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A Characterization of ω-Limit Sets for Continuous Flows on Surfaces.

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Sunto. – *Si dà una descrizione topologica esplicita degli insiemi ω-limite dei flussi continui in superfici compatte senza frontiera. Alcuni risultati si possono estendere a varietà di dimensione maggiore.*

Summary. – *An explicit topological description of ω-limit sets of continuous flows on compact surfaces without boundary is given. Some of the results can be extended to manifolds of larger dimensions.*

1. – Introduction.

In what follows, M always denotes a topological *n-manifold* (or simply a *manifold*), that is, a second countable Hausdorff topological space for which any point has a homeomorphic neighbourhood to some open subset of the euclidean half-space $\mathbb{R}^{n-1} \times [0, \infty)$. The number n is called the *dimension* of M; 2-manifolds are called *surfaces* and we usually employ the symbol S to denote them. We always assume M to be connected. The set of points of M which are *not* locally homeomorphic to $\mathbb{R}^n$, denoted by ∂M, is called the *combinatorial boundary* of M. If we say that $N \subset M$ is a *submanifold* of M (or a *subsurface* when $n = 2$) of M then we mean that N is an n-manifold with the induced topology. In particular notice that the topological boundary $\operatorname{Bd} N$ of N and its combinatorial boundary ∂N need not coincide. In what follows we use the word "boundary" without further ado: it will be clear from the context what kind of boundary we are referring to. Every manifold M is metrizable; $d(\cdot, \cdot)$ denotes a distance compatible with its topological structure.

It is well known that all n-manifolds with $n \leq 3$ admit a unique (up to diffeomorphisms) smooth (C^∞) structure compatible with the topological one, but it higher dimensions this is no longer true. Thus, when speaking about a *smooth n-manifold*, we implicitly assume it endowed with a (fixed) smooth structure, which is inherited by its *smooth submanifolds*.

A *continuous (resp. smooth) flow* on a topological (resp. smooth) manifold M

is a continuous (resp. smooth) map $\Phi : \mathbb{R} \times M \to M$ satisfying: (i) $\Phi(0, x) = x$ for every $x \in M$; (ii) $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ for every $t, s \in \mathbb{R}$ and any $x \in M$.

The asymptotic behaviour of the *orbits* $\{\Phi_x(t) = \Phi(t, x) : t \in \mathbb{R}\}$ of points $x \in M$ can be understood, up to some extent, via their *ω-limit sets* $\omega_\Phi(x) = \{y \in M : \exists (t_n)_n \to \infty; (\Phi_x(t_n))_n \to y\}$. In particular, the problem of giving an explicit topological description of the subsets of M which can be ω-limit sets for continuous flows on M arises in a natural way and has been undertaken by the authors in a series of papers [3], [4], [9], [5], [6], [10]. The aim of this note is to complete this description in the setting of compact surfaces with empty boundary. For the sake of completeness we also recall the main results in those papers.

Some notational remarks. An *arc* (resp. a *circle*, an *annulus*) is a topological space homeomorphic to $[0, 1]$ (resp. $\mathbb{S}^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$, $\mathbb{R}^2 \setminus \{(0, 0)\}$). If A is an arc and we write $A = [u; v]$, then we mean that u and v are its endpoints. If A, B are arcs, then the *Hausdorff distance* between A and B is defined by $d_H(A, B) = \max\{\max_{x \in A} \min_{y \in B} d(x, y), \max_{y \in B} \min_{x \in A} d(x, y)\}$. A *curve* is either a circle or a continuous one-to-one image of $\mathbb{R}$. A circle $C \subset S$ is called *orientable* if it has an annular neighbourhood, and *null homotopic* if it can be homotopically deformed onto a single point. A surface S is *simply connected* if all circles $C \subset S$ are null homotopic (equivalently, S is homeomorphic to $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\} \setminus P$, with P being a closed subset of the circle $\mathbb{S}^1$). A *region* is an open connected set. $\operatorname{Int} A$, $\operatorname{Cl} A$ and $\operatorname{diam} A$ will denote the interior, the closure and the diameter of a set A.

The topological structure of an ω-limit set will strongly depend on its dynamical nature. By far the most complicated case is that of *recurrent* points (those satisfying $x \in \omega_\Phi(x)$; here we also say that the orbit of x is *recurrent*), especially in the case when the interior of the ω-limit set is empty. We analyze these possibilities separately.

2. – The nonrecurrent case.

This is, relatively, the simplest one.

DEFINITION 2.1. – *We say that an annulus $R \subset S$ is regular if* $\operatorname{Bd} R$ *has two connected components.*

THEOREM 2.2 ([5]). – *Let S be a compact surface, $\partial S = \emptyset$. Let Φ be a continuous flow on S and let $x \in S$. Assume that x is nonrecurrent or that* $\operatorname{Int} \omega_\Phi(x) = \emptyset$ *and $S \setminus \omega_\Phi(x)$ has a finite number of components. Then $\omega_\Phi(x)$ is a boundary component of a regular annulus in S.*

Conversely, if Ω is a boundary component of a regular annulus in S then there are a smooth flow Φ on S and a point $x \in S$ such that $\Omega = \omega_\Phi(x)$.

REMARK 2.3. – Theorem 2.2 remains true for arbitrary surfaces provided (in the direct statement) that there is a compact set K such that $\Phi_x([0,\infty)) \subset K \subset S \setminus \partial S$. This can be proved using some ideas from Sections 2 and 3 in [6].

For surfaces admitting no flows with recurrent points except the trivial ones (fixed points and those belonging to periodic orbits), namely the sphere $\mathbb{S}^2$, the projective plane $\mathbb{P}^2$ and the Klein bottle $\mathbb{B}^2$, Theorem 2.2 then provides a complete characterization of ω-limit sets. In the sphere case ω-limit sets had been more simply described (as boundaries of simply connected regions) by Vinograd [12] as early as 1952, see also [2] for a detailed proof. In the case of $\mathbb{P}^2$ the problem was posed by Anosov in 1996 [1], where it is emphasized that Vinograd's characterizacion no longer works. For $\mathbb{P}^2$ and $\mathbb{B}^2$, alternative formulations to that of Theorem 2.2 were given in [3] and [9]. Next we recall these results:

THEOREM 2.4 ([12], [3], [9]). – *Let $S = \mathbb{S}^2$, $\mathbb{P}^2$ or $\mathbb{B}^2$ and let $\Omega \subset S$. Then Ω is the ω-limit set for some continuous (or, equivalently, some smooth) flow if and only if it is the boundary of a region $\emptyset \subsetneq O \subsetneq S$ such that:*

(a) *O is simply connected (in the case $S = \mathbb{S}^2$);*

(b) *$S \setminus O$ is connected (in the case $S = \mathbb{P}^2$);*

(c) *$S \setminus O$ is connected and either O is simply connected or there is a nonnull homotopic circle $C \subset O$ such that Ω is included in the boundary of one of the components of $O \setminus C$ (in the case $S = \mathbb{B}^2$).*

REMARK 2.5. – In $\mathbb{S}^2$ simply connected regions are of course those having connected complementary so (a) could be rewritten as in (b). On the other hand notice that, for instance, $\mathbb{P}^2$ minus a single point is a region with connected complementary which is not simply connected.

Characterizing ω-limit sets of manifolds of larger dimensions in a satisfactory way is much more difficult, particularly in the recurrent case, because no classification theorem as that of surfaces is available. Yet it is worth stressing that in the n-dimensional sphere $\mathbb{S}^n$, ω-limit sets of nonrecurrent orbits are still the boundaries of regions with connected complementary, see [4] for a proof. As in the converse part of Theorem 2.2 the corresponding flows can be got smooth. In [4] only the standard smooth structure is considered; in [10] the construction was extended to the rest of smooth structures of $\mathbb{S}^n$. In the last paper ω-limit sets of nonrecurrent orbits for an arbitrary compact manifold M with empty boundary were in fact characterized as the boundaries of those regions $O \subset M$ with the property that for every $\varepsilon > 0$ there is a component O_ε of the set $\{x \in M : d(x, \operatorname{Bd} O) < \varepsilon\}$ such that $\operatorname{Bd} O \subset \operatorname{Bd} O_\varepsilon$.

3. – The recurrent, nonempty interior case.

As far as ω-limit sets of recurrent orbits are concerned, two possibilities must be separately analyzed. The nonempty interior case is closely related to the notion of *transitivity*. We say that a manifold (resp. smooth manifold) is *transitive* (resp. *smooth transitive*) if it admits a continuous (resp. smooth) flow having a dense orbit. It turns out, see [6], that a set $\Omega \subset M$ with nonempty interior is an ω-limit set for a continuous (resp. smooth) flow on M if and only it is the closure of a transitive submanifold (resp. smooth submanifold) of M. The study of transitive manifolds has a long tradition, see [6] for a list of relevant references. We completely characterize them in [6] (in particular answering some questions in [7] and [8]). If $n \geq 3$ the situation is fairly simple: all (smooth) n-manifolds are (smooth) transitive. To deal with the two-dimensional case we need an additional notion.

DEFINITION 3.1. – *Let $C, D \subset S$ be orientable circles. We say that C and D are crossing if they intersect at exactly one point p and, moreover, if A (resp. B) is a small arc in C (resp. D) containing p but not having it as an endpoint, then both components of $A \setminus \{p\}$ (resp. $A \setminus \{p\}$) lie at opposite sides of C (resp. D).*

THEOREM 3.2 ([6]). – *Let S be a surface (resp. an orientable surface). Then the following statements are equivalent:*

(i) *S is smooth transitive;*

(ii) *S is transitive;*

(iii) *S is not homeomorphic to $\mathbb{S}^2$, $\mathbb{P}^2$, nor to any subsurface of $\mathbb{B}^2$ (resp. is not homeomorphic to any subsurface of $\mathbb{S}^2$);*

(iv) *S has two crossing circles.*

4. – The recurrent, empty interior case.

Now we complete our description of ω-limit sets for compact surfaces with empty boundary by characterizing those sets with empty interior which are ω-limit sets of some recurrent orbit (but of no nonrecurrent orbit). A proof of Theorem 4.4, the new result we announce in this note, will appear elsewhere.

DEFINITION 4.1. – *Let $\mathcal{B}$ be a family of curves in S, let $p \in B \in \mathcal{B}$ and let O be a neighbourhood of p. We say that O is a Whitney regular neighbourhood of p if for every $\varepsilon > 0$ there is a $\delta > 0$ such that:*

(a) *if $[p'; q'] \subset O \cap B'$ for some $B' \in \mathcal{B}$ and $d(p', q') < \delta$, then* $\operatorname{diam}([p'; q']) < \varepsilon$;

(b) *if $[p; q] \subset O \cap B$, $p' \in O \cap B'$ for some $B' \in \mathcal{B}$ and $d(p, p') < \delta$, then there is some $[p'; q'] \subset O \cap B'$ such that $d_H([p; q], [p'; q']) < \varepsilon$.*

We say that $\mathcal{B}$ is *Whitney regular* if every point of every $B \in \mathcal{B}$ has a Whitney regular neighbourhood.

DEFINITION 4.2. – *Let R be a subsurface of S and let $C \subset S$ be an orientable non-null homotopic circle. We say that R twists around C if there is a continuous one-to-one map $\phi : [0,1] \times [0,1] \to R$ such that:*

(i) *$\phi((0,1) \times (0,1)) \subset R \setminus (\partial R \cup C)$;*

(ii) *$\phi([0,1] \times \{0,1\}) \subset \partial R$;*

(iii) *$\phi(\{0,1\} \times [0,1]) \subset C$;*

(iv) *if $\varepsilon > 0$ is small enough then $\phi((0,\varepsilon] \times [0,1])$ and $\phi([1-\varepsilon,1) \times [0,1])$ lie at opposite sides of C.*

We call the set $\phi([0,1] \times [0,1])$ a *twisting section* of R and denote the union set of all twisting sections of R by $\Upsilon(R)$.

DEFINITION 4.3. – *Let $\{R_m\}_{m=1}^{\infty}$ be a family of pairwise disjoint simply connected subsurfaces of S and let $C \subset S$ be an orientable non-null homotopic circle. We call $\{R_m\}_m$ a feasible family for C if the following properties hold:*

(i) *all R_m twist around C and, for every k, the set $R_k \cup \mathrm{Cl}\big(\bigcup_m \Upsilon(R_m)\big)$ is a neighbourhood of ∂R_k;*

(ii) *the family of the components of all ∂R_m is Whitney regular;*

(iii) *for every $x, y \in \bigcup_{m=1}^{\infty} \mathrm{Bd}\, R_m$ and every $\varepsilon > 0$ there is an arc A in $\bigcup_{m=1}^{\infty} \partial R_m$ satisfying $d(x,A) < \varepsilon$, $d(y,A) < \varepsilon$.*

THEOREM 4.4. – *Let $\Omega \subset S$ with $\mathrm{Int}\, \Omega = 0$. Then Ω is the ω-limit of a recurrent point for some continuous flow on S (but of no nonrecurrent point for any continuous flow on S) if and only if there is a feasible family $\{R_m\}_m$ such that $\Omega = \mathrm{Bd} \bigcup_m R_m$. Moreover, in this case there is a C^∞-flow on S having a homeomorphic set to Ω as the ω-limit set of one of its recurrent points.*

The statement of Theorem 4.4 is rather complicated when compared to that of Theorems 2.2 and 3.2, but clear improvements are hard to suggest. In fact there are families $\{R_m\}_{m=1}^{\infty}$ of pairwise disjoint simply connected surfaces in the torus $\mathbb{T}^2$ satisfying two of the three conditions in Definition 4.3 and such that $\mathrm{Bd} \bigcup_n R_m$ is not an ω-limit set for any flow on $\mathbb{T}^2$. More precisely, a counterexample can be found satisfying (ii), (iii) and even with all R_m twisting around some curve C, and also a counterexample (even in the sphere $\mathbb{S}^2$) satisfying (ii) and (iii) and such that $R_k \cup \mathrm{Cl}\big(\bigcup_m \Upsilon(R_m)\big)$ is a neighbourhood of ∂R_k for every k. Similarly, there are families satisfying (i), (iii) and with just (a) or (b) failing in Definition 4.1 for which the second statement of Theorem 4.4 does not work. One could interpret (iii) as a strong form of connectedness, since there is also a counterexample for our statement where (i) and (ii) are satisfied and the set $\mathrm{Bd}\bigcup_n R_n$ is connected. Notice finally that the smoothness part of the statement is weaker than those of previous theorems. This cannot be helped, for it is easy to provide examples of sets Ω as those in Theorem 4.4 which are ω-limit sets of no smooth flows. Precise constructions of the examples above can be found in [11, pp. 131-140].

Gluing together the previous results we arrive to the desired topological description of ω-limit sets.

COROLLARY. – *Let S be a compact surface, $\partial S = 0$, and let $\Omega \subset S$. Then Ω is an ω-limit set for some continuous flow on S if and only if one of the following alternatives hold:*

(i) *Ω is one of the boundary componentes of a regular annulus;*

(ii) *Ω is the closure of a region having two crossing circles;*

(iii) *Ω is the boundary of the union of the sets from some feasible family.*

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