme $\mathbb{X}' = \mathbb{X} \setminus P_{23}$, then its matrix is

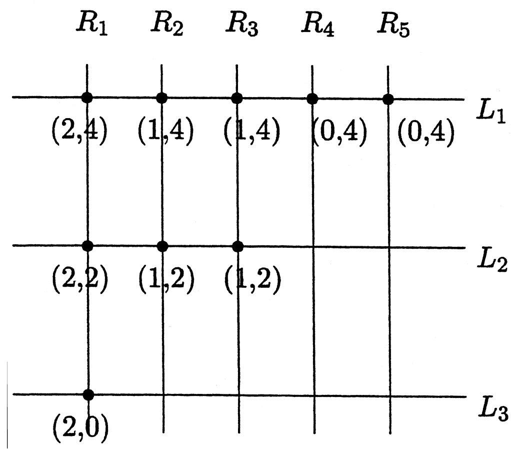
M'=

	0	1	**2**	3	4	5	...
0	1	2	3	4	5	5	
1	2	4	(5)	**6**	**7**	**7**	
2	3	5	**6**	**7**	**8**	**8**	
3	3	5	**6**	**7**	**8**	**8**	
4	3	5	**6**	**7**	**8**	**8**	
5	...	...	...	...	...	...	

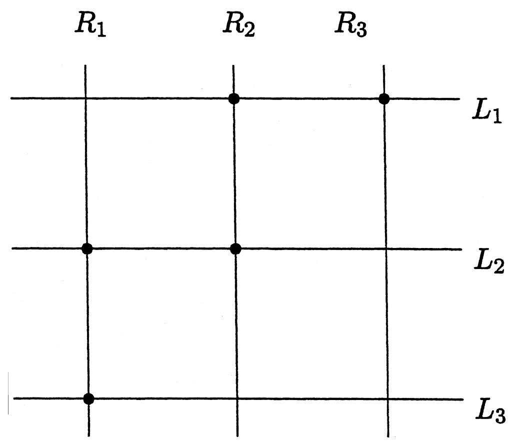
We see that

$$s.deg_{\mathbb{X}} P_{23} = \{(1,2)\}.$$

In the following diagram we give, for each point, a pair of numbers that represents that its separating degrees.



EXAMPLE 7.7. – In this example we consider an $\mathbb{X}$ non *ACM* 0-dimensional scheme. We see that the set $s.deg_{\mathbb{X}}P$ of particular points P of $\mathbb{X}$ has (at least) two pairs of numbers. Let us consider the following diagram, with the above notation



The 0-dimensional ACM scheme $\mathbb{X}$ represented by the marked points has Hilbert matrix

	0	1	2	3	...
0	1	2	3	3	
1	2	4	5	5	
2	3	5	5	5	
3	3	5	5	5	

Let

$$P_{ij} = L_i \cap R_j$$

be the points of $\mathbb{X}$.

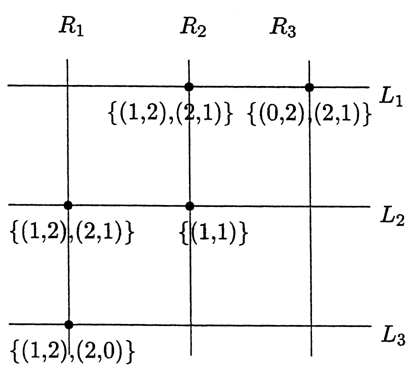
If, for example we consider the scheme $\mathbb{X}' = \mathbb{X} \setminus P_{13}$, then its matrix is

	0	1	2	3	...
0	1	2	**2**	**2**	
1	2	4	**4**	**4**	
2	3	**4**	**4**	**4**	
3	3	**4**	**4**	**4**	

We see that

$$s.deg_{\mathbb{X}} P_{13} = \{(0,2),(2,1)\}.$$

In the following diagram we give the set of separating degrees of each point in $\mathbb{X}$



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Department of Mathemathics, University of Catania,
Viale A. Doria 6, 95125 Catania, Italy
E-mail address: lmarino@dmi.unict.it