Luciano A. Lomonaco

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On Singer's Algebra and Coalgebra Structures.

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Sunto. – *Recentemente W. M. Singer ha introdotto le nozioni di algebra con coprodotti e di coalgebra con prodotti, indebolendo, in un certo senso, la nozione di algebra di Hopf (cfr [6]). Nel presente lavoro si considerano alcune particolari algebre di invarianti e si dimostra che esse costituiscono infatti ulteriori esempi di algebre con coprodotti e di coalgebre con prodotti. Si discute inoltre la stretta relazione fra tali algebre e le strutture considerate da W. M. Singer, fornendo, in particolare, una descrizione di queste ultime in termini di invarianti modulari.*

Summary. – *Recently W. M. Singer has introduced the notion of algebra with coproducts (and the dual notion of coalgebra with products) by somehow weakening the notion of Hopf algebra (see [6]). In this paper we consider certain algebras of invariants and show that they are, in fact, further examples of algebras with coproducts and coalgebras with products. Moreover, we discuss the close relation between such algebras and the structures considered in Singer's paper.*

Introduction.

W. M. Singer has recently introduced two new algebraic structures: the structure of algebra with coproducts and the dual structure of coalgebra with products (see [6]). In that paper he deals, in particular, with two typical examples. He shows that the algebra $\mathcal{B}$ of Steenrod operations in the E_2-term of the classical Adams spectral sequence is in fact an $\mathbb{F}_2$-algebra with products, and explicitly constructs its bigraded dual $\mathcal{H}$, a coalgebra with products. In the present work we consider some algebras of invariants. More precisely, for each $s \geq 1$, we let P_s be the polynomial ring on indeterminates $t_1, \dots, t_s$ (of degree one) on the field $\mathbb{F}_2$, and write G_s for the general linear group $GL(s, \mathbb{F}_2)$ and T_s for its upper triangular subgroup. There is a standard action of G_s on P_s and the element e_s, the product of all the nonzero elements of degree one in P_s, is fixed by such action. We set

$$\Phi_s = P_s[e_s^{-1}]$$

and consider the rings of invariants

$$\Delta_s = \Phi_s^{T_s}\,;\ \Gamma_s = \Phi_s^{G_s}\,.$$

With the notation of [7] (slightly modified though), we have

$$\Delta_s = \mathbb{F}_2[v_{s,1}^{\pm 1}, \dots, v_{s,s}^{\pm 1}]\,;\ \Gamma_s = \mathbb{F}_2[Q_{s,0}^{\pm 1}, Q_{s,1}, \dots, Q_{s,s-1}]$$

where, in particular, $Q_{s,0} = e_s$. We further consider the graded objects

$$\Delta = \{\, \Delta_s\ |s \geq 0 \,\}\,;\ \Gamma = \{\, \Gamma_s\ |s \geq 0 \,\}$$

where $\Delta_0 = \Gamma_0 = \mathbb{F}_2$ by convention and show that Δ, Γ and a certain subalgebra Γ^- of Γ can be given a structure of coalgebra with products. Moreover we explicitely construct the dual $\hat{Q}$ of Γ, an example of algebra with coproducts. Finally we discuss the relation between Γ, Γ^-, $\hat{Q}$ and the structures $\mathcal{B}$ and $\mathcal{H}$ considered by Singer. In particular we introduce a monomorphism

$$a : \mathcal{H} \longrightarrow \Gamma \subseteq \Delta$$

which suggests that each generator ξ_j of the dual of the Steenrod algebra $\mathcal{A}^*$ is related to a certain family of elements of Γ, namely $\{\, Q_{s,0}^{-1} Q_{s,j}\,,\ s \geq j \,\}$. The geometric motivation for defining the monomorphism a will be discussed in a future paper.

1. – Algebras with coproducts and coalgebras with products.

In order to make this paper reasonably self contained, we recall the two main definitions of [6]. We fix $\mathbb{F}_2$ as ground field, and start with the definition of $\mathbb{F}_2$-algebra with coproducts.

DEFINITION 1.1. – *An $\mathbb{F}_2$-algebra with coproducts is a bigraded algebra $\mathcal{C}$ together with degree preserving maps $\varepsilon_s : \mathcal{C}_{*,s} \to \mathbb{F}_2$ and $\psi_s : \mathcal{C}_{*,s} \to \mathcal{C}_{*,s} \otimes \mathcal{C}_{*,s}$, for each $s \geq 0$, such that*

(i) *$\mathcal{C}_{*,s}$ is a graded coalgebra with counit ε_s and coproduct ψ_s, for each $s \geq 0$;*
(ii) *the algebra unit $\eta : \mathbb{F}_2 \to \mathcal{C}_{*,0}$ is a map of coalgebras;*
(iii) *the multiplication $\mu : \bigoplus_{p+q=s} (\mathcal{C}_{*,p} \otimes \mathcal{C}_{*,q}) \to \mathcal{C}_{*,s}$ preserves the coalgebra structure.*

The definition of algebra with coproducts is formally dual to the above definition.

DEFINITION 1.2. – *An $\mathbb{F}_2$-coalgebra with products is a bigraded coalgebra $\mathcal{K}$ together with degree preserving maps $\eta_s : \mathbb{F}_2 \to \mathcal{K}_{*,s}$ and $\mu_s : \mathcal{K}_{*,s} \otimes \mathcal{K}_{*,s} \to \mathcal{K}_{*,s}$, for each $s \geq 0$, such that*

(i) $\mathcal{K}_{*,s}$ *is a graded algebra with unit* η_s *and product* μ_s*, for each* $s \geq 0$*;*
(ii) *the algebra counit* $\varepsilon : \mathcal{K}_{*,0} \to \mathbb{F}_2$ *is a map of algebras;*
(iii) *the comultiplication* $\psi : \mathcal{K}_{*,s} \to \prod_{p+q=s} \left(\mathcal{K}_{*,p} \otimes \mathcal{K}_{*,q}\right)$ *preserves the algebra structure.*

In [6] two typical examples of such structures are provided. The first one is the algebra $\mathcal{B}$ of secondary Steenrod operations in the E_2-term of the Adams spectral sequence: $\mathcal{B}$ is generated by symbols $Sq^0, Sq^1, Sq^2, \ldots$, subject to the classical Adem relations, but *not* to the relation $Sq^0 = 1$. $\mathcal{B}$ is bigraded by setting $|Sq^k| = (k, 1)$ and is an example of algebra with coproducts. In particular we have

$$\psi_1(Sq^k) = \sum_{j+\ell=k} Sq^j \otimes Sq^\ell \quad (\text{with } j, \ell \geq 0).$$

The second example is the algebra $\mathcal{H}$. He defines, for each $s > 0$,

$$\mathcal{H}_{*,s} = \mathbb{F}_2[\xi_{1,s}, \xi_{2,s}, \ \ldots\ , \xi_{s,s}]$$

with $|\xi_{k,s}| = (2^k - 1, s)$ for each $1 \leq k \leq s$, and

$$\mathcal{H} = \left\{\, \mathcal{H}_{*,s} \ |\{s \geq 0 \,\right\}$$

where $\mathcal{H}_{*,0}$ is conventionally taken to be $\mathbb{F}_2$ and the unit in $\mathcal{H}_{*,s}$ is $\xi_{0,s}$ for each $s \geq 0$. Singer proves that $\mathcal{H}$ is a coalgebra with products and that $\mathcal{B} \cong \hat{\mathcal{H}}$, the bigraded dual of $\mathcal{H}$.

2. – Modular invariants.

In this section we recall some standard facts of modular invariant theory. With the notation already used in the introduction, we grade each Δ_s by assigning degree 1 to the element $v_{s,j}$ for each j. Consequently, Δ is naturally bigraded. As in [2], we define a graded multiplication

$$\mu : \Delta \otimes \Delta \longrightarrow \Delta$$

by juxtaposition: if

$$I = (i_1, \ldots, i_h) \quad , \ J = (j_1, \ldots, j_k)$$

are multi-indices and we let v^I, v^J denote the monomials $v_{h,1}^{i_1} \ldots v_{h,h}^{i_h} \in \Delta_h$ and $v_{k,1}^{j_1} \ldots v_{k,k}^{j_k} \in \Delta_k$, we set

$$v^I * v^J = \mu(v^I \otimes v^J) = v_{h+k,1}^{i_1} \cdots v_{h+k,h}^{i_h} v_{h+k,h+1}^{j_1} \cdots v_{h+k,h+k}^{j_k} \in \Delta_{h+k} \ .$$

With respect to this algebra structure, we consider the two-sided ideal (Γ_2) in Δ

and the quotient

$$\bar{Q} = \Delta/(\Gamma_2) \ .$$

In [5] P. May considered an algebra, which we call *the universal Steenrod algebra* and indicate by Q, with generators x_k, $k \in \mathbb{Z}$ and generalized Adem relations

$$x_{2k-1-n}x_k = \sum \binom{n-1-j}{j} x_{2k-1-j}x_{k+j-n} \quad (k \in \mathbb{Z},\ n \in \mathbb{N}_0)$$

and we have

$$Q \cong \bar{Q} \ ,$$

an isomorphism being induced by the map $x_k \mapsto v_{1,1}^{k-1}$ (see [2]). In [7] a coalgebra structure has been considered in Δ by introducing a coproduct

$$\psi : \Delta \longrightarrow \Delta \otimes \Delta$$

induced by the maps

$$\psi_{h,k} : \Delta_{h+k} \longrightarrow \Delta_h \otimes \Delta_k$$

where $\psi_{h,k}(v^I * v^J) = v^I \otimes v^J$. In fact, Γ is a subcoalgebra of Δ. More explicitely, we have

$$\psi_{h,k}(Q_{h+k,i}) = \sum_{j \geq 0} Q_{h,0}^{2^k - 2^j} Q_{h,i-j}^{2^j} \otimes Q_{k,j} \ .$$

3. – Some new examples.

Let us consider the coalgebra Δ. More precisely, we consider the comultiplication ψ in Δ, but not the graded multiplication μ. Moreover, for each $s \geq 0$, we let

$$\mu_s : \Delta_s \otimes \Delta_s \longrightarrow \Delta_s$$

be the usual multiplication. In other words, if $v_{s,1}^{\ell_1} \dots v_{s,s}^{\ell_s}$ and $v_{s,1}^{m_1} \dots v_{s,s}^{m_s}$ are in Δ_s, we consider the product

$$\mu_s(v_{s,1}^{\ell_1} \dots v_{s,s}^{\ell_s} \otimes v_{s,1}^{m_1} \dots v_{s,s}^{m_s}) = v_{s,1}^{\ell_1} \dots v_{s,s}^{\ell_s} \cdot v_{s,1}^{m_1} \dots v_{s,s}^{m_s} = v_{s,1}^{\ell_1+m_1} \dots v_{s,s}^{\ell_s+m_s}.$$

Finally, we define $\varepsilon : \Delta_0 \to \mathbb{F}_2$ and, for each $s \geq 0$, $\eta_s : \mathbb{F}_2 \to \Delta_s$ in the obvious way: ε is the identity map and η_s is the natural embedding.

THEOREM 3.1. – *With the above operations, Δ is a coalgebra with products (over the field $\mathbb{F}_2$).*

PROOF. – The first two axioms of Definition 1.2 are trivially verified. As far as Axiom (iii) is concerned, we notice that it is enough to check that for each $h, k \geq 0$

$$\psi_{h,k} : \Delta_{h+k} \longrightarrow \Delta_h \otimes \Delta_k$$

is a map of graded algebras. If $v^I, v^{I'} \in \Delta_h$, and $v^J, v^{J'} \in \Delta_k$, with

$$I = (i_1, \ldots, i_h)\,,\ I' = (i'_1, \ldots, i'_h)\,,\ J = (j_1, \ldots, j_k)\,,\ J' = (j'_1, \ldots, j'_k)$$

we write

$$I + I' = (i_1 + i'_1, \ldots, i_h + i'_h)\,,\ J + J' = (j_1 + j'_1, \ldots, j_k + j'_k)$$

and

$$(I, J) = (i_1, \ldots, i_h, j_1, \ldots, j_k)\,,\ (I', J') = (i'_1, \ldots, i'_h, j'_1, \ldots, j'_k)\,.$$

In particular $v^{(I,J)} = v^I * v^J$. We have

$$\begin{aligned}\psi_{h,k}\big(v^{(I,J)} \cdot v^{(I',J')}\big) &= \psi_{h,k}\big(v^{(I+I',J+J')}\big)\\ &= v^{I+I'} \otimes v^{J+J'}\\ &= \big(v^I \otimes v^J\big) \cdot \big(v^{I'} \otimes v^{J'}\big)\\ &= \psi_{h,k}\big(v^{(I,J)}\big) \cdot \psi_{h,k}\big(v^{(I',J')}\big)\end{aligned}$$

■

It has been observed that Γ is a subcoalgebra of Δ. Therefore

COROLLARY 3.2. – *Γ is a sub-(coalgebra with products) of Δ.*

PROOF. – An easy consequence of the above theorem and the fact that Γ is a subcoalgebra of Δ. ■

We would like to introduce a further example of coalgebra with products. For each $s \geq 0$, we set

$$\Gamma_s^- = \mathbb{F}_2[Q_{s,0}^{-1}Q_{s,1}, Q_{s,0}^{-1}Q_{s,1}, \ldots, Q_{s,0}^{-1}Q_{s,s-1}, Q_{s,0}^{-1}]$$

and

$$\Gamma^- = \big\{\, \Gamma_s^- \;|\{s \geq 0\,\big\}.$$

COROLLARY 3.3. – *Γ^- is a sub-(coalgebra with products) of Δ and Γ.*

PROOF. – Again, the only nontrivial fact to check is that the coproduct in Γ restricts to a coproduct in Γ^-. To this extent, we observe that, for each $h, k \geq 0$

and for each nonnegative $i \leq h+k$, we have

$$\begin{aligned}\psi_{h+k}(Q_{h+k,0}^{-1}Q_{h+k,i}) &= \left(Q_{h,0}^{2^k} \otimes Q_{k,0}\right)^{-1} \cdot \sum_{0\leq j\leq i} Q_{h,0}^{2^k-2^j} Q_{h,i-j}^{2^j} \otimes Q_{k,j} \\ &= \sum_{0\leq j\leq i} Q_{h,0}^{-2^j} Q_{h,j}^{2^j} \otimes Q_{k,0}^{-1}Q_{k,j} \\ &= \sum_{0\leq j\leq i} \left(Q_{h,0}^{-1}Q_{h,j}\right)^{2^j} \otimes Q_{k,0}^{-1}Q_{k,j} \in \Gamma_h^- \otimes \Gamma_k^-\end{aligned}$$

as we wanted. ■

Now we would like to exhibit an example of algebra with coproducts, and, at the same time, show the connection between our examples and those introduced in [6]. First we construct a homomorphism

$$\alpha : \mathcal{H} \longrightarrow \Delta.$$

We set

$$\alpha(\xi_{j,s}) = Q_{s,0}^{-1}Q_{s,j}.$$

The map α is a bidegree preserving homomorphism (up to a sign). In fact $|\xi_{j,s}| = (2^j-1, s)$, while $|Q_{s,0}^{-1}Q_{s,j}| = (1-2^j, s)$. Moreover, α is clearly a monomorphism and we have

$$\operatorname{im}\alpha = \Gamma^-.$$

Therefore Γ^- can be seen as an invariant theoretic description of $\mathcal{H}$. Obviously, we find examples of algebras with coproducts by looking at the bigraded dual structures $\hat{\Delta}$, $\hat{\Gamma}$, $\hat{\Gamma}^-$ of Δ, Γ, Γ^- respectively, equipped with epimorphisms onto $\mathcal{B}$, as $\mathcal{B}$ has been proven to be isomorphic to the dual $\hat{\mathcal{H}}$ of $\mathcal{H}$. We want to explicitly describe $\hat{\Gamma}$. We already know that the algebra Q maps epimorphically onto $\mathcal{B}$, as $\mathcal{B}$ is isomorphic to a quotient of Q described in [3] (see also below). But Q is not the dual of Γ, and there is no consistent structure of Q involving a comultiplication. Instead, we consider a completion of Q, as described in [4]. We set, for each $k \in \mathbb{Z}$,

$$J_k = (x_{k-1}, x_{k-2}, \dots)$$

the two-sided ideal of Q generated by $x_{k-1}, x_{k-2}, \dots$ Then $\mathcal{B} \cong Q/J_0$, an isomorphism ω being obtained by setting $\omega(Sq^j) = [x_j]$, the class of x_j modulo J_0. We indicate by

$$\pi_k : \frac{Q}{J_k} \longrightarrow \frac{Q}{J_{k+1}}$$

the epimorphism induced by the identity map $Q \to Q$. The announced completion is then

$$Q\hat{} = \lim_{\leftarrow} \left(Q/J_k, \pi_k\right).$$

In [1] it has been proven that $D_*(Q) \cong \Gamma$, where D_* indicates the diagonal homology functor, while in [4] the diagonal cohomology of Q has been computed and it turns out that $D^*(Q) \cong Q^\wedge$. As we are working with coefficients in a field, namely $\mathbb{F}_2$, the cohomology of Q is the bigraded dual of the homology of Q. Hence we have shown the following

THEOREM 3.4. – *The completion $Q^\wedge$ of the algebra Q described above is isomorphic to the bigraded dual of Γ and such isomorphism gives $Q^\wedge$ the structure of algebra with coproducts.*

We remark that, if we write ψ_h for the coproducts in $Q^\wedge$, the coproduct formula for each generator x_k is given by an infinite sum:

$$\psi_1(x_k) = \sum_{j+\ell=k} x_j \otimes x_\ell,$$

as it is immediate to check. In particular, for $k \geq 0$ we have

$$\psi_1(x_k) = x_0 \otimes x_k + x_1 \otimes x_{k-1} + \cdots + x_k \otimes x_0 \\ + \text{ terms involving generators } x_j \text{ with } j < 0.$$

and we see how this corresponds to the usual coproduct in $\mathcal{B}$ through ω.

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Dipartimento di Matematica e Applicazioni
Università di Napoli "Federico II",
Complesso Universitario Monte S. Angelo, Via Cintia - 80126 Napoli Italy
E-mail address: lomonaco@unina.it

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