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Local Regularity of Solutions to Quasilinear Subelliptic Equations in Carnot Caratheodory Spaces.

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Sunto. – *In questa nota proviamo la disuguaglianza di Harnack per le soluzioni deboli di una equazione sub-ellittica quasilineare del tipo*

$$(*) \qquad \sum_{j=1}^{m} X_j^* A_j(x, u(x), Xu(x)) + B(x, u(x), Xu(x)) = 0,$$

dove $X_1, \dots, X_m$ *denotano un sistema non commutativo di campi vettoriali localmente lipschitziani. Come conseguenza otteniamo la continuità delle soluzioni deboli della* $(*)$.

Summary. – *We prove Harnack inequality for weak solutions to quasilinear subelliptic equation of the following kind*

$$(*) \qquad \sum_{j=1}^{m} X_j^* A_j(x, u(x), Xu(x)) + B(x, u(x), Xu(x)) = 0,$$

where $X_1, \dots, X_m$ *are a system of non commutative locally Lipschitz vector fields. As a consequence, the weak solutions of* $(*)$ *are continuous.*

1. – Introduction.

The regularity properties of weak solutions to many kind of PDE's have been very well studied in the past decades (see e. g. [13], [12], [10], [16], [17], [18], [11] and [20]). To better introduce the problem we are going to study, we outline some previous results in the literature. Let us consider the following quasilinear elliptic equation

$$\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = 0 \tag{1.1}$$

where A and B are measurable functions satisfying the structure assumptions

$$\begin{cases} |A(x,u,\xi)| \leq a|\xi|^{p-1} + b|u|^{p-1} + e \\ |B(x,u,\xi)| \leq c|\xi|^{p-1} + d|u|^{p-1} + f \\ \xi A(x,u,\xi) \geq |\xi|^{p} - d|u|^{p} - g \end{cases} \tag{1.2}$$

for a.e. $x \in \Omega \subset \mathbb{R}^n$, $\forall u \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^n$.

Regularity properties of weak solutions to (1.1), in the case when the lower order terms belong to Lebesgue classes, were settled out in the celebrated work [14] (see also the classic book [9]). However, it is easy to realize that Lebesgue classes are not natural to study regularity properties as continuity or Hölder continuity. In this direction, the results in [14] and [9] have been greatly improved (see e.g. [12] and [13]) assuming the lower order terms in some classes of Morrey type which have been found to be the natural classes were to put the lower order terms in order to obtain hölder continuity of the weak solutions. Indeed, hölder continuity of weak solutions is proved for equations of the kind

$$\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = \mu$$

where μ is an appropriate Radon measure and, following the classical pattern of [9], the L^p assumptions have been replaced by the strictly more general Morrey type conditions.

In [18], one of us, using a technique inspired by [14], gave some improvements of the results contained in [13] and [12]. In [18] an invariant Harnack inequality and the local hölder continuity for weak solutions of equation (1.1) were proved. We stress that the assumptions in [18] are sharp in the Morrey scale.

It is worthwhile to recall [10], where, assuming the a priori boundedness of the solutions, Harnack inequality and hölder continuity for the solutions of (1.1) are proved. In [10] the Author considered structural assumptions which are more general than those in [18], at least in some instances.

All the above mentioned papers are devoted to the study of the hölder continuity of the solutions of equation (1.1). We like to recall the work [20] (see also [11]) where the coefficients in (1.2) are taken in a suitable class of functions of Stummel-Kato type. These classes properly contain the Morrey spaces where the coefficients in the above listed works are assumed to belong. In [20], following the pattern of [14], Harnack inequality is proved for the non negative solutions of (1.1) and, as a consequence, it is obtained the continuity of the solutions of (1.1). A crucial role in [20] is played by a Fefferman-Phong embedding, established in [19], used to estimate products of coefficients times test functions. The reader is referred to the enlightening paper of F. Chiarenza [3] for further details.

Very recently, this kind of results have been generalized to the setting of Carnot – Caratheodory spaces. These are metric spaces whose distance is generated by the sub-unit curves associated to a system of non-commuting vector fields. The non-commutativity of the vector field is one of the main difficulties in this new context. In this direction we quote the papers [2], [5], [6] , [8] and also the bibliography therein, where the above mentioned problems are studied in the subelliptic context.

In order to state the results we need some notations. Let us consider a system $X = (X_1, \dots, X_m)$ of non commutative vector fields in an open set $\Omega \subseteq \mathbb{R}^n$ with locally Lipschitz continuous coefficients. We write

$$X_j = \sum_{k=1}^{n} b_{jk} \frac{\partial}{\partial x_k}, \quad b_{jk} \in Lip_{loc}(\Omega) \quad j = 1, \ldots, m, \;\; k = 1, \ldots, n,$$

and denote by $X_j^* = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(b_{jk} \;\; \cdot)$ its formal adjoint. Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a function. We set $Xu = (X_1 u, \ldots, X_m u)$ and $|Xu| = \left(\sum_{j=1}^{m} (X_j u)^2 \right)^{\frac{1}{2}}$ where, as usual, $X_j u(x) = \langle X_j, \nabla u(x) \rangle$ identifying the X_j's with the first order differential operator that acts on $u \in Lip(\Omega)$ via the above formula.

In the sequel we set

$$\mathcal{L}^{1,p}(\Omega) = \{ u \in L^p(\Omega) \; : \; X_j u \in L^p(\Omega), \; j = 1, \ldots, m \}, \quad 1 \leq p < \infty,$$

and for any $u \in \mathcal{L}^{1,p}(\Omega)$

$$\|u\|_{1,p} \equiv \|u\|_p + \| \, |Xu| \, \|_p. \tag{1.3}$$

The completion of the set $C_0^\infty(\Omega)$ with respect to the norm (1.3) will be denoted by $S_0^{1,p}(\Omega)$.

A piecewise C^1 curve $\gamma : [0, T] \to \mathbb{R}^n$ is called sub-unit, with respect to the system X, if whenever $\gamma'(t)$ exists one has

$$\langle \gamma'(t), \xi \rangle^2 \leq \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2 \qquad \forall \xi \in \mathbb{R}^n.$$

The sub-unit lenght of γ is by definition $l_S(\gamma) = T$. Given $x, y \in \mathbb{R}^n$, we denote by $\Phi(x, y)$ the collection of all sub-unit curves connecting x to y. Then $d(x, y) = \inf\{ l_S(\gamma) : \gamma \in \Phi(x, y) \}$ defines a distance, usually called the Carnot Caratheodory distance generated by the system X. We will denote $B(x, r) = \{ y \in \mathbb{R}^n : d(x, y) < r \}$ the metric ball centered at x of radius r and whenever x is not relevant we shall write B_r. We shall denote with $d_e(x, y) = |x - y|$ the Euclidean distance in $\mathbb{R}^n$.

We now introduce the relevant quantitative assumptions we will need in the sequel.

(A1). *$i : (\mathbb{R}^n, d_e) \to (\mathbb{R}^n, d)$ is continuous.*
(A2). (Doubling condition). – *For every bounded set $\Omega \subset \mathbb{R}^n$ there exist costants C_D, $R_D > 0$ such that for $x_0 \in \Omega$ and $0 < r < R_D$ one has*

$$|B(x_0, 2r)| \leq C_D |B(x_0, r)|.$$

(A3). (Weak-L^1 Poincarè type inequality). – *Given Ω as in* (A2), *there exist positive constants C_P and $a \geq 1$ such that for any $x_0 \in \Omega$, $0 < r < R_D$ and*

$u \in C^1(B(x_0, ar))$, one has

$$\sup_{\lambda>0} [\lambda|\{x \in B(x_0, r) \ : \ |u(x) - u_{B(x_0,r)}| > \lambda\}|] \leq C_P R \int_{B(x_0,ar)} |Xu| dx,$$

where $u_{B(x_0,r)}$ denotes the integral average $|B(x_0, r)|^{-1} \int_{B(x_0,r)} u(y)\, dy$.

Finally we put

$$Q = \log_2 C_D,$$

and we shall call it the homogeneous dimension of Ω.

Let Ω be a bounded open subset of $\mathbb{R}^n$ with homogeneous dimension Q. The equation we consider has the form

$$\sum_{j=1}^{m} X_j^* A_j(x, u(x), Xu(x)) + B(x, u(x), Xu(x)) = 0, \tag{1.4}$$

where the functions

$$A(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$$

and

$$B(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$$

satisfy the structural assumptions (1.2) with p a fixed number in $]1, Q[$.

Concerning the regularity properties of the solutions of (1.4) we recall the paper [2] where Harnack inequality and local hölder continuity are proved in the case when the vector fields X satisfy Hörmander condition and the coefficients are taken in suitable Lebesgue spaces.

Later, in [5] e [6] similar results are obtained in the setting of Carnot Caratheodory spaces with more general assumptions involving Morrey spaces related to Carnot Caratheodory metric (see Definition 1.4). However, their assumptions do not recover the sharper conditions of the classical euclidean case (see e. g. [18]). Despite of this we stress that the assumptions in [8] give back the classical sharp ones. To introduce our assumptions we state the following definitions.

DEFINITION 1.1. (Stummel-Kato class). – *Let $V \in L^1_{loc}(\Omega)$, $r > 0$ and $p \in\ ]1, Q[$. We set*

$$\phi_V(r) \equiv \sup_{x\in\Omega} \left(\int_{B(x,r)\cap\Omega} \frac{d(x,y)}{|B(x,d(x,y))|} \left(\int_{B(x,r)\cap\Omega} |V(z)| \frac{d(z,y)}{|B(z,d(z,y))|} dz \right)^{\frac{1}{p-1}} dy \right)^{p-1}.$$

We say that a function $V \in L^1_{loc}(\Omega)$ belongs to the space $(M_X)_p(\Omega)$ if and only if $\phi_V(r)$ is finite for any $r > 0$ and, in addition, $\lim_{r\to 0+} \phi_V(r) = 0$.

We will call $\phi_V(r)$ the $(M_X)_p - modulus$ of V. It is not difficult to prove that $\phi_V(r)$ is a continuous function.

REMARK 1.2. – We wish to point out that the function $\phi_V(r)$ satisfies the doubling property, i.e.

$$\exists C > 1 : \phi_V(2r) \leq C\phi_V(r), \quad r > 0.$$

DEFINITION 1.3. – *Let* $1 < p < Q$. *We say that* $V \in (M_X)_p(\Omega)$ *belongs to the class* $(M_X)'_p(\Omega)$ *if*

$$\exists \delta > 0 : \int_0^\delta \frac{\phi_V(t)^{\frac{1}{p}}}{t} dt < +\infty.$$

REMARK 1.4. – We wish to point out that in the euclidean case $X_j = \partial_{x_j}$, then $(M_X)_p$ gives back the class M_p introduced in [20], which, for $p = 2$, coincides with the classical Stummel Kato class.

Referring to (1.2) we assume that $a = constant$ and $b(x)$, $c(x)$, $d(x)$, $e(x), f(x)$ and $g(x)$ are measurable functions such that

$$b^{\frac{p}{p-1}}(x),\ e^{\frac{p}{p-1}}(x),\ c^p(x),\ d(x), f(x),\ g(x) \in (M_X)'_p(\Omega). \tag{1.5}$$

Following the classical Moser's iteration technique, as adapted by J. Serrin in [14] to the quasilinear case, with the aid of the Fefferman-Phong type inequality (Theorem 2.1) and related Corollary , we are able to prove the local boundedness of the solutions (Theorem 3.1) and the Harnack inequality for non negative solutions of equation (1.4) (Theorem 3.2). As a consequence of the Harnack inequality we obtain the continuity of the solutions of (1.4) (Theorem 3.3).

Now, in order to compare our hypotheses with those in our previous paper [8], we need one more definition.

DEFINITION 1.5. (Morrey spaces) – *Let* $p \in [1, +\infty[$. *We say that* $V \in L^p_{loc}(\Omega)$ *belongs to* $L^{p,\lambda}_X(\Omega)$, *for some* $\lambda > 0$, *if*

$$\|V\|_{L^{p,\lambda}_X(\Omega)} = \sup_{\substack{x\in\Omega \\ 0<r<d_0}} \left(\frac{r^\lambda}{|B(x,r)\cap\Omega|} \int_{B(x,r)\cap\Omega} |V(y)|^p \, dy \right)^{\frac{1}{p}} < +\infty,$$

where $d_0 = \min(\text{diam}(\Omega), R_D)$.

REMARK 1.6. – It is trivial that in the euclidean case $L^{p,\lambda}_X$ gives back the classical Morrey space $L^{p,n-\lambda}$.

In [8] we assumed

$$b^{\frac{p}{p-1}}(x),\ e^{\frac{p}{p-1}}(x),\ c^{p}(x),\ d(x), f(x),\ g(x) \in L_X^{1,p-\varepsilon}(\Omega), \qquad \text{for some } \varepsilon > 0. \tag{1.6}$$

We can give a comparison between the classes introduced above. Indeed,

PROPOSITION 1.7. – *Let* $1 < p < Q$ *and* $0 < \varepsilon < p$. *If* $V \in L_X^{1,p-\varepsilon}(\Omega)$ *we have*

$$\phi_V(r) \leq C(C_D, p, \varepsilon)\|V\|_{L_X^{1,p-\varepsilon}} r^{\varepsilon}$$

for any $0 < r < R_D$ *and then*

$$L_X^{1,p-\varepsilon}(\Omega) \subseteq (M_X)'_p(\Omega).$$

PROOF. – Setting

$$A(r_1, r_2, x) = \{y \in \Omega \ : \ r_1 \leq d(x,y) < r_2\},$$

we have

$$\int\limits_{\{y\in\Omega\,:\,d(x,y)<r\}} \frac{d(x,y)}{|B(x,d(x,y))|}\left(\int\limits_{\{z\subset\Omega\,:\,d(z,x)<r\}} |V(z)|\,\frac{d(z,y)}{|B(z,d(z,y))|}\,dz\right)^{\frac{1}{p-1}} dy$$

$$= \sum_{j=1}^{+\infty} \int\limits_{A(\frac{r}{2^{j+1}},\frac{r}{2^j},x)} \frac{d(x,y)}{|B(x,d(x,y))|}\left(\sum_{k=0}^{j}\int\limits_{A(\frac{r}{2^{j-k+1}},\frac{r}{2^{j-k}},x)} |V(z)|\,\frac{d(z,y)}{|B(z,d(z,y))|}\,dz\right)^{\frac{1}{p-1}} dy$$

$$\leq C(C_D,p)\sum_{j=1}^{+\infty} \int\limits_{A(0,\frac{r}{2^j},x)} \frac{\frac{r}{2^j}}{|B(x,\frac{r}{2^j})|}\left(\sum_{k=0}^{j}\frac{\frac{r}{2^{j-k}}}{|B(x,\frac{r}{2^{j-k}})|}\int\limits_{A(0,\frac{r}{2^{j-k}},x)} |V(z)|\,dz\right)^{\frac{1}{p-1}} dy$$

$$\leq C(C_D,p)\|V\|^{\frac{1}{p-1}}_{L_X^{1,p-\varepsilon}} \sum_{j=1}^{+\infty}\frac{r}{2^j}\left(\sum_{k=0}^{j}\frac{r}{2^{j-k}}^{\varepsilon-p+1}\right)^{\frac{1}{p-1}}$$

$$= C(C_D,p,\varepsilon)\|V\|^{\frac{1}{p-1}}_{L_X^{1,p-\varepsilon}} r^{\frac{\varepsilon}{p-1}}. \qquad \square$$

2. – Preliminary results.

This section contains some useful tools to obtain our results.

THEOREM 2.1. – (see [7]) *Suppose* (A1)-(A3) *hold true, and consider a bounded open set* $\Omega \subset \mathbb{R}^n$, *with homogeneous dimension* Q. *Let* $1 < p < Q$. *Let*

$V \in (M_X)_p(\Omega)$. Then it exists a positive constant c independent of u such that

$$\int_\Omega |V(x)|\,|u(x)|^p\,dx \le c\phi_V(2r)\int_\Omega |Xu(x)|^p\,dx$$

for any $u \in S_0^{1,p}(\Omega)$ supported in B_r.

The next corollary is an easy consequence of the previous Theorem. It can be obtained via a standard partition of unity (see e.g. [7]).

COROLLARY 2.2. – *Under the same assumptions of Theorem* 2.1 *we have that for any $\sigma > 0$ there exists a positive function $K(\sigma) \sim \frac{\sigma}{\left[\phi_V^{-1}(\sigma)\right]^{Q+p}}$ such that*

$$\int_\Omega |V(x)|\,|u(x)|^p\,dx \le \sigma\int_\Omega |Xu(x)|^p\,dx + K(\sigma)\int_\Omega |u(x)|^p\,dx,$$

for all $u \in S_0^{1,p}(\Omega)$.

The following lemma will be useful during the iteration procedure.

LEMMA 2.3. – *Let $\mu(r)$ a continuous positive increasing function defined in $]0,+\infty[$ such that $\lim_{r\to 0}\mu(r) = 0$, $0 < \theta < 1$.*

The series

$$\sum_{i=0}^{+\infty} \theta^i \log \mu^{-1}(\theta^{qi}),$$

where $q > 0$, is convergent if and only if there exists $\rho > 0$ such that

$$\int_0^\rho \frac{\mu^{\frac{1}{q}}(t)}{t}\,dt < +\infty.$$

PROOF. – We claim that

$$\int_0^\rho \frac{\mu^{\frac{1}{q}}(t)}{t}\,dt < +\infty$$

if and only if the serie

$$\sum_{i=0}^{+\infty} (\theta a_i - a_{i+1})$$

is convergent, where

$$a_i = \theta^i \log \mu^{-1}(\mu(\rho)\theta^{qi}).$$

Indeed

$$\int_0^{\rho} \frac{\mu^{\frac{1}{q}}(t)}{t}\,dt = \int_0^{\mu(\rho)} \frac{s^{\frac{1}{q}}}{\mu^{-1}(s)} \frac{1}{\mu'(\mu^{-1}(s))}\,ds$$

$$= \sum_{i=0}^{+\infty} \int_{\mu(\rho)\theta^{q(i+1)}}^{\mu(\rho)\theta^{qi}} \frac{s^{\frac{1}{q}}}{\mu^{-1}(s)} \frac{1}{\mu'(\mu^{-1}(s))}\,ds$$

$$< \sum_{i=0}^{+\infty} \Big\{ \mu^{\frac{1}{q}}(\rho)\theta^{i} \log \mu^{-1}\{(\mu(\rho)\theta^{qi})$$

$$- \frac{1}{\theta} \mu^{\frac{1}{q}}(\rho)\theta^{i+1} \log \mu^{-1}\big(\mu(\rho)\theta^{q(i+1)})\big) \Big\}$$

$$= \frac{\mu^{\frac{1}{q}}(\rho)}{\theta} \sum_{i=0}^{+\infty} (\theta a_i - a_{i+1}).$$

In analogous way it is shown that

$$\int_0^{\rho} \frac{\mu^{\frac{1}{q}}(t)}{t}\,dt > \sum_{i=0}^{+\infty} (\theta a_i - a_{i+1})$$

and the claim is proved.

Then, since the two series $\sum_{i=0}^{+\infty} (\theta a_i - a_{i+1})$ and $\sum_{i=0}^{+\infty} a_i$ have the same behaviour, the conclusion is obtained. □

The following lemma will play a crucial role to obtain the continuity of the solutions of (1.4).

LEMMA 2.4. – (see [11]) *Let* $0 < \gamma < 1$, $h :]0, +\infty[\to]0, +\infty[$ *a non decreasing function with* $\lim_{t \to 0} h(t) = 0$, *such that*

$$h(t) \le C h(t/2) \qquad (C > 1) \tag{2.1}$$

and $\omega :]0, +\infty[\to]0, +\infty[$ *a non decreasing function.*

If

$$\omega(\rho) \le \gamma \omega(4\rho) + h(\rho) \qquad \forall \rho < \rho_0 < 1 \tag{2.2}$$

then there exist $\overline{\rho} \le \rho_0$, $0 < \sigma \le 1$ *and a positive constant* K *such that*

$$\omega(\rho) \le K h^{\sigma}(\rho) \qquad \forall \rho < \overline{\rho}.$$

PROOF. – Let $\tilde{\rho} > 0$ be such that $h(\rho) < 1$ for $\rho < \tilde{\rho}$. Set $\overline{\rho} = \min(\tilde{\rho}, \rho_0)$, we choose $R > 0$ such that $R < \overline{\rho}$.

If $\rho \in \left[\frac{R}{4}, R\right]$, letting

$$M = \sup_{[R/4,R]} \frac{\omega(\rho)}{h(\rho)},$$

we have

$$\omega(\rho) \leq Mh(\rho). \tag{2.3}$$

If $\rho \in \left[\frac{R}{4^2}, \frac{R}{4}\right]$, by (2.2) and (2.3) we have

$$\omega(\rho) \leq \gamma Mh(4\rho) + h(\rho) \leq \gamma Mh^\sigma(4\rho) + h^\sigma(\rho)$$

for every $\sigma : 0 < \sigma \leq 1$. Fixing σ such that

$$\gamma C^{2\sigma} = a < 1$$

where C is the constant in (2.2), we obtain

$$\omega(\rho) \leq (Ma + 1)h^\sigma(\rho).$$

Iterating this procedure, if $\rho \in \left[\frac{R}{4^{i+1}}, \frac{R}{4^i}\right]$, we have

$$\omega(\rho) \leq \left(Ma^i + \sum_{k=0}^{i-1} a^k\right)h^\sigma(\rho) \leq \left[M + \frac{1}{1-a}\right]h^\sigma(\rho),$$

and the conclusion is obtained with $K = M + \frac{1}{1-a}$. □

Finally we recall the following definition.

DEFINITION 2.5. – *A function $u \in \mathcal{L}^{1,p}_{loc}(\Omega)$ is said to be a weak solution of* (1.4) *in Ω if*

$$\sum_{j=1}^{m} \int_\Omega A_j(x, u(x), Xu(x))X_j\varphi(x)\,dx + \int_\Omega B(x, u(x), Xu(x))\varphi(x)\,dx = 0 \tag{2.4}$$

for every $\varphi \in S^{1,p}_0(\Omega)$.

Definition 2.5 is meaningful by Theorem 2.1.

3. – Main results.

The purpose of this section is to provide an Harnack's inequality for non negative weak solutions of equation (1.4) and, as a consequence, to obtain the continuity of its solutions. We will follow the classical Moser's iteration technique, as adapted by J. Serrin in [14] to the quasilinear case.

We begin showing that weak solutions of equation (1.4) are locally bounded.

THEOREM 3.1. – *Suppose* (A1)-(A3) *hold true. Let* Ω *be a bounded open set with local homogeneous dimension* Q *and* $u \in \mathcal{L}^{1,p}_{loc}(\Omega)$, *with* $1 < p < Q$, *be a weak solution of* (1.4). *Assume that conditions* (1.2) *and* (1.5) *hold. Then, there exists a positive constant* C, *independent of* u, *such that, for any* $B_r = B(x_0, r)$ *for which* $B(x_0, 4r) \subset \Omega$ *and* $r < R_D$, *we have*

$$\|u\|_{L^\infty(B_r)} \leq C\left\{ \left(\frac{1}{|B_r|} \int_{B_{2r}} |u|^p \, dx \right)^{\frac{1}{p}} + h(r) \right\}$$

where

$$h(r) = \left[\phi_{e^{\frac{p}{p-1}}}(2r) + \phi_g(2r)\right]^{\frac{1}{p}} + [\phi_f(2r)]^{\frac{1}{p-1}}.$$

PROOF. – Set

$$v = |u| + h$$

where $h = h(r)$ is as in the statement of the theorem, by (1.2) we deduce

$$\begin{cases} |A(x,u,\xi)| \leq a|\xi|^{p-1} + b_1|v|^{p-1} \\ |B(x,u,\xi)| \leq c|\xi|^{p-1} + d_1|v|^{p-1} \\ \xi A(x,u,\xi) \geq |\xi|^p - d_1|v|^p \end{cases} \tag{3.1}$$

where b_1 and d_1 are defined by

$$b_1 = b + h^{1-p}e \quad , \quad d_1 = d + h^{1-p}f + h^{-p}g.$$

It is easy to see that $b_1^{\frac{p}{p-1}}$ and d_1 belong to the class $(M_X)'_p(B_{4r})$. Moreover

$$\phi_{b_1^{\frac{p}{p-1}}}(\rho) \leq C(p)\left[\phi_{b^{\frac{p}{p-1}}}(\rho) + h^{-p}\phi_{e^{\frac{p}{p-1}}}(\rho)\right] \leq C(p)\left[\phi_{b^{\frac{p}{p-1}}}(\rho) + 1\right]$$

and

$$\phi_{d_1}(\rho) \leq C(p)\left[\phi_d(\rho) + h^{1-p}\phi_f(\rho) + h^{-p}\phi_g(\rho)\right] \leq C(p)\left[\phi_d(\rho) + 2\right],$$

for any $0 < \rho < 2r$.

For fixed numbers $q \geq 1$ and $l > h$, we consider the functions

$$F(v) = \begin{cases} v^q & \text{if} \quad h \leq v \leq l \\ ql^{q-1} - (q-1)l^q & \text{if} \quad l \leq v \end{cases}$$

and

$$G(u) = \quad \operatorname{sign} u\{F(v)[F'(v)]^{p-1} - q^{p-1}h^\beta\} \quad u \in]-\infty, +\infty[,$$

where β such that $pq = p + \beta - 1$.

As a test function in (2.1) we take

$$\varphi(x) = \eta^p(x)G(u),$$

where $\eta(x)$ is a non negative smooth function with support in B_{2r}.

Substituting $\varphi(x)$ in (2.4), by conditions (3.1), we obtain as in [14]

$$\int_{B_{2r}} \eta^p |Xw|^p \, dx \le ap \int_{B_{2r}} |(X\eta)w||\eta(Xw)|^{p-1} \, dx + q^{p-1}p \int_{B_{2r}} b_1 |(X\eta)w||\eta w|^{p-1} \, dx + \int_{B_{2r}} c|\eta w||\eta(Xw)|^{p-1} \, dx + (1+p)q^{p-1} \int_{B_{2r}} d_1 |\eta w|^p \, dx \tag{3.2}$$

where $w = w(x) = F(v)$.

With the aid of the inequality

$$ab^{p-1} \le \frac{1}{p}\varepsilon^{1-p}a^p + \left(1 - \frac{1}{p}\right)\varepsilon b^p, \quad \forall \varepsilon > 0$$

we have

$$\int_{B_{2r}} \eta^p |Xw|^p \, dx \le C(p,a)q^{p-1} \left\{ \int_{B_{2r}} |w(X\eta)|^p \, dx + \int_{B_{2r}} V|\eta w|^p \, dx \right\}, \tag{3.3}$$

where V is defined by

$$V = b_1^{\frac{p}{p-1}} + c^p + d_1.$$

This implies that $V \in (M_X)'_p(B_{4r})$.

Moreover $\phi_V(\rho)$ is such that

$$\phi_V(\rho) \le C(p) \left\{ \phi_{b_1^{\frac{p}{p-1}}}(\rho) + \phi_{c^p}(\rho) + \phi_{d_1}(\rho) \right\} \le C(p) \left\{ \phi_{b^{\frac{p}{p-1}}}(\rho) + \phi_{c^p}(\rho) + \phi_d(\rho) + 3 \right\},$$

for any $0 < \rho < 2r$.

Using Corollary 2.2 we get

$$\int_{B_{2r}} \eta^p |Xw|^p \, dx \le Cq^{p-1} \left\{ (1+\sigma) \int_{B_{2r}} |w(X\eta)|^p \, dx + \sigma \int_{B_{2r}} \eta^p |Xw|^p \, dx + K(\sigma) \int_{B_{2r}} \eta^p w^p \, dx \right\} \quad \forall \sigma > 0,$$

where

$$K(\sigma) \sim \frac{\sigma}{\left[\phi_V^{-1}(\sigma)\right]^{Q+p}},$$

and C is a positive constant independent of w.

Fixing $\sigma = \frac{1}{2Cq^{p-1}}$, we obtain

$$\int_{B_{2r}} \eta^p |Xw|^p \, dx \leq C\left\{ q^{p-1} \int_{B_{2r}} |X\eta|^p w^p \, dx + q^{p-1} K\left(\frac{1}{2Cq^{p-1}}\right) \int_{B_{2r}} \eta^p w^p \, dx \right\}.$$

By Sobolev inequality we have

$$(3.4) \quad \left(\int_{B_{2r}} |\eta w|^{p*} \, dx \right)^{\frac{p}{p*}}$$

$$\leq C \frac{r^p}{|B_r|^{\frac{p}{Q}}} \left\{ q^{p-1} \int_{B_{2r}} |X\eta|^p w^p \, dx + q^{p-1} K\left(\frac{1}{2Cq^{p-1}}\right) \int_{B_{2r}} \eta^p w^p \, dx \right\},$$

where $p^* = \frac{pQ}{Q-p}$ and C is a positive constant independent of w.

Let now r_1 and r_2 satisfy $r \leq r_1 < r_2 \leq 2r$ and choose $\eta(x)$ so that $\eta(x) = 1$ in $B_{r_1} = B(x_0, r_1)$, $0 \leq \eta(x) \leq 1$ in B_{r_2} and $|X\eta| \leq \frac{C}{r_2 - r_1}$. Substituting this function in (3.4) we have

$$\left(\int_{B_{r_1}} w^{p*} \, dx \right)^{\frac{p}{p*}} \leq C \frac{r^p}{|B_r|^{\frac{p}{Q}}} \frac{1}{(r_2 - r_1)^p} q^{p-1} K\left(\frac{1}{2Cq^{p-1}}\right) \int_{B_{r_2}} w^p \, dx,$$

that is

$$\left(\int_{B_{r_1}} w^{p\chi} \, dx \right)^{\frac{1}{\chi}} \leq C \frac{r^p}{|B_r|^{\frac{p}{Q}}} \frac{1}{(r_2 - r_1)^p} \left[\frac{1}{\phi_V^{-1}\left(\frac{1}{Cq^{p-1}}\right)} \right]^{Q+p} \int_{B_{r_2}} w^p \, dx,$$

where $\chi = \frac{p*}{p} = \frac{Q}{Q-p}$.

Letting $l \to +\infty$, $w \to v^q$ and then we have

$$\left(\int_{B_{r_1}} v^{pq\chi} \, dx \right)^{\frac{1}{pq\chi}} \leq C^{\frac{1}{pq}} \frac{r^{\frac{1}{q}}}{|B_r|^{\frac{1}{qQ}}} \frac{1}{(r_2 - r_1)^{\frac{1}{q}}} \left[\frac{1}{\phi_V^{-1}\left(\frac{1}{Cq^{p-1}}\right)} \right]^{\frac{Q+p}{pq}} \left(\int_{B_{r_2}} v^{pq} \, dx \right)^{\frac{1}{pq}}.$$

Set $\gamma = pq$, we have

$$\|v\|_{L^{\chi\gamma}(B_{r_1})} \leq C^{\frac{1}{\gamma}} \frac{r^{\frac{p}{\gamma}}}{|B_r|^{\frac{p}{\gamma Q}}} \left(\frac{1}{r_2 - r_1}\right)^{\frac{p}{\gamma}} \left[\frac{1}{\left(\phi_V^{-1}\left(\frac{1}{C\left(\frac{\gamma}{p}\right)^{p-1}}\right)\right)^{Q+p}}\right]^{\frac{1}{\gamma}} \|v\|_{L^{\gamma}(B_{r_2})}.$$

Set

$$\gamma_i = p\chi^i$$

and

$$r_i = r + \frac{r}{2^i},$$

where $i = 1, 2, \ldots$, we have

$$\|v\|_{L^{\gamma_{i+1}}(B_{r_{i+1}})} \leq C^{\frac{1}{p\chi^i}} \left(\frac{2^{i+1}}{|B_r|^{\frac{1}{Q}}}\right)^{\frac{1}{\chi^i}} \left[\frac{1}{\left(\phi_V^{-1}\left(\frac{1}{C\chi^{(p-1)i}}\right)\right)^{Q+p}}\right]^{\frac{1}{p\chi^i}} \|v\|_{L^{\gamma_i}(B_{r_i})}.$$

Iterating the previous inequality, we obtain

$$\|v\|_{L^{\infty}(B_r)} \leq C|B_r|^{-\frac{1}{p}} \prod_{i=0}^{+\infty} \left[\frac{1}{\left(\phi_V^{-1}\left(\frac{1}{C\chi^{(p-1)i}}\right)\right)^{Q+p}}\right]^{\frac{1}{p\chi^i}} \|v\|_{L^p(B_{2r})}.$$

We stress that

$$\prod_{i=0}^{+\infty} \left[\frac{1}{\left(\phi_V^{-1}\left(\frac{1}{C\chi^{(p-1)i}}\right)\right)^{Q+p}}\right]^{\frac{1}{p\chi^i}} < +\infty$$

if and only if the series

$$\sum_{i=0}^{+\infty} \frac{1}{\chi^i} log\, \phi_V^{-1}\left(\frac{1}{\chi^{(p-1)i}}\right)$$

is convergent. The conclusion thus follows by Lemma 2.3. □

THEOREM 3.2. – *Suppose* (A1)-(A3), (1.2) *and* (1.5) *hold true. Let Ω be a bounded connected open set with homogeneous dimension Q. If $u \in \mathcal{L}^{1,p}_{\text{loc}}(\Omega)$ with $1 < p < Q$ is a nonnegative weak solution of* (1.4), *then there exists a positive constant C, independent of u, such that, for any $B_r = B(x_0, r)$ for which $B_{4r} \subset \Omega$ and $r < R_D$, we have*

$$\max_{B_r} u \leq C\left\{\min_{B_r} u + h(r)\right\}$$

where

$$h(r)=\left[\phi_{e^{\frac{p}{p-1}}}(2r)+\phi_g(2r)\right]^{\frac{1}{p}}+\left[\phi_f(2r)\right]^{\frac{1}{p-1}}.$$

PROOF. – Proceeding as in Theorem 3.1, setting $v=|u|+h$, with h defined by

$$h=\left[\phi_{e^{\frac{p}{p-1}}}(3r)+\phi_g(3r)\right]^{\frac{1}{p}}+\left[\phi_f(3r)\right]^{\frac{1}{p-1}},$$

we deduce the new conditions (3.1) and then, taking as a test function in (2.4)

$$\varphi(x)=\eta^p(x)v^\beta(x),$$

where $\eta(x)$ is a non negative smooth function with support in B_{3r} and $\beta\in\mathbb{R}$, we have

$$\int_{B_{3r}}|Xv|^p\eta^p v^{\beta-1}\,dx\le C_1(p,a)(1+|\beta|^{-1})^p\left\{\int_{B_{3r}}|X\eta|^p v^{p+\beta-1}\,dx+\int_{B_{3r}}V\eta^p v^{p+\beta-1}\,dx\right\},\tag{3.5}$$

where V is defined by

$$V=b_1^{\frac{p}{p-1}}+c^p+d_1.$$

Setting

$$w(x)=\begin{cases}v^q(x) & \text{where}\quad pq=p+\beta-1\quad\text{if}\quad\beta\neq 1-p\\ \log v(x) & \text{if}\quad\beta=1-p\end{cases}$$

by (3.5) we have

$$\int_{B_{3r}}\eta^p|Xw|^p\,dx\le C_1|q|^p(1+|\beta|^{-1})^p\left\{\int_{B_{3r}}|X\eta|^p w^p\,dx+\int_{B_{3r}}V\eta^p w^p\,dx\right\}\qquad\text{if }\beta\neq 1-p\tag{3.6}$$

and

$$\int_{B_{3r}}\eta^p|Xw|^p\,dx\le C_1\left\{\int_{B_{3r}}|X\eta|^p\,dx+\int_{B_{3r}}V\eta^p\,dx\right\}\qquad\text{if }\beta=1-p.\tag{3.7}$$

We consider first the (3.7). By Theorem 2.1, we have

$$\int_{B_{3r}} V\eta^p\,dx \le C_2\phi_V(1)\int_{B_{3r}} |X\eta|^p\,dx$$

and then, from (3.7), we have

$$\int_{B_{3r}} \eta^p|Xw|^p\,dx \le C_3(p,a,\phi_V,\,\mathrm{diam}\,\Omega)\int_{B_{3r}} |X\eta|^p\,dx. \tag{3.8}$$

Let B_h an arbitrary open ball contained in B_{2r}. Choosing $\eta(x)$ so that $\eta(x)=1$ in B_h, $0\le\eta\le 1$ in $B_{3r}\setminus B_h$ and $|X\eta|\le\frac{3}{h}$, by (3.8) we have

$$\|Xw\|_{L^p(B_h)} \le C_4(p,a,\phi_V,\,\mathrm{diam}\,\Omega)\frac{|B_h|^{\frac{1}{p}}}{h}.$$

Therefore, by Poincaré and John-Nirenberg lemmas (see [1]), there exist two positive constants p_0 and C_5 depending on the same arguments of C_4, such that

$$\left(\frac{1}{|B_{2r}|}\int_{B_{2r}} e^{p_0w}\,dx\right)^{\frac{1}{p_0}}\left(\frac{1}{|B_{2r}|}\int_{B_{2r}} e^{-p_0w}\,dx\right)^{\frac{1}{p_0}} \le C_5. \tag{3.9}$$

Set

$$\Phi(p,h) = \left(\int_{B_h} |v|^p\,dx\right)^{\frac{1}{p}}$$

for any real number $p\ne 0$; by (3.9), recalling that $w=\log v$ we have

$$\frac{1}{|B_{2r}|^{\frac{1}{p_0}}}\Phi(p_0,2r) \le C_5|B_{2r}|^{\frac{1}{p_0}}\Phi(-p_0,2r). \tag{3.10}$$

We consider now (3.6). By Corollary 2.2 we obtain

$$\int_{B_{3r}} |Xw|^p\eta^p\,dx \le C\left\{(|q|^p+1)\left(1+\frac{1}{|\beta|}\right)^p\int_{B_{3r}} |X\eta|^pw^p\,dx\right.$$
$$\left.+\left[\frac{1}{\phi_V^{-1}\left(|q|^{-p}\left(1+\frac{1}{|\beta|}\right)^{-p}\right)}\right]^{Q+p}\int_{B_{3r}}\eta^pw^p\,dx\right\}.$$

By Sobolev inequality we have

$$\left(\int_{B_{3r}} |\eta w|^{p*}\,dx\right)^{\frac{p}{p*}} \le C\frac{r^p}{|B_r|^{\frac{p}{Q}}}\left\{(|q|^p+2)\left(1+\frac{1}{|\beta|}\right)^p\int_{B_{3r}}|X\eta|^p w^p\,dx \right.$$
$$\left. + \left[\frac{1}{\phi_V^{-1}\left(|q|^{-p}\left(1+\frac{1}{|\beta|}\right)^{-p}\right)}\right]^{Q+p}\int_{B_{3r}}\eta^p w^p\,dx\right\}, \tag{3.11}$$

where $p^* = \frac{pQ}{Q-p} = p\chi$ and C is a positive constant independent of w.

Let r_1 and r_2 be real numbers such that $r \le r_1 < r_2 \le 2r$. Let the function η be chosen so that $\eta(x)=1$ in B_{r_1}, $0\le \eta(x)\le 1$ in B_{r_2}, $\eta(x)=0$ outside B_{r_2}, $|X\eta|\le \frac{C}{r_2-r_1}$. By (3.11) we have

$$\left(\int_{B_{r_1}} w^{p*}\,dx\right)^{\frac{p}{p*}} \le C\frac{r^p}{|B_r|^{\frac{p}{Q}}}\frac{1}{(r_2-r_1)^p}(|q|^p+2)$$
$$\times\left(1+\frac{1}{|\beta|}\right)^p\left[\frac{1}{\phi_V^{-1}\left(|q|^{-p}\left(1+\frac{1}{|\beta|}\right)^{-p}\right)}\right]^{Q+p}\int_{B_{r_2}} w^p\,dx.$$

Putting $\gamma = pq = p+\beta-1$ and recalling that $w(x)=v^q(x)$, we have

$$\Phi(\chi\gamma, r_1) \le C^{\frac{1}{\gamma}}\left(\frac{r}{|B_r|^{\frac{1}{Q}}}\right)^{\frac{1}{q}}(|q|^p+2)^{\frac{1}{\gamma}}\left(1+\frac{1}{|\beta|}\right)^{\frac{1}{q}}$$
$$\times\left[\frac{1}{\phi_V^{-1}\left(|q|^{-p}\left(1+\frac{1}{|\beta|}\right)^{-p}\right)}\right]^{\frac{Q+p}{\gamma}}\frac{1}{(r_2-r_1)^{\frac{1}{p}}}\Phi(\gamma, r_2), \tag{3.12}$$

for positive $\gamma \ne p-1$, and

$$\Phi(\chi\gamma, r_1) \ge C^{\frac{1}{\gamma}}\left(\frac{r}{|B_r|^{\frac{1}{Q}}}\right)^{\frac{1}{q}}(|q|^p+2)^{\frac{1}{\gamma}}\left[\frac{1}{\phi_V^{-1}(|q|^{-p})}\right]^{\frac{Q+p}{\gamma}}\frac{1}{(r_2-r_1)^{\frac{1}{p}}}\Phi(\gamma, r_2), \tag{3.13}$$

for negative γ. These are the inequalities which we wish to iterate. In order that (3.12) be applicable at each stage, we choose an initial value $p_0' \le p_0$ in such a way that the point $p=1$ lies midway between two consecutive iterates of p_0' and for $i=0,1,\dots,$ we let

$$p_i = \chi^i p_0'$$

and

$$r_i = r + \frac{r}{2^i}.$$

Thus we also obtain

$$|\beta| \geq \frac{\chi - 1}{1 + \chi}.$$

Iterating (3.12) and using Lemma 2.3 to prove the convergence of the iteration procedure, we have

$$\Phi(\infty, r) \leq C(p, a, \phi_V, \operatorname{diam} \Omega)|B_r|^{\frac{-1}{p_0}} \Phi(p_0, 2r). \tag{3.14}$$

Now if $\gamma_i = \chi^i p_0$ and $r_i = r + \frac{r}{2^i}$, the iteration of (3.13) yields

$$\Phi(-\infty, r) \geq C(p, a, \phi_V, \operatorname{diam} \Omega)|B_r|^{\frac{1}{p_0}} \Phi(-p_0, 2r). \tag{3.15}$$

Therefore, from (3.10), (3.14), (3.15) and noting that from Hölder's inequality

$$\Phi(p_0', 2r) \leq \Phi(p_0, 2r)|B_r|^{\frac{1}{p_0'} - \frac{1}{p_0}},$$

we obtain

$$\Phi(\infty, r) \leq C\Phi(-\infty, r)$$

where $C \equiv C(p, a, \phi_V, \operatorname{diam} \Omega)$, that is

$$\max_{B_r} u \leq C\left\{\min_{B_r} u + h\right\}. \qquad \square$$

REMARK 3.3. – We wish to note that the proof of Theorem 3.1 works also with weak subsolutions of (1.4) and the proof of of Theorem 3.2 provides a weak Harnack inequality for non negative weak supersolutions (see [15]).

By a standard argument Harnack's inequality implies that weak solutions of (1.4) are continuous with respect to the Carnot Caratheodory metric $d(x, y)$.

THEOREM 3.4. – *Suppose* (A1)-(A3), (1.2) *and* (1.5) *hold true. Let Ω be a bounded connected open set with local homogeneous dimension Q. Let $u \in \mathcal{L}^{1,p}(\Omega)$, with $1 < p < Q$, be a weak solution of* (1.4) *with* $\sup_{\Omega} |u| = L < +\infty$. *Then u is continuous.*

PROOF. – By Theorem 3.1 we have that

$$|u(x)| \leq L$$

in every arbitrary open subset Ω' of Ω, where L is a positive constant depending on $n, p, a, \phi_{\frac{p}{e^{p-1}}}(1), \phi_f(1), \phi_g(1)$ and Ω'.

Let B_r be an arbitrary ball contained in Ω', the functions

$$M(r) = \max_{B_r} u, \qquad m(r) = \min_{B_r} u$$

are then well defined in B_r.

Set

$$\overline{u} = M(r) - u,$$

it follows that $\overline{u}$ is a non negative weak solution in B_r of the differential equation

$$\sum_{j=1}^{m} X_j^* A_j(x, \overline{u}, X\overline{u}) + \tilde{B}(x, \overline{u}, X\overline{u}) = 0, \tag{3.16}$$

where $\tilde{A}(x, \overline{u}, \overline{\xi})$ and $\tilde{B}(x, \overline{u}, \overline{\xi})$ are defined by

$$\tilde{A}(x, \overline{u}, \overline{\xi}) = -A(x, M - \bar{u}, -\overline{\xi})$$
$$\tilde{B}(x, \overline{u}, \overline{\xi}) = B(x, M - \overline{u}, -\overline{\xi}),$$

and satisfy

$$\begin{cases} |\tilde{A}(x, \overline{u}, \overline{\xi})| \leq a|\overline{\xi}|^{p-1} + \overline{b}|\overline{u}|^{p-1} + \overline{e} \\ |\tilde{B}(x, \overline{u}, \overline{\xi})| \leq c|\overline{\xi}|^{p-1} + \overline{d}|\overline{u}|^{p-1} + \overline{f} \\ \overline{\xi}\tilde{A}(x, \overline{u}, \overline{\xi}) \geq |\overline{\xi}|^{p} - \overline{d}|\overline{u}|^{p} - \overline{g} \end{cases} \tag{3.17}$$

where $\overline{b}(x)$, $\overline{d}(x)$, $\overline{e}(x)$, $\overline{f}(x)$ and $\overline{g}(x)$ are measurable functions belonging to $(M_X)'_p$ defined by

$$\begin{cases} \overline{b}(x) = 2^p b(x) \\ \overline{d}(x) = 2^p d(x) \\ \overline{e}(x) = 2^p b(x) L^{p-1} + e(x) \\ \overline{f}(x) = 2^p d(x) L^{p-1} + f(x) \\ \overline{g}(x) = 2^p d(x) L^{p-1} + g(x). \end{cases} \tag{3.18}$$

By Theorem 3.2 we have

$$\max_{B_{\frac{r}{3}}} \overline{u}(x) \leq C\left(\min_{B_{\frac{r}{3}}} \overline{u}(x) + \overline{h}\right) \tag{3.19}$$

where

$$C \equiv C\left(p, a, \phi_{\frac{p}{b^{p-1}}}(1), \phi_{c^p}(1), \phi_d(1)\right)$$

$$\overline{h} \equiv \overline{h}(r) = \left[\phi_{\frac{p}{\overline{e}^{p-1}}}(r) + \phi_{\overline{g}}(r)\right]^{\frac{1}{p}} + \left[\phi_{\overline{f}}(r)\right]^{\frac{1}{p-1}}.$$

Note that $\overline{h}$ is a positive non decreasing function with $\lim_{r \to 0} \overline{h}(r) = 0$, such that

$$\overline{h}\left(\frac{r}{2}\right) \geq K\overline{h}(r) \quad , \quad 0 < K < 1.$$

By (3.19) we have

$$M(r) - m\left(\frac{r}{3}\right) \leq C\left[M(r) - M\left(\frac{r}{3}\right) + \overline{h}(r)\right]. \tag{3.20}$$

In the same way, setting

$$\overline{\overline{u}} = u - m(r)$$

we obtain

$$M\left(\frac{r}{3}\right) - m(r) = \max_{B_{\frac{r}{3}}} \overline{\overline{u}} \leq C\left(\min_{B_{\frac{r}{3}}} \overline{\overline{u}} + \overline{h}\right) = \tag{3.21}$$

$$= C\left[m\left(\frac{r}{3}\right) - m(r) + \overline{h}(r)\right].$$

Adding (3.20) and (3.21) we have

$$M\left(\frac{r}{3}\right) - m\left(\frac{r}{3}\right) \leq \frac{C-1}{C+1}[M(r) - m(r)] + \frac{2C}{C+1}K^2\overline{h}\left(\frac{r}{4}\right).$$

Set, for $\rho > 0$

$$\omega(\rho) = M(\rho) - m(\rho),$$

$$\gamma = \frac{C-1}{C+1}$$

and

$$h(r) = \frac{2C}{C+1}K^2\overline{h}(r),$$

then

$$\omega\left(\frac{r}{4}\right) \leq \omega\left(\frac{r}{3}\right) \leq \gamma\omega(r) + h\left(\frac{r}{4}\right).$$

The conclusion follows by Lemma 2.4. □

REMARK 3.5. – Using Proposition 1.7, it is trivial to note that the results listed in this Section come back the analogous ones in [8].

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