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Artinian Automorphisms of Infinite Groups.

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Sunto. – *Un automorfismo a di un gruppo G è detto artiniano se per ogni catena strettamente decrescente $H_1 > H_2 > \cdots > H_n > \cdots$ di sottogruppi di G esiste un intero positivo m tale che $(H_n)^a = H_n$ per ogni $n \geq m$. In questa nota si dimostra che in molti casi il gruppo di tutti gli automorfismi artiniani di G coincide con il gruppo di tutti gli automorfismi potenza di G.*

Summary. – *An automorphism a of a group G is called an artinian automorphism if for every strictly descending chain $H_1 > H_2 > \cdots > H_n > \cdots$ of subgroups of G there exists a positive integer m such that $(H_n)^a = H_n$ for every $n \geq m$. In this paper we show that in many cases the group of all artinian automorphisms of G coincides with the group of all power automorphisms of G.*

1. – Introduction.

An automorphism a of a group G is called a *power automorphism* if $H^a = H$ for each subgroup H of G. The set $PAutG$ of all power automorphisms of G is an abelian residually finite normal subgroup of the full automorphism group $AutG$ of G, and its structure has been described by Cooper in [2]. Later, Curzio, Franciosi and de Giovanni [3] investigated properties of the group $IAutG$ consisting of all I-automorphisms of the group G (i.e. of all automorphisms a of G such that $H^a = H$ for each infinite subgroup H of G).

An automorphism a of a group G is called an *artinian automorphism* (or an *A-automorphism*) if for every strictly descending chain

$$H_1 > H_2 > \cdots > H_n > \cdots$$

of subgroups of G there exists a positive integer m such that $H_n^a = H_n$ for every $n \geq m$.

The set $AAutG$ of all artinian automorphisms of a group G is a normal subgroup of $AutG$ containing $IAutG$, and clearly $AAutG = AutG$ if G satisfies the minimal condition on subgroups.

The aim of this paper is to show that in many natural cases $AAutG = PAutG$

when G is non-artinian. Among the results, we will prove that this equality holds if the group G contains a locally radical non-artinian normal subgroup. As a consequence of this result, we observe that, if G is any infinite group with $AAutG \neq PAutG$, then all locally radical Černikov normal subgroups of G are soluble Černikov groups, and the same arguments used in [1] can be applied to give a description of groups with $AAutG \neq PAutG$.

Most of our notation is standard and can for instance be found in [6].

2. – General properties.

Recall that the *norm* $N(G)$ of a group G is the intersection of the normalizers of the subgroups of G; of course, $N(G)$ is the set of all elements of G inducing by conjugation a power automorphism on G. In our consideration we will use the set $AN(G)$ of all elements of G inducing an artinian automorphism on G (the *A-norm* of G); it turns out that $AN(G)$ is a characteristic subgroup of G playing a relevant role in the study of A-automorphisms. Recall also that a group G is *locally graded* if each finitely generated non-trivial subgroup of G contains a proper subgroup of finite index.

LEMMA 2.1. – *If G is any group, the set $AAutG$ is a normal subgroup of $AutG$.*

PROOF. – Let α, β be elements of $AAutG$, and consider in G a descending chain of subgroups

$$H_1 > H_2 > \cdots > H_n > \cdots (\star)$$

Then there exists a positive integer m such that $H_n^{\alpha} = H_n^{\beta} = H_n$ for every $n \geq m$, so that $H_n^{\alpha\beta^{-1}} = H_n$ for $n \geq m$ and $\alpha\beta^{-1} \in AAutG$. Therefore $AAutG$ is a subgroup of $AutG$. In order to prove that $AAutG$ is normal in $AutG$, let $\varphi \in AutG$, $\alpha \in AAutG$, and consider a descending chain of subgroups $(\star)$ as above. The hypothesis applied to the chain

$$H_1^{\varphi^{-1}} > H_2^{\varphi^{-1}} > \cdots > H_n^{\varphi^{-1}} > \cdots$$

yields that there is $k \in \mathbb{N}$ such that $H_n^{\varphi^{-1}\alpha} = H_n^{\varphi^{-1}}$ for each $n \geq k$; thus $H_n^{\varphi^{-1}\alpha\varphi} = H_n$ for $n \geq k$ and $\varphi^{-1}\alpha\varphi \in AAutG$. ■

LEMMA 2.2. – *Let G be a group, and let α be an A-automorphism of G. If H is a subgroup of G and $H^{\alpha} \leq H$, then $H^{\alpha} = H$.*

PROOF. – Assume by contradiction that $H^{\alpha} < H$. Then $H^{\alpha^{n+1}} < H^{\alpha^n}$ for each non-negative integer n, and hence

$$H > H^a > \cdots > H^{a^n} > \cdots$$

is a strictly descending chain of subgroups which are not fixed by a. ■

LEMMA 2.3. – *If G is any group, the set $AN(G)$ is a characteristic subgroup of G and AAutG) acts trivially on $G/AN(G)$. Moreover, if G is locally graded, then $AN(G)$ either is abelian or locally finite.*

PROOF. – It follows from Lemma 2.1 that $AN(G)$ is a characteristic subgroup of G. Consider elements $g \in G$ and $a \in AAutG$, and let

$$H_1 > H_2 > \cdots > H_n > \cdots$$

be a descending chain of subgroups of G; then there exists a positive integer m such that $H_n^a = H_n$ and $(H_n^g)^a = H_n^g$ for each $n \geq m$. It follows that

$$H_n^g = (H_n^g)^a = (H_n^a)^{g^a} = H_n{}^{g^a},$$

and hence $H_n^{g^{-1}g^a} = H_n$ for $n \geq m$. Thus $g^{-1}g^a$ belongs to $AN(G)$, and a acts trivially on $G/AN(G)$.

Suppose now that G is locally graded. Clearly $AN(G)$ locally satisfies the minimal condition on non-normal subgroups, so that every finitely generated subgroup of $AN(G)$ either is finite or abelian (see [5]). The lemma is proved. ■

LEMMA 2.4. – *Let G be a group, and let a be an A-automorphism of G. If x is any element of infinite order of G, then either $x^a = x$ or $x^a = x^{-1}$.*

PROOF. – Let p and q be different primes. Since $a \in AAutG$, there exists a positive integer m such that $\langle x^{p^n} \rangle^a = \langle x^{p^n} \rangle$ and $\langle x^{q^n} \rangle^a = \langle x^{q^n} \rangle$ for each $n \geq m$. Then $\langle x \rangle = \langle x^{p^n}, x^{q^n} \rangle$ is fixed by a, and hence either $x^a = x$ or $x^a = x^{-1}$. ■

COROLLARY 2.5. – *If G is any torsion-free group, then $AAutG = PAutG$. In particular, $|AAutG| = 1$ if G is non-abelian and $|AAutG| = 2$ if G is abelian.*

3. – Main results.

The results in this section show that there are many structural restrictions on a group G when $AAutG \neq PAutG$. Moreover, in many cases $AAutG$ induces a group of power automorphisms on $AN(G)$, so that in particular $AAutG$ is metabelian.

LEMMA 3.1. – *Let G be a group containing an infinite residually finite subgroup H. If x is an element of $N_G(H)$, then $\langle x\rangle^a = \langle x\rangle$ for each A-automorphism a of G.*

PROOF. – By Lemma 2.4 it can be assumed that x has finite order, so that also the subgroup $\langle H, x\rangle$ is residually finite. Consider in H a descending chain

$$H_1 > H_2 > \cdots > H_n > \cdots$$

of normal subgroups of finite index of $\langle H, x\rangle$ such that $H_1 \cap \langle x\rangle = \{1\}$ and

$$\bigcap_{n\in\mathbb{N}} H_n = \{1\}.$$

Clearly,

$$\langle H_1, x\rangle > \langle H_2, x\rangle > \cdots > \langle H_n, x\rangle > \cdots$$

is a descending chain of subgroups of G, and hence there exists a positive integer m such that $\langle H_n, x\rangle^a = \langle H_n, x\rangle$ for each $n \geq m$. Since

$$\bigcap_{n\geq m} \langle H_n, x\rangle = \langle x\rangle,$$

it follows that $\langle x\rangle^a = \langle x\rangle$. ■

COROLLARY 3.2. – *Let G be a group, and let H be an infinite residually finite subgroup of G. Then H is fixed by $AAutG$, and $AAutG$ acts on H as a group of power automorphisms.*

Our next result shows in particular that $AAutG = PAutG$ when G is an infinite residually finite group.

THEOREM 3.3. – *Let G be a group containing an infinite residually finite normal subgroup H. Then $AAutG = PAutG$.*

COROLLARY 3.4. – *Let G be a group containing an infinite residually finite subgroup H. Then $AAutG$ acts as a group of power automorphisms on $AN(G)$; in particular, $AAutG$ is metabelian.*

PROOF. – Let x be any element of $AN(G)$. By hypotesis there is in G an infinite descending chain of subgroups

$$H_1 > H_2 > \cdots > H_n > \cdots$$

such that the index $|H_n : H_{n+1}|$ is finite for every n. Since x induces an A-automorphism on G, there exists a positive integer m such that ${H_n}^x = H_n$ for each

$n \geq m$. In particular, x normalizes the infinite residually finite subgroup H_m, and so it follows from Lemma 3.1 that $\langle x \rangle^a = \langle x \rangle$ for any A-automorphism a of G. Therefore $AAutG$ acts as a group of power automorphisms on $AN(G)$; in particular, the commutator subgroup $(AAutG)'$ acts trivially on both $AN(G)$ and $G/AN(G)$, so that $(AAutG)'$ is abelian and $AAutG$ is metabelian. ■

Recall that a group G is *radical* if it has an ascending (normal) series with locally nilpotent factors.

LEMMA 3.5. – *Let G be a group containing a locally radical non-artinian subgroup H. If x is an element of $N_G(H)$, then $\langle x \rangle^a = \langle x \rangle$ for each A-automorphism a of G.*

PROOF. – By Lemma 2.4 it can be assumed that x has finite order. Suppose first that H is periodic, so that $\langle x, H \rangle$ is a locally soluble group; then $\langle x, H \rangle$ contains an abelian non-artinian subgroup A such that $A^x = A$ (see [8]). Clearly, the socle S of A is an infinite residually finite subgroup with $S^x = S$, and it follows from Lemma 3.1 that $\langle x \rangle$ is fixed by a. Assume now that H contains an element h of infinite order. As $\langle x, h \rangle$ is a radical non-artinian group on which $\langle x \rangle$ induces a finite group of automorphisms, there exists an abelian non-artinian subgroup B of $\langle x, h \rangle$ such that $B^x = B$ (see [3], Lemma 2.3). If B is periodic, the above argument can be used to prove that $\langle x \rangle^a = \langle x \rangle$. Suppose finally that B contains an element b of infinite order; the infinite subgroup $\langle x, b \rangle$ is metabelian and so also residually finite (see [6] Part 2, Theorem 9.51), and hence $\langle x \rangle^a = \langle x \rangle$ by Lemma 3.1. ■

The main result of this section is a consequence of the previous lemma. It shows in particular that $AAutG = PAutG$ for any locally radical non-artinian group G.

THEOREM 3.6. – *Let G be a group containing a locally radical non-artinian normal subgroup H. Then $AAutG = PAutG$.*

COROLLARY 3.7. – *Let G be a group containing a locally radical non-artinian subgroup H. Then $AAutG$ acts as a group of power automorphisms on $AN(G)$, and in particular $AAutG$ is metabelian.*

PROOF. – The proof of this corollary is similar to that of Corollary 3.4. ■

In our next theorem we will need the following result due to B. Hartley [4].

LEMMA 3.8. – *Let G be a locally finite group admitting an automorphism φ of prime-power order such that $C_G(\varphi)$ is a Černikov group. Then G contains a locally soluble subgroup of finite index.*

THEOREM 3.9. – *Let G be a group containing a locally finite non-artinian normal subgroup H. Then $AAutG = PAutG$.*

PROOF. – It is clearly enough to prove that, if x is any element of prime power order of G, then $\langle x \rangle^a = \langle x \rangle$ for each A-automorphism a of G. Assume first that the centralizer $C_H(x)$ is not a Černikov group. Then $C_H(x)$ does not satisfy the minimal condition on abelian subgroups (see [7]), and hence it contains an abelian non-artinian subgroup A; it follows from Lemma 3.5 that $\langle x \rangle^a = \langle x \rangle$. Suppose now that $C_H(x)$ is a Černikov group. Applying Lemma 3.8 to the automorphism φ induced by x on H, we obtain that H contains a locally soluble subgroup K of finite index, and of course K can be chosen to be normal in $\langle x, H \rangle$. Then $\langle x \rangle^a = \langle x \rangle$ by Lemma 3.5, and the theorem is proved. ■

The same argument used in the proof of Corollary 3.4 gives the following result.

COROLLARY 3.10. – *Let G be a group containing a locally finite non-artinian subgroup H. Then $AAutG$ acts as a group of power automorphisms on $AN(G)$, and in particular $AAutG$ is metabelian.*

4. – Non-periodic groups.

In section 2 we proved that $AAutG = PAutG$ for any torsion-free group; this can be proved in a more general situation. If G is any group, we shall denote by $W(G)$ the subgroup generated by all elements of infinite order of G; a group G is said to be *weak* if $W(G) = G$.

LEMMA 4.1. – *Let G be a group, and x, y be elements of infinite order of G. If a is any A-automorphism of G, then either $x^a = x$ and $y^a = y$ or $x^a = x^{-1}$ and $y^a = y^{-1}$.*

PROOF. – By Lemma 2.4 the subgroups $\langle x \rangle$ and $\langle y \rangle$ are fixed by a. Assume by contradiction that $x^a = x$ and $y^a = y^{-1}$, so that in particular $\langle x \rangle \cap \langle y \rangle = \{1\}$. It follows from Lemma 2.3 that $y^2 \in AN(G)$, so that y^2 normalizes $\langle x \rangle$ and hence $[x, y^4] = 1$. Thus $\langle x, y^4 \rangle$ is a torsion-free abelian group and $\langle x, y^4 \rangle^a = \langle x, y^4 \rangle$, and a induces on $\langle x, y^4 \rangle$ a power automorphism by Corollary 2.5; this is a contradiction, since $x^a = x$ and $(y^4)^a = y^{-4}$. ■

THEOREM 4.2. – *If G is a weak group, then $AAutG = PAutG$. In particular, $|AAutG| = 2$ if G abelian and $|AAutG| = 1$ if G is non-abelian.*

PROOF. – If G is abelian the statement follows from Theorem 3.6. Assume that G is not abelian, so that there exist elements of infinite order x, y of G such that $xy \neq yx$. Let a be any non-trivial A-automorphism of G; it follows from Lemma 4.1 that $h^a = h^{-1}$ for each element of infinite order h of G. In particular, $x^a = x^{-1}$ and $y^a = y^{-1}$, so that $(xy)^a = x^{-1}y^{-1} \neq (xy)^{-1}$ and hence xy has finite order. Since $y^2 \in AN(G)$ by Lemma 2.3, we have that $[x, y^4] = 1$ and so $[xy, y^4] = 1$. In particular, $\langle xy \rangle^a = \langle xy \rangle$ by Lemma 3.1, so that the non-periodic abelian subgroup $\langle xy, y^4 \rangle$ is fixed by a, and a acts as a power automorphism on $\langle xy, y^4 \rangle$ by Theorem 3.6, a contradiction since $PAut(\langle xy, y^4 \rangle) = \{1, -1\}$. It follows that G admits no non-trivial A-automorphisms, and so $AAutG = PAutG = \{1\}$. ■

Finally, we consider the case of *strong groups*, i.e. groups G such that $1 \neq W(G) \neq G$.

LEMMA 4.3. – *Let G be a group, and let K be a normal subgroup of G. If a is an automorphism of G acting as a universal power automorphism on K, then $[G, a] \leq C_G(K)$.*

PROOF. – Let n be an integer such that $x^a = x^n$ for each $x \in K$. If g is any element of G, we have

$$g^{-1}x^n g = (g^{-1}xg)^a = (g^a)^{-1}x^n g^a$$

for all $x \in K$, and hence $g^a g^{-1} \in C_G(K)$. ■

THEOREM 4.4. – *Let G be a strong group, and let a be an A-automorphism of G. Then a acts trivially on $W(G)$ and on $G/W(G)$. In particular, the group $AAutG$ is abelian.*

PROOF. – Since $W(G)$ is a weak group, it follows from Theorem 4.2 that a acts on $W(G)$ as the identity or as the inversion. Therefore a acts trivially on $G/C_G(W(G))$ by Lemma 4.3. But $C_G(W(G))$ is obviously contained in $W(G)$, and hence a acts trivially on $G/W(G)$. Assume by contradiction that a acts non-trivially on $W(G)$, so that $W(G)$ is abelian and $x^a = x^{-1}$ for each $x \in W(G)$. Thus $AAutG = PAutG$ by Theorem 3.6, and in particular a acts trivially on $G/Z(G)$ (see [2], Theorem 2.2.1). Let h be an element of infinite order of G. Then $h^{-1}h^a = h^{-2}$ belongs to $Z(G)$, and so $Z(G)$ is not periodic; it follows that $W(G) = G$, and the contradiction completes the proof of the theorem. ■

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