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Cyclic Phenomena for Composition Operators on Weighted Bergman Spaces.

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Sunto. – *In questo lavoro diamo una generalizzazione alla famiglia di Spazi di Bergman con peso G, A^2_G, di alcuni risultati ottenuti in* [4] *per lo spazio di Hardy H^2. In particolare studiamo il comportamento ciclico e iperciclico, nello spazio A^2_G, di operatori di composizione indotti da una funzione olomorfa φ del disco unitario $\Delta \subset \mathbb{C}$ in sé.*

Summary. – *In the present paper we give a generalization to the family of Bergman Spaces with weight G, A^2_G, of several results, obtained in* [4] *for the Hardy space H^2, concerning the cyclic and hypercyclic behaviour of composition operators C_φ induced by a holomorphic self map φ of the open unit disc $\Delta \subset \mathbb{C}$.*

Introduction and preliminaries.

Let φ be a holomorphic self map of the open unit disc Δ in the complex plane $\mathbb{C}$. Consider a space X of holomorphic functions on Δ. We want to investigate how the "behavior" of the composition operator C_φ, acting on X, which associates to each function $f \in X$ the function $C_\varphi f := f \circ \varphi$ changes, according to the dynamical properties of the inducing map φ.

In the Hardy space H^2, the link between function theory and theory of composition operators is suggested by the Littlewood Subordination Principle, which guarantees C_φ to be bounded. In that case it becomes natural to take up the problem of relating properties of C_φ with those of its inducing map φ. We can then investigate which conditions on φ determine whether C_φ is bounded, compact or cyclic. Recall that an operator T on a linear topological space X is said to be *cyclic* if there is a vector x in the space (called *cyclic vector* for T) whose orbit $Orb(T, x) = \{T^n x \,:\, n = 0, 1, 2...\}$ has dense linear span. When, without additional help from the linear span, the orbit of x is dense in X, then T is called *hypercyclic* and x is an *hypercyclic vector*.

The study of cyclicity is particularly interesting for composition operators because the n-th power of C_φ is the composition operator induced by the n-th iterate of φ. This simple property suggests that the cyclic behavior of a composition operator should be strongly influenced by the dynamical properties of its inducing map, and that this behavior should be strictly related with the kind of fixed points of φ, according to the derivative of φ in the fixed points, and after all with the type of convergence of the iterates of φ to the Wolff point [4].

In the present paper, in particular, we show how the position of the fixed points determine the cyclic behavior of the composition operator C_φ. In this context we have found a generalization of several theorems proved for the Hardy space [4], to a different family of spaces of functions called Bergman Spaces with weight G, denoted by $A^2_G(\Delta)$. We have easily extended to this family all the results which guarantee the cyclicity on H^2, proving the density of H^2 in $A^2_G(\Delta)$, and established conditions on G that permit to extend also some "negative" results, where "negative" means results in which one proves non cyclicity or non hypercyclicity of an operator.

In the first part of the work we recall known results, due to B. McCluer and T. Kriete [8] concerning how the weight G and the map φ determine whether C_φ in $A^2_G(\Delta)$ is bounded. In the second part we characterize the cyclic behavior of the linear fractional composition operators in A^2_G, according to the position of the fixed points of its inducing linear fractional map (see for example Propositions 1.1 and 1.3). One can transfer the obtained results to a wider setting by using linear fractional self map to represent more general maps through "linear fractional models"(see [4]).

We here introduce some more notations. Let dA denote the area measure on Δ and suppose G is a positive continuous function on (0,1) such that $\int_0^1 G(r)\, rdr < \infty$. The Weighted Bergman Space with weight G, $A^2_G(\Delta)$, is defined to be the collection of all holomorphic functions f on Δ for which

$$\|f\|^2_G = \int_\Delta |f(z)|^2 G(|z|)\, dA < \infty. \tag{1}$$

$A^2_G(\Delta)$ is an Hilbert space with norm $\|\cdot\|_G$ defined by (1).

It is easy to check that f, denoting with $p_n = \int_0^1 G(r) r^{2n+1} dr$, can be expanded as $f(z) = \sum_{n=0}^\infty \widehat{f}(n) z^n$ and $\|f\|^2_G = 2\pi \sum_{n=0}^\infty |\widehat{f}(n)|^2 p_n$, since the functions $\{\frac{z^n}{\sqrt{2\pi p_n}}\}_{n=0}^\infty$ are an orthonormal basis in $A^2_G(\Delta)$.

The weights $G(r) = (1-r)^a$, $a > -1$, are one version of *standard* weights; other examples, which give equivalent norms, will be described later (see also [12]). The quantities p_n are called *moments*.

Recall that (see also [8]) a function G, continuous and positive on (0,1), is called *fast weight* if

$$\lim_{r\to 1}\frac{G(r)}{(1-r)^a}=0\ \ \forall a>0.$$

We will always assume that $G(r)$ is non increasing on (0,1). However our results can be applied to any weight comparable to a non increasing weight near $r=1$.

Another well known property of the weighted Bergman space is that it admits a reproducing kernel $K_w(z)$ given by

$$K_w(z)=\frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{\overline{w}^n z^n}{p_n} \tag{2}$$

where $w,z\in\Delta$. Moreover the point evaluation functionals are continuous (for more details see e.g. [12]).

When $G(r)=(1-r)^a$ a result similar to the Subordination Principle holds in $A^2_G(\Delta)$ [11], hence any composition operator C_φ is bounded on $A^2_G(\Delta)$, with $\|C_\varphi\|\le(\frac{1+\varphi(0)}{1-\varphi(0)})^{\frac{a+2}{2}}$.

Unfortunately the procedure used for standard weights fails in the case of fast weights; indeed it turns out that if φ is an automorphism of Δ, other than a rotation, then C_φ is unbounded on $A^2_G(\Delta)$ for any fast weight G [8]. In a positive direction Kriete and McCluer have shown that if

$$\limsup_{|z|\to 1}\frac{G(|z|)}{G(|\varphi(z)|)}<\infty \tag{3}$$

C_φ is bounded on $A^2_G(\Delta)$.

T.L. Kriete [7] has proved that condition (3), together with further hypotheses on the regularity of G and on its way of decreasing for $r\to 1$, is not only sufficient but also necessary for the boundedness of the operator C_φ in $A^2_G(\Delta)$.

I thank Professor P. Bourdon for precious suggestions concerning the example which appears in section 1.4.

1.1 – *Preliminary results.*

Relations between H^2 and $A^2_G(\Delta)$.

Our aim here is the extension of some results which are valid for the composition operator C_φ in H^2, to the cases of the space $A^2_G(\Delta)$. In order to give this generalization we need to develop relations between these spaces. We will always consider weights G depending only on the modulus of z, unless otherwise specified, (for a more general theory, where the weight function G can be any function of z, we refer to Mergelyan [10]).

In what follows $\|\cdot\|_2$ denotes the H^2 norm and $\|\cdot\|_G$ the norm in $A^2_G(\Delta)$. In order to simplify the notation we always denote the Bergman Space with A^2_G, without specifying the domain, whenever it is not misleading.

We remark that, since $p_n = \int_0^1 G(r) r^{2n+1} dr$ and $r \leq 1$, we have: $p_n \leq \int_0^1 G(r) r dr$; for the hypotheses on G this last integral is bounded by a constant K. It then follows that

$$\|f\|^2_G \leq \|f\|^2_2 K$$

and we can conclude that the space H^2 is contained in A^2_G. As further consequences we can also say that the convergence in $\|\cdot\|_2$ implies the convergence in $\|\cdot\|_G$ and that the topology induced by $\|\cdot\|_2$ (denoted by τ_2) is finer than the one induced by $\|\cdot\|_G$ (τ_G).

REMARK 1.1. – We claim that polynomials are dense in A^2_G. Indeed, take f in A^2_G, since f is holomorphic in Δ, it can be written as $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ and $\|f\|^2_G = \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 p_n < \infty$. Therefore for all $\varepsilon > 0$ there exists n_0 such that for all $n > n_0$, the series $\sum_{n=n_0+1}^{\infty} |\widehat{f}(n)|^2 p_n < \varepsilon$. Let $p(z) = \sum_{n=o}^{n_0} \widehat{f}(n) z^n$ then

$$\|p(z) - f(z)\|^2_G = \|\sum_{n=0}^{n_0} \widehat{f}(n) z^n - \sum_{n=0}^{\infty} \widehat{f}(n) z^n\|^2_G$$

$$= \|\sum_{n=n_0+1}^{\infty} \widehat{f}(n) z^n\|^2_G = \sum_{n=n_0+1}^{\infty} |\widehat{f}(n)|^2 p_n < \varepsilon.$$

Now, since the topology τ_2 is finer than τ_G we can conclude that *the space H^2 is dense in A^2_G in the topology τ_G of A^2_G*. This remark has two important consequences.

THEOREM 1.1 (Generalized Walsh Theorem). – *Suppose $\mathcal{D} \subseteq \mathbb{C}$ is a simply connected domain whose boundary is a Jordan curve. Let the holomorphic function φ map Δ univalently onto $\mathcal{D}$. Then polynomials in φ are dense in A^2_G with the natural topology τ_G.*

PROOF. – The result holds in H^2 [4]; it follows that the polynomials in φ, denoted by $C_\varphi(\mathcal{P})$, are dense in H^2 in the topology τ_2. Then $C_\varphi(\mathcal{P}) \subseteq H^2 \subseteq A^2_G$ and $\overline{C_\varphi(\mathcal{P})} = H^2$ in τ_2 which is finer than τ_G. It follows that $\overline{C_\varphi(\mathcal{P})} = H^2$ also in τ_G; using the density of H^2 in A^2_G the theorem is proved. ■

With a similar proof we obtain, whenever C_φ is bounded on A^2_G, THEOREM 1.2. – *If the operator C_φ is cyclic (resp. hypercyclic) on H^2 it is also cyclic (resp. hypercyclic) on A^2_G.*

The same argument can be applied to any linear operator T bounded on H^2 and on A^2_G. We can then always assert that the operators cyclic (or hypercyclic) on H^2 are cyclic (or hypercyclic) also on A^2_G.

Other versions of Standard Weights.

Here we recall that in [11] new versions of standard weights are given and there is proven that they induce equivalent norms.

The two weights $\overline{G}(r) = \frac{2^a}{\pi\Gamma(a+1)}(\log\frac{1}{r})^a$ where Γ is the Euler function and $\widetilde{G}(r) = \frac{a+1}{\pi}(1-r)^a$, with $a > -1$ in both cases, are characterized by the property that they give norms equivalent to the one given by $G(r) = (1-r)^a$ with $a \geq -1$, but they have, respectively, simpler moments $\overline{p}_n$ and reproducing kernel $\{\widetilde{k_w}\}$. For example, for the weight $\overline{G}(r)$, we have $\overline{p}_n = \frac{1}{2\pi(n+1)^{a+1}}$ and if we consider the weight $\widetilde{G}(r)$ we can prove that the associated reproducing kernel has the simple form

$$\widetilde{k}_w(z) = \frac{1}{(1-\overline{w}^n z^n)^{a+2}}.$$

One can show the equivalence of the norms and some other useful relations between the reproducing kernel associated to G , $\overline{G}$ and to $\widetilde{G}$. For G as above, one can easily prove that

$$A^2_{\overline{G}} = A^2_{\widetilde{G}} = A^2_G. \tag{4}$$

Moreover there exist positive constants c and C such that:

$$c\|\widetilde{k}_w\|_{\widetilde{G}} \leq \|k_w\|_G \leq C\|\widetilde{k}_w\|_{\widetilde{G}} \tag{5}$$

where the same kind of relation holds for G and $\overline{G}$.

Let $\beta = \beta_n$ be a positive, non increasing sequence of real numbers, for which

$$\sigma = \sup\left\{\frac{\beta_k}{\beta_{k+1}} \;\Big|\; k \geq 0\right\} < \infty$$

We denote by $H^2(\beta)$ the weighted Hardy Space, which is the space of power series such that the norm defined by

$$\|f\|^2_\beta = \sum_{k=0}^{\infty} |\widehat{f}(k)|^2 \beta_k$$

is finite. The space $A^2_G(\Delta)$ corresponds to the particular case in which $\beta_n = 2\pi p_n$.

We finally recall that the norm of the Backward Shift $B: f(z) = \sum_{n=0}^{\infty}\widehat{f}(n)z^n \mapsto \sum_{n=0}^{\infty}\widehat{f}(n+1)z^n$ in a weighted Hardy space $H^2(\beta)$ is given by $\|B\| = \sup_n \frac{\beta_n}{\beta_{n+1}}$.

As a consequence, from the equivalence of the norms induced by $G, \overline{G}$ and by $\widetilde{G}$ and since (5) holds, we find an estimate of $|f(w)|$ for $w \in \Delta$. Indeed

$$|f(w)|^2 \leq \|f\|_G^2\|k_w\|_G^2 \leq C\|f\|_G^2\|\widetilde{k}_w\|_{\widetilde{G}}^2 = C\|f\|_G^2\frac{1}{(1-|w|^2)^{a+2}},$$

where, in the last step, we have used the general fact that, for every G, $\|k_w\|_G^2 = k_w(w)$.

We now want to find an estimate also for $|f'(w)|$ with $w \in \Delta$. We proceed in three steps. We firstly prove that for $w \in \Delta$, the function $h = \sum_{n=1}^{\infty}\frac{n\overline{w}^{n-1}z^{n-1}}{2\pi\overline{p}_{n-1}}$ belongs to $A^2_{\overline{G}}$. Indeed

$$\|\sum_{n=1}^{\infty}\frac{n\overline{w}^{n-1}z^{n-1}}{2\pi\overline{p}_{n-1}}\|_{\overline{G}}^2$$

$$= \|\sum_{m=0}^{\infty}\frac{(m+1)\overline{w}^m z^m}{2\pi\overline{p}_m}\|_{\overline{G}}^2 = \sum_{m=0}^{\infty}\frac{(m+1)^2|w|^{2m}}{2\pi\overline{p}_m}$$

$$= \sum_{m=0}^{\infty}(m+1)^{a+3}|w|^{2m}$$

(since $\overline{p}_m = \frac{1}{2\pi(m+1)^{a+1}}$). The last series converges, and then the result holds.

We now prove that

$$f'(w) = <\sum_{n=1}^{\infty}\widehat{f}(n)z^{n-1}, h> = <Bf, h> .$$

To this aim, thanks to the uniform convergence, we can differentiate the series $\sum_{n=0}^{\infty}\widehat{f}(n)z^n$ and hence we have:

$$<\sum_{m=0}^{\infty}\widehat{f}(m+1)z^m, \sum_{m=0}^{\infty}\frac{(m+1)\overline{w}^m z^m}{2\pi\overline{p}_m}>_{\overline{G}}$$

$$= \sum_{m=o}^{\infty}\frac{\widehat{f}(m+1)(m+1)}{\overline{p}_m}\int_0^1 r^{2m+1}w^m\overline{G}(r)dr$$

$$= \sum_{m=0}^{\infty}\widehat{f}(m+1)(m+1)w^m = f'(w).$$

Finally we get, applying the Cauchy-Schwartz inequality to $f'(w)$

$$|f'(w)|^2 \leq \|Bf\|^2_{\overline{G}}\|h\|^2_{\overline{G}}.$$

Now since $\beta_n = 2\pi\overline{p}_n$ it follows that $\|B\| = \sup\frac{(n+2)^{a+1}}{(n+1)^{a+1}} \leq 2^{a+1}$. Then

$$\|Bf\|_{\overline{G}} \leq \|B\|\|f\|_{\overline{G}} \leq 2^{a+1}\|f\|_{\overline{G}} \tag{6}$$

Recall that there exists $D > 0$ such that $\|\overline{k}_w\|^2_{\overline{G}} \leq D\|\widetilde{k}_w\|^2_{\widetilde{G}}$ and hence the last inequality can be written explicitly as

$$\sum_{m=0}^{\infty}(m+1)^{a+1}|w|^{2m} \leq \frac{D}{(1-|w|^2)^{a+2}}.$$

Since $\|h\|^2_{\overline{G}} = \sum_{m=0}^{\infty}(m+1)^{a+3}|w|^{2m}$, by assuming $\gamma = a+2$ we obtain

$$\|h\|^2_{\overline{G}} \leq \frac{D}{(1-|w|^2)^{\gamma+2}} = \frac{D}{(1-|w|^2)^{a+4}} \tag{7}$$

and inequalities (6) and (7) yield:

$$|f'(w)| \leq 2^{a+1}\|f\|_{\overline{G}}\ \frac{\sqrt{D}}{(1-|w|^2)^{\frac{a+4}{2}}}. \tag{8}$$

Note that the same inequality holds for $f \in H^2$.

We have already observed that the Bergman space can be viewed as a particular case of a weighted Hardy space $H^2(\beta)$ with $\beta_n = 2\pi p_n$, and similarly H^2, with $\beta_n = 1$. We introduce here a particular sequence a^{a+1} with $a \geq -1$, where a denotes the harmonic sequence $\frac{1}{n+1}$.We denote the Hardy space associated to a^{a+1} with $H^2(a^{a+1})$. Note that for $a = -1$ this space coincides with H^2 and for $a > -1$ with $A^2_{\overline{G}} = A^2_G$, where G is the standard weight. The space $H^2(a^{a+1})$ is an Hilbert space. In the sequel we will prove our results in $H^2(a^{a+1})$. Note that, with this notation, if a composition operator C_φ is cyclic in $H^2(a^{a+1})$ for $a = -1$ then it is cyclic for every $a \geq -1$.

Holomorphic self maps of the open unit disc.

In a certain sense, every holomorphic self map of Δ has an attractive fixed point: if there is not a fixed point in Δ, then there is a unique boundary point that serves the purpose. This is the content of the famous Denjoy-Wolff theorem. We call a point $p \in \partial\Delta$ a *boundary fixed point of* φ if φ has non tangential limit p in p. The fixed point p, to which the iterates of φ converge is called *Denjoy-Wolff point* of φ. For a proof of the Wolff-Denjoy theorem, see [11]. Motivated by the classification of linear fractional self maps of Δ, and encouraged by the restrictions given by the Wolff-Denjoy theorem on the values the derivative of an arbitrary holomorphic self map of Δ can take at its Denjoy-Wolff point, we introduce the following general classification: a holomorphic self map φ of Δ is of *dilation type* if it has a fixed point in Δ; *hyperbolic type* if it has no fixed point in Δ and has derivative strictly smaller than 1 at its Denjoy-Wolff point, and of *parabolic type* if it has no fixed point in Δ and it has derivative equal to 1 at its Denjoy-Wolff point. The maps of parabolic type fall in two subclasses: *automorphic type,* those having an orbit that is hyperbolically separated (meaning that, for each $z \in \Delta$, the hyperbolic distance, [11], between subsequent points of the orbit $\varphi_n(z)$ stay bounded away from zero) and *non automorphic type,* those for which no orbit is hyperbolically separated.

In the following proposition, one can find a simple illustration of how dynamical properties of an inducing map, as those determined in this case by the fixed point position, can influence the hypercyclic behaviour of the induced composition operator. We omit the proof in A_2^G since it is quite similar to the one given for the Hardy space (Prop. 0.1 in [4]); here one have to use the continuity of point evaluation functionals on A_G^2.

PROPOSITION 1.1. – *Let G be a standard weight function and let φ be a holomorphic self map of Δ that fixes a point z_0 in Δ. Then C_φ is not hypercyclic in A_G^2. Moreover, if φ is not an elliptic automorphism, then for each $f \in A_G^2$ the only limit point of $Orb(C_\varphi, f)$ is the constant function $f(z_0)$.*

REMARK 1.2. – This result holds in the case of standard weights G. For fast weights we must request the boundedness of C_φ. If φ fixes the origin then it induces a bounded composition operator (in fact from the Schwarz lemma and from the hypothesis that G is not increasing, condition (3) is always satisfied). We can then observe that in A_G^2, with G fast weight, every elliptic automorphism with the corresponding composition operator C_φ bounded, must necessarily fix the origin, that is it must be a rotation. We then conclude, by the previous proposition, that every function in the closure of the orbit along f must be equal to 0 at the origin. If instead φ is not an elliptic automorphism, the proof is the same as the one given for standard weights.

1.2 – *The linear fractional case*

In the study of cyclic and hypercyclic behaviour of the composition operator C_φ, the linear fractional self maps of Δ play a fundamental role. We first study the cyclic and hypercyclic composition operators induced by linear fractional self maps of Δ; one can then transfer the obtained results to a wider setting by using linear fractional maps to represent more general ones. The possibility of such a connection is suggested by a remarkable theorem of classical function theory: the *Linear Fractional Model Theorem* [5], [4].

Hence, as a first step, we study composition operators induced by linear fractional maps.

Recall that, by Proposition 1.1 the possibility of hypercyclicity arises only for self maps of Δ without interior fixed point; for the rest, the issue is confined to cyclicity.

Simply observing that every automorphism φ of Δ is conjugated, by the automorphism $\gamma = \frac{z-z_0}{1-\bar{z}_0 z}$ (where z_0 is a fixed point of φ), to a rotation $\psi(z) = \lambda z$, we can conclude that if $C_{\lambda z}$ is cyclic so is C_φ. Conversely, if $C_{\lambda z}$ is not cyclic, nor is C_φ. Recall that the positive result holds in H^2 [4]: if λ is an irrational multiple of π then C_φ is cyclic in H^2. By Theorem 1.2 it follows that, in this case, the composition operator C_φ is cyclic in A^2_G for any weight function G. Suppose, conversely, that λ is a rational multiple of π. We must distinguish the standard case and non standard case. For the first one let f be in A^2_G, its orbit, under the $C_{\lambda z}$ action, is made of a finite number of points, then f cannot be a cyclic vector. In the non standard case, since C_φ is bounded, by Remark 1.2, φ must be a rotation. We can then apply the same proof given in the standard case. We have then concluded once and for all the elliptic case:

PROPOSITION 1.2. – *Let G be any weight function. If φ is an elliptic automorphism of Δ then the associated composition operator C_φ, if it is bounded, is cyclic in A^2_G if, and only if, φ is conjugate with a rotation of an irrational multiple of π.*

Now, thanks to Theorem 1.2 and Theorem 2.2 in [4], we conclude that all the maps without interior fixed point, except for the parabolic non automorphism maps, induce hypercyclic composition operators, while the maps that are not automorphisms induce cyclic operators if they have an interior and an exterior fixed point. The case in which φ has an interior and a boundary fixed point is still open for weighted Bergman spaces. We have to analyze now the case in which φ is a linear fractional parabolic self map of Δ, that is not an automorphism. In the following proposition we prove that C_φ is strictly non hypercyclic on $H^2(a^{a+1})$ with $-1 \leq a < 0$. The proof is quite similar to the one given in H^2, but we enclose it here since we use the obtained estimate (8) for $|f'(w)|$.

PROPOSITION 1.3. – *Let φ be a linear fractional self map of Δ that is not an automorphism and which does not have an interior fixed point. If φ is parabolic, then C_φ is not hypercyclic on $H^2(a^{a+1})$ with $-1 \leq a < 0$; in fact the only possible limit points of the C_φ-orbit are constant functions.*

PROOF. – The map φ is parabolic, so it has a unique fixed point, necessarily on the unit circle. Without loss of generality we may assume this fixed point is 1. We explicitly compute φ by employing the Cayley trasformation $T(z) = \frac{z+1}{1-wz}$, which sends Δ to the right half plane $\Pi = \{w \in \mathbb{C} \ : \ \Re e w > 0\}$, the fixed point 1 to ∞ and φ to a traslation map $\psi{:}\Pi \to \Pi$, that fixes ∞. Then $\psi = T \circ \varphi \circ T^{-1}$ has the form $\psi(w) = w + a$, where $\Re e a > 0$, since ψ has range in Π and it is not an automorphism ($\Rightarrow \Re e a \neq 0$.) Pulling back to the unit disc we obtain:

$$\varphi(z) = \frac{(2-a)z + a}{-az + (2+a)}$$

And more generally, for $n = 0, 1, 2....$

$$\varphi_n(z) = \frac{(2-na) + na}{-naz + (2+na)}$$

Furthermore

$$1 - \varphi(z) = \frac{2(1-z)}{a(1-z) + 2}$$

and

$$\varphi(z) - \varphi(0) = \frac{4z}{(2+a)(2+a-az)}$$

Upon replacing a by na in these expressions and letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} n[1 - \varphi_n(z)] = \frac{2}{a}$$

$$\lim_{n\to\infty} n^2[\varphi_n(z) - \varphi_n(0)] = \frac{4z}{a^2(1-z)}$$

Now suppose $f, g \ \in \ H^2(a^{a+1})$ with g a cluster point of $Orb(C_\varphi, f)$. Our goal is to prove that g must be costant on Δ. Recall that for every $w \in \ \Delta$ the estimate 8 holds; hence by the equivalence of the norms induced, respectively, by A^2_G and by $A^2_{\tilde{G}}$ we obtain also

$$|f'(w)| \leq 2^{a+1}\|f\|_G \ \frac{M}{(1-|w|^2)^{\frac{a+4}{2}}} \leq 2^{a+1}\|f\|_G \ \frac{M}{(1-|w|)^{\frac{a+4}{2}}},$$

for an appropriate constant M. A similar inequality holds in H^2 and so in general for each $f \in H^2(a^{a+1})$. In the sequel we denote with D a costant indipendent of n. Thus for any z, w in Δ, with $|z| < |w| < 1$, upon integrating f' over the line segment from z to w, and using the above inequality for $H^2(a^{a+1})$, we obtain that:

$$|f(z) - f(w)| \leq C\|f\|_a \frac{|z-w|}{(1-|w|)^{\frac{a+4}{2}}}$$

By referring to the right half plane realization of φ_n as a traslation by na (where we recall that $\Re ea > 0$) we see that the φ orbit of any point in Δ converges non tangentially to 1. Now fix $z \in \Delta$ and write $s_n = \varphi_n(0)$ and $t_n = \varphi_n(z)$. By the non tangential convergence we can assert that there exists $C = \max\{C_1, C_2\}$ such that $|1 - s_n| \leq C_1(1 - |s_n|) \leq C(1 - |s_n|)$ and $|1 - t_n| \leq C_2(1 - |t_n|) \leq C(1 - |t_n|)$ for every $n \in \mathbb{N}$. For convenience let $\{u_n\}$ denote either $\{s_n\}$ or $\{t_n\}$: the one that converges faster to 1. Then

$$|f(t_n) - f(s_n)| \leq D\|f\|_a \frac{|t_n - s_n|}{(1 - |u_n|)^{\frac{a+4}{2}}} \leq D_1 \|f\|_a n^{\frac{a}{2}}.$$

Thus, for $-1 \leq a < 0$ and for $n \to \infty$, it follows that $\lim_{n\to\infty} [f(t_n) - f(s_n)] = 0$. Since the convergence in $H^2(a^{a+1})$ implies pointwise convergence then, if g is a cluster point of $f \circ \varphi_n$, we get

$$g(z) - g(0) = \lim_{n\to\infty} [f(\varphi_n(z)) - f(\varphi_n(0))] = \lim_{n\to\infty} [f(t_n) - f(s_n)] = 0$$

implying that $g(z) = g(0)$, regardless of our choice of $z \in \Delta$. So g is costant, as desired. ■

A proof of the same result, with complitely different techniques, is given by E.A. Gallardo-Gutierrez and A. Montes-Rodriguez in the paper *The Role of the Spectrum in the Cyclic Behaviour of Composition Operators* [6].

1.3 – *Transference Principle.*

Following word-for-word [4] one can also solve the cyclicity problem in the case of composition operators C_φ which are not induced by linear fractional maps. Recall that the linear fractional model theorem guarantees that each univalent self map of Δ is conformally similar to a linear fractional self map of Δ that is of the same type, but is now viewed as acting on a more complicated domain.
Using the same notation of [4] we get the following general result

THEOREM 1.3 (Transference Principle). – *Suppose that φ is a univalent self map of Δ of either dilation, hyperbolic or parabolic automorphism type. Let σ be the interwining map for φ obtained by the linear fractional model theorem, and let ψ be the linear fractional self map of Δ which represents φ. Suppose further that the set of polynomials in σ is dense in $H^2(a^{a+1})$. Then the cyclic behavior of the linear fractional composition operator C_ψ transfers to C_φ. More precisely:*

(i) *If φ is of hyperbolic type, or parabolic automorphism type, then C_φ is hypercyclic*

(ii) *If φ is of dilation type, then C_φ is cyclic, but not hypercyclic.*

The Transference Principle does not address the issue of whether the cyclic behavior of parabolic non automorphisms transfers to more general self maps of that type. However there are some results, also in that case, if one requests further hypoteses on the *regularity* of φ at its Wolff-Denjoy point (see [4]). In the dilation case we have:

THEOREM 1.4. – *Suppose that σ maps Δ univalently onto a domain $\mathcal{D} \subseteq \mathbb{C}$, and that there exists a complex number $\lambda \in \Delta$ such that $\lambda\mathcal{D} \subseteq \mathcal{D}$. Suppose further that the polynomials in σ are dense in $H^2(a^{a+1})$. Let $\varphi = \sigma^{-1} \circ \lambda\sigma$. Then the composition operator C_φ is cyclic. Furthermore the collection of cyclic vectors for this operator is dense in $H^2(a^{a+1})$.*

The proof of the previous theorem goes through word-for-word as the one given in H^2 [4].
One can apply the Transference Principle to show that the cyclicity problem for C_φ is related to a polynomial approximation problem for φ when $\|\varphi\|_\infty < 1$. Note that the condition $\|\varphi\|_\infty < 1$ implies, by the Brouwer theorem, that φ must have an interior fixed point.

THEOREM 1.5. – *Suppose that φ is holomorphic on Δ and that $\|\varphi\|_\infty < 1$. If polynomials in φ are dense in $H^2(a^{a+1})$, then C_φ is cyclic.*

A proof of the above theorem, in the standard case, easily follows the one given in [4] (Theorem 3.4), taking into account the density of polynomials and the generalized Littlewood principle in $H^2(a^{a+1})$. In their proof, Shapiro and Bourdon shows that, without loss of generality, one can assume $\varphi(0) = 0$. Note that, in the case of fast weight, one cannot arbitrarily choose the fixed point. Indeed if the fixed point is different from the origin, we may once again say that there exists an automorphism γ of Δ that takes this point to 0, but we cannot say that the cyclic behavior of the new composition operator, induced by $\gamma^{-1} \circ \varphi \circ \gamma$, is the same as the behavior of C_φ, since C_γ is not bounded. If we take φ such that $\varphi(0) = 0$, the composition operator C_φ is bounded and so is C_σ (where σ is the interwining map) since $\sigma(0) = 0$. We may then repeat the same proof given for the standard weight G; hence if G is fast we have to impose in Theorem 1.5 the additional condition $\varphi(0) = 0$.

By applying the generalized Walsh theorem, we can affirm that polynomials in φ are dense in $H^2(a^{a+1})$ whenever $\varphi(\Delta)$ is a Jordan domain; then we have the following:

COROLLARY 1. – *If the holomorphic function φ maps Δ univalently onto the interior of a Jordan curve lying in Δ, then C_φ is cyclic on $H^2(a^{a+1})$.*

1.4 – *Concluding remarks and open questions.*

In the previous part of the paper we have pointed out that all the positive cyclicity results on H^2 transfer to the weighted Bergman space. We here present an example of a composition operator induced by an opportunely chosen map φ which is cyclic on a weighted Bergman space and not cyclic on H^2. We have not been able to present an example in the unweighted Bergman space.

Recall that a region $\Omega \subseteq \mathbb{C}$ bounded by two Jordan curves which intersect in a single point, such that one of the Jordan curves is internal to the other, is usually called *crescent*. Let φ be a Riemann map onto a crescent domain Ω such that

- $0 \in \Omega$ and $\varphi(0) = 0$;
- The closure of Ω lies inside the unit circle so that $\|\varphi\|_\infty < 1$;
- The Toepliz operator T_φ on $H^2(\Delta)$, defined by $T_\varphi(f) := \varphi f$, is cyclic yet 1 is not a cyclic vector.

That such a Ω exists follows from Corollary 3.7 of [1]. Observe that the fact that 1 is not a cyclic vector for T_φ means that polynomials in φ cannot be dense in H^2 which by Corollary 1.6 of [4], shows that C_φ is not cyclic on H^2. Denote with f a cyclic vector for T_φ. We claim that C_φ is cyclic on the weighted Bergman space $A^2_{|f|^2}(\Delta)$. There are some details to check. First note that the closed graph theorem together with the fact that $\|\varphi\|_\infty < 1$ shows that C_φ is bounded. Now recall, from Remark 1.1 that polynomials are dense in A^2_G. By choice of φ, $\{p(\varphi)f :\ p \text{ is a polynomial}\}$ is dense in H^2. Hence this set is dense in the unweighted Bergman space, and it quickly follows that $\{p(\varphi) :\ p \text{ is a polynomial}\}$ is dense in $A^2_{|f|^2}(\Delta)$. Theorem 2.3 then implies that the composition operator is cyclic in $A^2_{|f|^2}(\Delta)$.

Note that the reasoning above can be applied to produce a cyclic composition operator on the unweighted Bergman space that is not cyclic on H^2 if, in the multiplication operators language there is a φ such that $T_\varphi : H^2 \to H^2$ is cyclic but does not have 1 as a cyclic vector, while $T_\varphi : A^2(\Delta) \to A^2(\Delta)$ is cyclic with 1 as cyclic vector. This is a tough problem, that is been open for some time. Note that the cyclic vector f can be taken to be bounded, in which case $A^2_{|f|^2}(\Delta)$ contains the unweighted Bergman space.

Necessary conditions for ciclicity.

In the Hardy space one can give several necessary conditions which guarantee the cyclicity of a composition operator. Shapiro and Bourdon have proved in [4] that if C_φ is cyclic in H^2 then φ must be univalent in Δ and univalent almost everywhere on $\partial\Delta$; however, in a more general framework we can similarly prove that if C_φ is cyclic in $A^2_G(\Delta)$ then φ must be univalent in Δ. We do not know whether the univalence a.e on $\partial\Delta$ is necessary for cyclicity on the Bergman space.

There are several results concerning necessary conditions for the cyclicity of the Shift operators M_z both in the Hardy and Bergman spaces ([1], [2]), and it would be very interesting to find connections between these two operators.

We recall that in H^2, a necessary condition for cyclicity is that polynomials in φ are dense in the space. Thus in H^2, the density of polynomials in φ is quite equivalent to the cyclicity of C_φ. In this context another open question is whether the density of polynomials in φ is necessary or not for cyclicity in $A^2_G(\varDelta)$. Indeed we cannot apply the argument used for H^2. Furthermore note that if φ is a Riemann map from the unit disc to an open domain $\varOmega$ and we consider the usual Bergman space $L^2(\varOmega)$ (see [1] for the precise definition) the equivalence between the density of polynomials in $L^2(\varOmega)$ and the density of polynomials in φ, contrary to what happens in H^2, does not hold. Indeed in [2] Akeroyd showed that, if $\varOmega$ is a crescent domain bounded by two internally tangent circles, then the shift M_z is cyclic on $H^2(\varOmega)$ and 1 is a cyclic vector; in other words, we have that polynomials in φ are dense in $H^2(\varDelta)$. Using the proved density of H^2 in the Bergman space we conclude that polynomials in φ are dense in $L^2(\varDelta) = A^2(\varDelta)$. In [2] it is also shown that 1 is not cyclic for M_z on $L^2(\varOmega)$, which means that polynomials are not dense in $L^2(\varOmega)$, and this proves our assertion.

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