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Well-Posedness of Optimization Problems and Hausdorff Metric on Partial Maps.

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Sunto. – *In questo lavoro si studiano alcune proprietà dello spazio* $(\mathcal{P}, H_\rho)$ *delle mappe parziali con dominio chiuso, munito della topologia della metrica di Hausdorff. Si prova un'equivalenza tra le definizioni di buona posizione secondo Tykhonov e Hadamard di problemi di minimizzazione continui e vincolati, dove la dipendenza continua è descritta dalla metrica di Hausdorff sulle mappe parziali. Lo studio della completezza della metrica di Hausdorff nello spazio delle multifunzioni usco con dominio variabile permette di individuare condizioni per la completa metrizzabilità di* $(\mathcal{P}, H_\rho)$.

Summary. – *The object of this paper is the Hausdorff metric topology on partial maps with closed domains. This topological space is denoted by* $(\mathcal{P}, H_\rho)$. *An equivalence of well-posedness of constrained continuous problems is proved. By using the completeness of the Hausdorff metric on the space of usco maps with moving domains, the complete metrizability of* $(\mathcal{P}, H_\rho)$ *is investigated.*

1. – Introduction.

Topologies and convergences on partial maps has been applied to different fields of mathematics, including differential equations, optimizations, convex analysis, mathematical economics, programming models, etc. ([1, 2, 4, 5, 9, 10, 11, 12, 13, 14, 15, 17, 18, 20, 24, 25, 27, 29, 32])

In our paper we study the Hausdorff metric topology on partial maps identified with their graphs. This topology seems to be particularly well-suited to the well-posedness of continuous minimizations problems.

In fact the main result of our paper claims that the Tykhonov well-posedness of a continuous minimization problem may be expressed as well-posedness of a Hadamard type where the continuous dependence on the data of the problem is described by the Hausdorff metric topology on graphs of constrained minimization problems.

Perhaps the first mention of the Hausdorff metric topology on the space of partial maps can be found in papers of Zaremba [38] and Kuratowski [28] who studied Hausdorff metric on partial maps with compact domains. Kuratowski in his paper studied the complete metrizability of this space.

In our paper we are interested in complete metrizability of the Hausdorff metric topology on partial maps with closed domains (which seems to be more difficult than the compact case).

An auxiliary step in our investigation is to study completeness of the Hausdorff metric on the space of usco maps with moving domains.

Some new characterizations of Atsuji spaces are also given in our paper.

2. – Well-posedness of minimization problems.

In what follows let (X, d) be a metric space. For basic notions and definitions the reader is referred to recent Beer's monograph [6]. Given two subsets A, B of a metric space (X, d), the excess or Hausdorff semi-distance of A over B is denoted by $e_d(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$ with the convention $e_d(A, \emptyset) = +\infty$ if $A \neq \emptyset$ and $e_d(\emptyset, B) = 0$.

It is well known that $H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}$ defines the Hausdorff distance between A and B.

Denote by $CL(X)$ the family of all non-empty closed subsets of X and by $K(X)$ the family of all compact sets in $CL(X)$.

The open (resp. closed) ball with center x and radius $r > 0$ will be denoted by $S(x, r)$ (resp. $B(x, r)$). The open (resp. closed) r−enlargement of A is the set $S(A, r) = \{x \in X : d(x, A) < r\}$ $(B(A, r) = \{x \in X : d(x, A) \leq r\})$, where by $d(x, A)$ we mean $\inf\{d(x, a) : a \in A\}$.

Let $LC(X)$ be the family of all lower semicontinuous functions $f : X \to \mathbb{R} \cup \{+\infty\}$ from X into the extended real line which are proper. By a proper function we mean, as usual, a function $f : X \to \mathbb{R} \cup \{+\infty\}$ for which its domain $dom f := \{x \in X : f(x) < +\infty\}$ is non-empty. An equivalent way to say that a function f is in $LC(X)$ is that its epigraph $epi f := \{(x, a) \in X \times \mathbb{R}: f(x) \leq a\}$ is a non-empty closed subset in $X \times \mathbb{R}$ considered with the product topology. For every $f \in LC(X)$ we denote by (X, f) the unconstrained problem to minimize f over X, that is, to find $x_0 \in X$ such that $f(x_0) = \inf\{f(x) : x \in X\} := \inf(X, f)$. By argmin (X, f) we mean the solution set for the problem (X, f). If $A \subset X$, we denote by (A, f) the constrained minimization problem $(A, f_{|A})$.

A sequence $(x_n) \subset X$ is said to be a minimizing sequence for (X, f) if $f(x_n) \to \inf(X, f)$. The minimization problem (X, f) is said to be Tykhonov well-posed if there is exactly one point $x_0 \in X$ such that $f(x_0) = \inf(X, f)$ and every minimizing sequence for (X, f) converges to x_0.

In dealing with a constrained minimization problem (A, f) one can also be interested in minimizing sequences that are close to A but not in A. Now, if (X, d)

is a metric space, we say that $(x_n) \subset X$ is a *Levitin-Polyak* generalized minimizing sequence for (A, f) if $d(x_n, A)$ converges to 0 and $f(x_n)$ converges to $\inf(A, f)$. Then, the minimization problem (A, f) is said to be *Levitin-Polyak* well-posed if it has a unique solution $x_0 \in A$ and every Levitin-Polyak minimizing sequence for (A, f) converges to x_0.

A more general notion of a minimizing sequence for (A, f) is the following one. A sequence $(x_n) \subset X$ is said to be a *generalized minimizing sequence* for (A, f) if $d(x_n, A) \to 0$ and $\limsup f(x_n) \leq \inf(A, f)$. So, we have the corresponding notion of strongly well-posedness for (A, f). Of course, this last well-posedness is stronger than the Levitin-Polyak one, and the Levitin-Polyak well-posedness is stronger than the Tykhonov one.

Another notion of well-posedness is the *Hadamard well-posedness*. It is based on the continuous dependence of the solution on the data of the problem.

Let τ_{hu} be the product topology on $CL(X) \times LC(X)$ induced by the Hausdorff metric topology H_d on $CL(X)$ and the uniform convergence topology τ_u on $LC(X)$. A minimization problem $(A, f) \in CL(X) \times LC(X)$ is said to be *Hadamard well-posed* if it has a unique solution $x_0 \in A$ and for every sequence $(A_n, f_n) \subset CL(X) \times LC(X)$ that τ_{hu}-converges to (A, f) and for every $(x_n) \subset X$ with $x_n \in \operatorname{argmin}(A_n, f_n)$ one has $x_n \to x_0$.

The following result is proved in [33].

THEOREM 1. – *Consider the following assertions for a minimization problem $(A, f) \in CL(X) \times LC(X)$:*

1) *(A, f) is Hadamard well-posed;*
2) *(A, f) is strongly well-posed;*
3) *(A, f) is Levitin-Polyak well-posed;*
4) *(A, f) is Tykhonov well-posed.*

Then $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 4$ and if f is uniformly continuous in $\operatorname{dom} f$ then $4 \Longrightarrow 1$.

In [33] a characterization of such metric spaces is given in which Tykhonov and Hadamard well-posedness (hence all four kinds) of continuous minimization problems coincide. These metric spaces, called Atsuji spaces, are those in which every continuous real valued function is uniformly continuous [3, 7]. This class of spaces has a number of beautiful characterizations [7]: (1) each pair of disjoint closed subsets of X lie a positive distance apart; (2) each open cover of X has a Lebesgue number; (3) every sequence (x_n) in X with $\lim_{n \to +\infty} d(x_n, \{x_n\}^c) = 0$ has a cluster point; (4) for each metric space Y the Hausdorff metric on $C(X, Y)$ induced by the box metric on $X \times Y$ yields the topology of uniform convergence.

Now, we denote, as usual, by $C(X)$ the set of all continuous real-valued functions. The following lemma shows that to verify Hadamard well-posedness of the continuous minimization problem it is sufficient to control only the data with continuous functions instead of lower semicontinuous ones.

LEMMA 1. – *The minimization problem* $(A, f) \in CL(X) \times C(X)$ *is Hadamard well-posed in* $CL(X) \times LC(X)$ *if and only if it is Hadamard well-posed in* $CL(X) \times C(X)$.

PROOF. – Let $(A, f) \in CL(X) \times C(X)$ be a minimization problem with a unique solution $x_0 \in A$. Suppose that for every sequence $(A_n, f_n) \subset CL(X) \times C(X)$, (A_n, f_n) τ_{hu}-converges to (A, f) and for every $(x_n) \subset X$ with $x_n \in$ argmin (A_n, f_n) then $x_n \to x_0$. Let $(A_n, f_n) \subset CL(X) \times LC(X)$, (A_n, f_n) τ_{hu}-converges to (A, f) and $x_n \in$ argmin (A_n, f_n) that is $f_n(x_n) = \inf (A_n, f_n)$.

We have $f_n(x_n) \to f(x_0)$; for every $n \in Z^+$, let $\varepsilon_n = | f_n(x_n) - f(x_0) |$ and $\eta_n = | f_n(x_n) - f(x_n) |$; we put $\varepsilon_n + \eta_n = a_n$. The continuity of f at x_n implies that there is an open neighbourhood U_n of x_n such that $f(z) \in (f(x_n) - a_n, f(x_n) + a_n)$ for every $z \in U_n$. Now let $g_n : X \to \mathbb{R}$ be a continuous function defined by

$$g_n(z) = \begin{cases} f(x_n) - a_n & \text{if } z = x_n \\ f(z) & \text{if } z \in X \setminus U_n \\ \text{Dugundji extension} & \text{otherwise} \end{cases}$$

and such that $g_n(\overline{U_n}) \subset [f(x_n) - a_n, f(x_n) + a_n]$. So $x_n \in$ argmin (A_n, g_n) and (A_n, g_n) τ_{hu}-converges to (A, f), thus $x_n \to x_0$. ■

Lemma 1 and Theorem 1 together give us the following result:

THEOREM 2. – *If* $A \in CL(X)$ *and* f *is uniformly continuous, then the following are equivalent:*

1) (A, f) *is Hadamard well-posed in* $CL(X) \times LC(X)$;
2) (A, f) *is Hadamard well-posed in* $CL(X) \times C(X)$;
3) (A, f) *is strongly well-posed;*
4) (A, f) *is Levitin-Polyak well-posed;*
5) (A, f) *is Tykhonov well-posed.*

REMARK 1. – Obviously, the equivalence of the five kinds of well-posedness for continuous minimization problems characterizes the Atsuji spaces.

The main result of this part claims that Tykhonov well-posedness of a continuous minimization problem can be expressed as well-posedness of a Hadamard type where the continuous dependence on the data of the problem is described by the Hausdorff metric topology on graphs of constrained minimization problems.

Let (X, d_X) and (Y, d_Y) be metric spaces and $C(X, Y)$ be the space of all continuous functions from X to Y. If $A \in CL(X)$ and $f \in C(A, Y)$, by $[A, f]$ we denote a partial map from A to Y. Put $\mathcal{P} = \{[A, f] : A \in CL(X), f \in C(A, Y)\}$ the set of all partial maps from X to Y. Consider in $X \times Y$ the box metric ρ of d_X and d_Y and by H_ρ we denote the Hausdorff distance on $CL(X \times Y)$. We can identify every partial map $[A, f]$ with its graph, which is of course a closed set in $X \times Y$. Under this identification we can consider $\mathcal{P}$ as a subset of $CL(X \times Y)$. Thus we can induce the Hausdorff metric H_ρ from $CL(X \times Y)$ to $\mathcal{P}$. We denote the resulting space by $(\mathcal{P}, H_\rho)$.

THEOREM 3. – *Let $(A, f) \in CL(X) \times C(X)$. Then (A, f) is Tykhonov well-posed if and only if*

i) *(A, f) has a unique solution x_0;*

ii) *for any $(A_n, f_n) \in CL(X) \times C(X)$ such that $[A_n, f_n|A_n] \to [A, f|A]$ in $(\mathcal{P}, H_\rho)$ and $(x_n) \subset X$ with $x_n \in$ argmin (A_n, f_n) one has $x_n \to x_0$.*

PROOF. – $\Longleftarrow$ Let $f(x_0) = \inf_{x \in A} f(x)$ and (x_n) be a minimizing sequence in A. Define $f_n : X \to \mathbb{R}$ by $f_n(x) = \max\{f(x), f(x_n)\}$. Obviously $f_n(x_n) = \inf_{x \in A} f_n(x)$. Moreover

$$\sup_{x \in A} |f_n(x) - f(x)| \leq f(x_n) - f(x_0).$$

Since $f(x_n) \to f(x_0)$, $(f_n|A)$ uniformly converges to $f|A$. Thus $(A, f_n|A) \to (A, f|A)$ in $(\mathcal{P}, H_\rho)$. The sequence (A, f_n) satisfies our assumptions; i.e. $x_n \to x_0$ and (A, f) is Tykhonov well-posed.

$\Longrightarrow$ Let f be continuous and (A, f) be Tykhonov well-posed with a unique solution x_0. Moreover, let $(A_n, f_n) \in CL(X) \times C(X)$ such that $[A_n, f_n \mid A_n]$ H_ρ-converges to $[A, f|A]$ and $(x_n) \subset X$ with $x_n \in$ argmin (A_n, f_n). Since $H_\rho([A_n, f_n|A_n], [A, f|A]) \to 0$, for every $n \in \mathbb{Z}^+$ there is $a_n \in A$ with

$$\max\{d(x_n, a_n), |f_n(x_n) - f(a_n)|\} \leq H_\rho([A_n, f_n|A_n], [A, f|A]) + \frac{1}{n}.$$

Thus $d(x_n, a_n) \to 0$ and also $|f_n(x_n) - f(a_n)| \to 0$. Since $f(x_0) = \inf_{x \in A} f(x)$, we have $f(a_n) \to f(x_0)$, otherwise a contradiction with $x_n \in$ argmin (A_n, f_n). Since (A, f) is Tykhonov well-posed, $x_n \to x_0$. ■

3. – Hausdorff metric topology on partial maps.

In this part we study the Hausdorff metric topology on the space of partial maps. The main aim of this paragraph is to find some sufficient conditions for the complete metrizability of $(\mathcal{P}, H_\rho)$. Notice that $(\mathcal{P}, H_\rho)$ need not be complete even if X and Y are compact ([22]).

We start with the following results:

PROPOSITION 1. – *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. Let* $[A_n, f_n]$, $[A, f] \in \mathcal{P}$ *and suppose* $[A_n, f_n] \to [A, f]$ *in* $(\mathcal{P}, H_\rho)$. *Then*

a) $A_n \to A$ *in* $(CL(X), H_{d_X})$;

b) *For every* $x \in A$ *and* $x_n \to x$, $x_n \in A_n \Longrightarrow f_n(x_n) \to f(x)$.

Moreover, if f *is uniformly continuous then for every* $B \in CL(X)$, *with* $B \subset A$ *and* $B \subset A_n$ *eventually, one has* $f_{n|B} \to f_{|B}$ *uniformly.*

PROOF. – Let $[A_n, f_n] \to [A, f]$ in $(\mathcal{P}, H_\rho)$. It is immediate to prove the property a). The property b) easily follows from Proposition 3.3 in [12].

Now suppose that f is uniformly continuous. Let B be a closed set of X with $B \subset A \cap A_n$ eventually. Without loss of generality we may assume that $B \subset A_n$ for every n. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ when $d_X(x, y) < \delta$. Moreover, if $\eta = \min\{\varepsilon, \delta\}$, let $\overline{n}$ be such that

$$\sup_{x \in A_n} \inf_{y \in A} \rho((x, f_n(x)), (y, f(y))) < \eta \qquad \text{for } n \geq \overline{n}.$$

Since $B \subset A_n$ for all n, then for every $x \in B$ there is $y_n \in A$ such that

$$d_X(x, y_n) < \eta \text{ and } d_Y(f_n(x), f(y_n)) < \eta \qquad \text{for every } n \geq \overline{n}.$$

By the uniform continuity of f, we have

$$d_Y(f_n(x), f(x)) \leq d_Y(f_n(x), f(y_n)) + d_Y(f(y_n), f(x)) \leq 2\varepsilon$$

for every $x \in B$ and for every $n \geq \overline{n}$. ■

THEOREM 4. – *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. TFAE:*

i) (X, d_X) *is an Atsuji space;*

ii) $[A_n, f_n]$ *converges to* $[A, f]$ *in* $(\mathcal{P}, H_\rho)$ *if and only if*

a) $A_n \to A$ in $(CL(X), H_{d_X})$;

b) *for every* $x \in A$ *and* $x_n \to x$, $x_n \in A_n \Longrightarrow f_n(x_n) \to f(x)$;

c) *for every* $B \in CL(X)$, *with* $B \subset A$ *and* $B \subset A_n$ *eventually, one has* $f_{n|B} \to f_{|B}$ *uniformly.*

PROOF. – $(i) \Rightarrow (ii)$ In view of Proposition 1 we only need to prove that if $a)$, $b)$, $c)$ hold, then $[A_n, f_n] \to [A, f]$ in $(\mathcal{P}, H_\rho)$. If it is not true, then there is an infinite subset I of $\mathbb{Z}^+$ and there exists $\varepsilon > 0$ such that

$$e([A_n, f_n], [A, f]) > \varepsilon \qquad \text{for every } n \in I$$

or

$$e([A, f], [A_n, f_n]) > \varepsilon \qquad \text{for every } n \in I.$$

In the first case, for every $n \in I$ there is $x_n \in A_n$ such that

$$\rho((x_n, f_n(x_n)), (y, f(y))) > \varepsilon \qquad \text{for all } y \in A. \tag{1}$$

The sequence $(x_n)_{n\in I}$ cannot have a cluster point. Suppose there is a cluster point x of the sequence $(x_n)_{n\in I}$. Then there is a subsequence (x_{n_k}) of $(x_n)_{n\in I}$ such that $x_{n_k} \to x \in A$. We can extend (x_{n_k}) to a sequence (y_n) convergent to x and such that $y_n \in A_n$ for every $n \in Z^+$. Then $f_n(y_n)$ converges to $f(x)$. This is a contradiction to (1). Since X is Atsuji there is $r > 0$ such that $S(x_n, r) = \{x_n\}$ for all $n \in I$ with $n \geq n_0$. From $e(A_n, A) \to 0$ we have that there is $n_1 \geq n_0$ such that for $n \in I$ with $n \geq n_1$ we have $x_n \in A$. Hence $(x_n)_{n\in I}$ is eventually contained in A. Put $B = \{x_n : n \in I, n \geq n_1\}$. By similar argument, from $e(A, A_n) \to 0$, one obtain that there is n_2 such that $B \subset A_n$ for every $n \geq n_2$. From c) we get $f_{n|B} \to f_{|B}$ uniformly.

But (1) implies that for all $n \in I$,we have $d_Y(f_n(x_n), f(x_n)) > \varepsilon$, that is a contradiction.

Now, we consider the case when

$$\sup_{y\in A} \inf_{x\in A_n} \rho((y, f(y)), (x, f_n(x))) > \varepsilon \qquad \text{for every } n \in I.$$

Then we can construct a sequence $(x_n)_{n\in I} \subset A$ such that for all $x \in A_n$, one has

$$\rho((x_n, f(x_n)), (x, f_n(x)) > \varepsilon. \tag{2}$$

The sequence $(x_n)_{n\in I}$ has no cluster point. Otherwise, there would be a subsequence of $(x_n)_{n\in I}$ that converges to a point $x \in A$. W.l.o.g. we can suppose that this subsequence is (x_n). Since $A_n \to A$, there is a sequence (y_n) with $y_n \in A_n$ such that $y_n \to x$ and so, from b), we have $f_n(y_n) \to f(x)$. Consider the sequence $(y_n)_{n\in I}$. Since $\lim d_X(x_n, y_n) = 0$ from (2) $d_Y(f(x_n), f_n(y_n)) > \varepsilon$ for all $n \in I$ sufficiently large. Contradiction, because

$$f(x_n) \to f(x) \text{ and } f_n(y_n) \to f(x).$$

The proof is then similar to the one in the first part.

(ii) $\Rightarrow$ *(i)* Suppose (X, d_X) is not an Atsuji space. Then there are sequences (x_n), (y_n) in X without cluster points and such that $d_X(x_n, y_n) \to 0$, and $x_n \neq y_n$ for every n. Put $A = \{x_n : n \in Z^+\}$. Then A is a closed set. For every $n \in Z^+$ put $A_n = A \cup \{y_n\}$. Then of course $(A_n) \to A$ in $(CL(X), H_{d_X})$. Let a, b be two different points in Y. Let $f \in C(A, Y)$ be defined as $f(x) = a$ for every $x \in A$. For every $n \in Z^+$ let $f_n \in C(A_n, Y)$ be defined as $f_n(x) = a$ for every $x \in A$ and $f_n(y_n) = b$. It is easy to verify that $[A_n, f_n]$ fails to converge to $[A, f]$ in $(\mathcal{P}, H_\rho)$. However all conditions (a), (b), (c) are satisfied.

PROPOSITION 2. – *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. If* (X, d_X) *is an Atsuji space then the map*

$$\eta : (CL(X) \times C(X,Y), \tau_{hu}) \longrightarrow (\mathcal{P}, H_\rho)$$

defined by $\eta(A,f) = [A,f]$ *is continuous.*

PROOF. – Suppose that (A_n, f_n) τ_{hu}-converges to (A, f) in $(CL(X) \times C(X, Y))$. Then, of course, by Theorem 4 (a) and (c) of (ii) hold. Moreover, since f_n uniformly converges to f, it also continuously converges to f ([26]) and thus also (b) of ii) of Theorem 4 is satisfied. Hence $[A_n, f_n]$ H_ρ-converges to $[A, f]$ and the continuity of η follows. ■

Moreover, we have the following result:

PROPOSITION 3. – *Let* (X, d_X) *be a metric space. If the map*

$$\eta : (CL(X) \times C(X), \tau_{hu}) \longrightarrow (\mathcal{P}, H_\rho)$$

is continuous then (X, d_X) *is an Atsuji space.*

PROOF. – To prove that (X, d_X) is an Atsuji space we show that, for continuous minimization problems, Tykhonov well-posedness implies Hadamard well-posedness.

Let (A, f) be Tykhonov well-posed with $f(x_0) = \inf_{x \in A} f(x)$ and let (A_n, f_n) τ_{hu}-converges to (A, f) with $x_n \in \operatorname{argmin}(A_n, f_n)$. Then continuity of η implies $[A_n, f_n]$ H_ρ-converges to $[A, f]$. Of course $f_n(x_n) \to f(x_0)$. There is a sequence (y_n) in A such that $\rho((y_n, f(y_n)), (x_n, f_n(x_n)) \to 0$, i.e. $d_X(y_n, x_n) \to 0$ and $d_Y(f_n(x_n), f(y_n)) \to 0$; i.e. $f(y_n) \to f(x_0)$. Thus $y_n \to x_0$, i.e. $x_n \to x_0$. Therefore (A, f) is Hadamard well-posed. ■

REMARK 2. – If (X, d_X) is an Atsuji space and (Y, d_Y) is a Fréchet space then we can prove (using the similiar idea as in [14]) that η is also onto and open. Thus $(\mathcal{P}, H_\rho)$ is completely metrizable, since a metrizable space which is a continuous and open image of completely metrizable space is completely metrizable too. However Atsujiness of X and Fréchetness of Y are strong requirements.

We will present weaker conditions on spaces to guarantee the complete metrizability of $(\mathcal{P}, H_\rho)$. To study the complete metrizability of $(\mathcal{P}, H_\rho)$ we first investigate the completeness of the Hausdorff metric of usco maps with moving domains and then by using of Alexandroff theorem we obtain the complete metrizability of $(\mathcal{P}, H_\rho)$.

For every $A \in CL(X)$ we denote by $U(A, Y)$ the space of all upper semicontinuous multifunctions from A to Y with non empty compact values (*usco maps*) [16] and we set

$$W(X,Y) = \{[A,F] : A \in CL(X),\ F \in U(A,Y)\}.$$

If $[A,F] \in W(X,Y)$ then the graph of $[A,F]$ is a closed set in $X \times Y$. Therefore if we identify an element of $W(X,Y)$ with its graph, we can consider $W(X,Y)$ as a subset of $(CL(X \times Y), H_\rho)$, where H_ρ is the Hausdorff distance in $CL(X \times Y)$ induced by the box metric ρ of d_X and d_Y on $X \times Y$.

In the next Lemma we denote by $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ the projections.

LEMMA 2. – *Let (X, d_X) be a locally compact metric space and (Y, d_Y) be a metric space. If $B \in \overline{W(X,Y)}$ in $(CL(X \times Y), H_\rho)$ then*

1) *for every $z \in X$ there is a compact neighbourood O_z of z such that*

$$\overline{p_Y((O_z \times Y) \cap B)}$$

is compact;

2) *$p_X(B)$ is closed in X.*

PROOF. – Let $z \in X$. Without loss of generality we may suppose that $z \in \overline{p_X(B)}$. The local compactness of X implies that there is $\delta > 0$ such that $B_{2\delta}[z]$ is a compact subset of X. Let $\varepsilon > 0$. Put $\eta = \min\{\varepsilon, \delta\}$. Let $[A,F] \in W(X,Y)$ be such that $H_\rho(B, [A,F]) < \eta$. Let $(u,v) \in ((B_\delta[z] \times Y) \cap B)$. Thus there must exist $(x,y) \in [A,F]$ such that $d_X(u,x) < \eta$ and $d_Y(v,y) < \eta$, i.e.

$$v \in S_\eta[y] \subset S_\eta[F(x)] \subset S_\varepsilon[F(B_{2\delta}[z] \cap A)].$$

The upper semicontinuity of F and compactness of values of F imply that $F(B_{2\delta}[z] \cap A) = L_\varepsilon$ is a compact set in Y.

Thus we proved that for every $\varepsilon > 0$ $p_Y((B_\delta[z] \times Y) \cap B) \subset S_\varepsilon[L_\varepsilon]$, i.e. $M_z = \overline{p_Y((B_\delta[z] \times Y) \cap B)}$ is a compact set in Y (since it is totally bounded closed set in a complete metric space). Thus 1. holds.

Now we prove that $p_X(B)$ is a closed set in X. Let $z \in \overline{p_X(B)}$. Thus there is a sequence z_n in $p_X(B)$ such that $\{z_n\} \to z$. By 1. there is $\delta > 0$ such that $M_z = \overline{p_Y((B_\delta[z] \times Y) \cap B)}$ is compact in Y. Without loss of generality we can suppose that $z_n \in B_\delta[z]$ for every $n \in Z^+$. Thus there is a sequence $\{y_n\}$ such that $(z_n, y_n) \in B$ for every $n \in Z^+$, i.e. $y_n \in M_z$ for every $n \in Z^+$. The compactness of M_z implies that $\{y_n\}$ has a cluster point $l \in M_z$, thus $(z, l) \in B$, i.e. $z \in p_X(B)$. ■

THEOREM 5. – *Let (X, d_X) be a metric space. The following are equivalent:*

1) *X is a locally compact space;*
2) *For each complete metric space (Y, d_Y), $W(X,Y)$ is a closed set in $(CL(X \times Y), H_\rho)$.*

PROOF. – 1. $\Rightarrow$ 2.

Let B be in the closure of $W(X, Y)$ in $(CL(X \times Y), H_\rho)$. By Lemma 2 $p_x(B)$ is closed. Define now a multifunction $H_B : p_X(B) \to Y$ as follows: $H_B(x) = \{y \in Y : (x, y) \in B\}$. Using Lemma 2 we have that for every $x \in p_X(B)$ there is a neighbourhood O_x of x and a compact set K_x in Y such that $H_B(z) \subset K_x$ for every $z \in O_x$. Thus H_B is an upper semicontinuous compact-valued multifunction; i.e. $[p_X(B), H_B] \in W(X, Y)$ and B is the graph of $[p_X(B), H_B]$. Thus $B \in W(X, Y)$, i.e. W(X,Y) is closed in $(CL(X \times Y), H_\rho)$.

2. $\Rightarrow$ 1.

Suppose that X is not locally compact. Let Y be an arbitrary non compact complete metric space. By the proof of Theorem 1 in [21] there is a sequence $\{F_n\}$ of upper semicontinuous compact-valued multifunctions such that $([X, F_n])$ H_ρ-converges to a multifunction F with a closed graph which is not upper semicontinuous, i.e. $[X, F] \notin W(X, Y)$. ■

The next theorem, concerning the complete metrizability of $W(X, Y)$, is a consequence of theorem 5. In fact, it is known that if X and Y are complete metric spaces, then $(CL(X \times Y), H_\rho)$ is complete. So, by theorem 5, if X is also locally compact, then $W(X, Y)$ is complete since it is closed in $(CL(X \times Y), H_\rho)$.

THEOREM 6. – *If (X, d_X) and (Y, d_Y) are complete metric spaces and (X, d_X) is locally compact, then $(W(X, Y), H_\rho)$ is also complete.*

We recall that a metric space (X, d_X) is boundedly Atsuji [8, 22] if every closed and bounded subspace of X is Atsuji.

Now we give a result concerning the complete metrizability of $(\mathcal{P}, H_\rho)$. By an easy modification of Theorem 4 in [21] we can prove the following result:

PROPOSITION 4. – *Let (X, d_X) be a boundedly Atsuji space and (Y, d_Y) be a metric space. Then $\mathcal{P}$ is a G_δ- subset of $(W(X, Y), H_\rho)$.*

By using of Alexandroff theorem we have:

THEOREM 7. – *If (X, d_X) is a locally compact boundedly Atsuji space and (Y, d_Y) is a complete metric space then $(\mathcal{P}, H_\rho)$ is completely metrizable.*

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