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Grażyna Anioł

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On the Rate of Convergence of the Bézier-Type Operators.

GRAŻYNA ANIOŁ

Sunto. – *Per le funzioni limitate f su un intervallo I, in particolare, per le funzioni con potenza p-sima a variazione limitata su I è stimato il rango di convergenza puntuale della modificazione di tipo Bézier degli operatori discreti di Feller. Nel teorema principale è stato usato il modulo di variazione di Chanturiya.*

Summary. – *For bounded functions f on an interval I, in particular, for functions of bounded p-th power variation on I there is estimated the rate of pointwise convergence of the Bézier-type modification of the discrete Feller operators. In the main theorem the Chanturiya modulus of variation is used.*

1. – Preliminaries.

Let $\{X_{k,x}\}_{k=1}^{\infty}$ be a family of sequences of independent and identically distributed random variables with expectation $EX_{k,x} = x$ for all $k \in N$ and finite variance $\sigma^2(x)$, where x is a real parameter taking values in a bounded or unbounded interval $I \subseteq [0, \infty)$. Consider the sum $S_{n,x} = X_{1,x} + X_{2,x} + \ldots + X_{n,x}$ and its distribution $\{p_{n,j}(x) : x \in I, j \in J_n\}$. Suppose that $E|f(S_{n,x}/n)| < \infty$ for all $x \in I, n \in N$ and that the weights $p_{n,j}$ are continuous on I. Assume, moreover, that J_n is of the form $\{0, 1, \ldots, m_n\}$ with some $m_n \in N$ and $m_n \leq m_{n+1}$ for all $n \in N$ or $J_n = N_0 := N \cup \{0\}$ for all $n \in N$.

Let $M(I)$ be the class of all real-valued functions bounded on an interval $I \subseteq [0, \infty)$. Introduce, for $f \in M(I)$, the Bézier-type discrete operators

$$L_n^{(a)} f(x) := \sum_{k \in J_n} f\left(\frac{k}{n}\right) Q_{n,k}^{(a)}(x), \tag{1}$$

where $a > 0$ and

$$\begin{aligned} Q_{n,k}^{(a)} &:= q_{n,k}^a(x) - q_{n,k+1}^a(x), \\ q_{n,k}(x) &:= \sum_{j \in J_n, j \geq k} p_{n,j}(x) \quad \text{for } k \in J_n. \end{aligned} \tag{2}$$

If $J_n = \{0, 1, \ldots, m_n\}$ then $q_{n,l}(x) = 0$ for all $l > m_n$.

Recently, several authors studied some approximation properties of the special operators (1), in which $Q_{n,k}^{(a)}$ are the Bézier basis functions defined by (2) with $p_{n,j}(x) := \binom{n}{j} x^j (1-x)^{n-j}$, $x \in I = [0,1]$, $j \in J_n = \{0, 1, \ldots, n\}$ (see [2]). Zeng and Piriou [10,11] gave estimates for the rate of pointwise convergence of these operators for functions f of bounded variation in the Jordan sense on $I = [0,1]$. In this paper we present an extension and generalization of their results to a general class of operators (1) with $0 < a < 1$ and to the wide class of function $f \in M(I)$ possesing the one-sided limits $f(x+)$, $f(x-)$ at a fixed point x. We prove that the rates of the pointwise convergence of the operators (1) in the above case are as good as in case $a \geq 1$, which can be found in [8]. In our estimates we use the so-called modulus of variation of a function g on an interval $Y = [c,d]$ defined as in [3]: if $k \in N$ then

$$v_k(g;Y) \equiv v_k(g;c,d) := \sup_{\prod_k} \left\{ \sum_{i=1}^{k} |g(t_i) - g(\tau_i)| \right\}$$

over all systems $\prod_k$ of k non-overlapping intervals (τ_i, t_i) contained in Y. We take $v_0(g;Y) = 0$.

2. – Results.

Let $f \in M(I)$ and let at a fixed point $x \in \operatorname{Int} I$ the one-sided limits $f(x+)$, $f(x-)$ exist. It is easy to verify that for all $t \in I$,

$$\begin{aligned} f(t) = {} & 2^{-a} f(x+) + (1 - 2^{-a}) f(x-) + g_x(t) + 2^{-a}(f(x+) - f(x-)) \operatorname{sgn}_x^{(a)}(t) \\ & + (f(x) - 2^{-a} f(x+) - (1 - 2^{-a}) f(x-)) \delta_x(t), \end{aligned}$$

where

$$g_x(t) := \begin{cases} f(t) - f(x+) & \text{if } t > x, \\ 0 & \text{if } t = x, \\ f(t) - f(x-) & \text{if } t < x \end{cases} \qquad \operatorname{sgn}_x^{(a)}(t) := \begin{cases} 2^a - 1 & \text{if } t > x, \\ 0 & \text{if } t = x, \\ -1 & \text{if } t < x \end{cases}$$

and $\delta_x(x) := 1, \delta_x(t) := 0$ if $t \neq x$ (see [11, p. 381]). Therefore

$$L_n^{(a)} f(x) - 2^{-a} f(x+) - (1 - 2^{-a}) f(x-) = L_n^{(a)} g_x(x) + \Delta_n^{(a)}(f;x) \tag{3}$$

with

$$\begin{aligned} \Delta_n^{(a)}(f;x) = {} & 2^{-a}(f(x+) - f(x-)) L_n^{(a)} \operatorname{sgn}_x^{(a)}(x) \\ & + (f(x) - 2^{-a} f(x+) - (1 - 2^{-a}) f(x-)) L_n^{(a)} \delta_x(x). \end{aligned} \tag{4}$$

To obtain the estimate of the term $\Delta_n^{(a)}(f;x)$ in (3) we consider only the points $x \in I$ at which

$$\sigma^2(x) > 0 \text{ and } \beta(x) := \sum_{j \in J_1} |j-x|^3 p_{1,j}(x) < \infty. \tag{5}$$

LEMMA. – *Under assumptions* (5) *and* $0 < a < 1$ *we have*

$$\begin{aligned} |\Delta_n^{(a)}(f;x)| \leq & |f(x+) - f(x-)| \frac{\tau\beta(x)}{\sqrt{n}\sigma^3(x)} \\ & + e_n(x)|f(x) - f(x-)| \left(\frac{2\tau\beta(x)}{\sqrt{n}\sigma^3(x)} + \frac{1}{\sqrt{2\pi n}\sigma(x)} \right) \end{aligned}$$

for $n \geq n_0(x)$, *where* $n_0(x) = (4\beta(x)/\sigma^3(x))^2$, $0 < \tau \leq 0.82$ *and* $e_n(x) = 0$ *if* $x \neq k/n$ *for all* $k \in J_n$, $e_n(x) = 1$ *if there exists a* $k' \in J_n$ *such that* $x = k'/n$.

PROOF. – For the sake of brevity we use the notation

$$\sum_{k \in J_n, k \geq r} p_{n,k}(x) = \sum_{k \geq r} p_{n,k}(x).$$

It is easy to see (as in [11]) that

$$\begin{aligned} L_n^{(a)} \mathrm{sgn}_x^{(a)}(x) &= 2^a \sum_{k > nx} Q_{n,k}^{(a)}(x) - 1 + e_n(x) Q_{n,k'}^{(a)}(x) \\ &= 2^a \sum_{k > nx} (q_{n,k}^a(x) - q_{n,k+1}^a(x)) - 1 + e_n(x) Q_{n,k'}^{(a)}(x) \\ &= 2^a \left(\sum_{j > nx} p_{n,j}(x) \right)^a - 1 + e_n(x) Q_{n,k'}^{(a)}(x) \end{aligned}$$

and

$$L_n^{(a)} \delta_x(x) = e_n(x) Q_{n,k'}^{(a)}(x).$$

Hence, in view of (4),

$$\begin{aligned} |\Delta_n^{(a)}(f;x)| \leq & 2^{-a} |f(x+) - f(x-)| \left| 2^a \left(\sum_{j > nx} p_{n,j}(x) \right)^a - 1 \right| \\ & + |f(x) - f(x-)| e_n(x) Q_{n,k'}^{(a)}(x) \\ = & |f(x+) - f(x-)| \left| \left(\sum_{j > nx} p_{n,j}(x) \right)^a - \frac{1}{2^a} \right| \\ & + |f(x) - f(x-)| e_n(x) Q_{n,k'}^{(a)}(x). \end{aligned}$$

By the mean value theorem we have

$$\left|\left(\sum_{j>nx} p_{n,j}(x)\right)^a - \frac{1}{2^a}\right| = a(\xi_{n,j}(x))^{a-1}\left|\sum_{j>nx} p_{n,j}(x) - \frac{1}{2}\right|,$$

where $\xi_{n,j}(x)$ lies between $\frac{1}{2}$ and $\sum\limits_{j>nx} p_{n,j}(x)$. In view of the Berry-Esséen theorem [4, p. 515; 5, p. 93],

$$\left|\sum_{j-nx\le t\sigma(x)\sqrt{n}} p_{n,j}(x) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}\exp(-u^2/2)du\right| \le \frac{\tau\beta(x)}{\sqrt{n}\sigma^3(x)}$$

for all $n \in N, t \in R$, where $0 < \tau \le 0.82$. From this it follows that

$$\left|\sum_{j>nx} p_{n,j}(x) - \frac{1}{2}\right| \le \frac{\tau\beta(x)}{\sqrt{n}\sigma^3(x)}. \tag{6}$$

In view of (6) we have $\sum\limits_{j>nx} p_{n,j}(x) \ge \frac{1}{4}$ for all $n \ge n_0(x) = (4\beta(x)/\sigma^3(x))^2$. Hence $(\xi_{j,n}(x))^{a-1} \le 4^{1-a}$ and

$$\left|\left(\sum_{j>nx} p_{n,j}(x)\right)^a - \frac{1}{2^a}\right| \le \frac{\tau\beta(x)}{\sqrt{n}\sigma^3(x)} \quad \text{for all } n \ge n_0(x), \tag{7}$$

since $a4^{1-a} \le 1$.

Further

$$Q^{(a)}_{n,k'}(x) = q^a_{n,k'}(x) - q^a_{n,k'+1}(x) = a(\zeta_{n,k'}(x))^{a-1}p_{n,k'}(x),$$

where $q_{n,k'+1}(x) < \zeta_{n,k'}(x) < q_{n,k'}(x)$. But, in view of (6),

$$\zeta_{n,k'}(x) > q_{n,k'+1}(x) = \sum_{j\ge k'+1} p_{n,j}(x) \ge \frac{1}{4} \quad \text{for all } n \ge n_0(x).$$

Hence

$$\begin{aligned} Q^{(a)}_{n,k'}(x) < a4^{1-a}p_{n,k'}(x) &\le \sum_{j\le k'} p_{n,j}(x) - \sum_{j\le k'-1} p_{n,j}(x) \\ &\le \frac{2\tau\beta(x)}{\sqrt{n}\sigma^3(x)} + \frac{1}{\sqrt{2\pi n}\sigma(x)} \end{aligned} \tag{8}$$

for all $n \ge n_0(x)$ (see [9, the proof of Lemma 3]).

Collecting the results we get our estimate. ∎

Let us introduce the moments

$$\mu_{n,\gamma}(x) := \sum_{k \in J_n} \left| \frac{k}{n} - x \right|^{\gamma} p_{n,k}(x),$$

where $n \in N, \gamma > 0$.

THEOREM 1. – *Let $f \in M(I)$ and let at a fixed point $x \in \operatorname{Int} I$ the one-sided limits $f(x+), f(x-)$ exist. Let a, b be two arbitrary positive numbers and let $0 < a < 1$. Then*

$$\begin{aligned} &|L_n^{(a)} f(x) - 2^{-a} f(x+) - (1-2^{-a}) f(x-)| \\ &\leq 2\big(1 + 4(a^{-2} + b^{-2}) n(\mu_{n,2/a}(x))^a\big) \\ &\times \left(\sum_{j=1}^{m-1} \frac{1}{j^3} v_j(g_x; Y(ja/\sqrt{n}, jb/\sqrt{n})) + \frac{1}{m^2} v_m(g_x; Y(a,b)) \right) \\ &+ \vartheta_x(a,b) \frac{(\mu_{n,2/a}(x))^a}{c^2} v_1(g_x; I) + |\varDelta_n^{(a)}(f; x)|, \end{aligned}$$

for $n \geq \max\{4, n_0(x)\}$, where $m = [\sqrt{n}]$, $n_0(x) = (4\beta(x)/\sigma^3(x))^2$, $Y_x(h,\eta) = [x-h, x+\eta] \cap I$ if $h > 0, \eta > 0$, $\vartheta_x(a,b) = 0$ if neither of the points $x-a, x+b$ belongs to $\operatorname{Int} I$, $\vartheta_x(a,b) = 1$ otherwise, and $|\varDelta_n^{(a)}(f;x)|$ is estimated via our Lemma.

PROOF. – First we write the term $L_n^{(a)} g_x(x)$ of (3) in the form

$$L_n^{(a)} g_x(x) = \sum_{k \in A_x(a,b)} g_x\left(\frac{k}{n}\right) Q_{n,k}^{(a)}(x) + \vartheta_x(a,b) \sum_{k \in D_x(a,b)} g_x\left(\frac{k}{n}\right) Q_{n,k}^{(a)}(x), \tag{9}$$

where $A_x(a,b) = \left\{k \in J_n : \frac{k}{n} \in Y_x(a,b)\right\}, D_x(a,b) = J_n \setminus A_x(a,b)$ and $\vartheta_x(a,b) = 0$ if neither of the points $x-a, x+b$ belongs to $\operatorname{Int} I$, $\vartheta_x(a,b) = 1$ otherwise. In order to estimate the terms of the right-hand side of (9) let us observe that $Q_{n,k}^{(a)}(x) \geq 0$ and

$$\sum_{k \in J_n} Q_{n,k}^{(a)}(x) = \left(\sum_{j \in J_n} p_{n,j}(x) \right)^a = 1.$$

Following the proof of Lemma 4 in [10] we have for $t < x$, $t, x \in I$,

$$\sum_{k \leq nt} Q_{n,k}^{(a)}(x) = q_{n,0}^a(x) - q_{n,[nt]+1}^a(x) = 1 - \left(\sum_{k \geq [nt]+1} p_{n,k}(x) \right)^a.$$

Note that $0 < a < 1$ and $\sum_{k \geq [nt]+1} p_{n,k}(x) \leq 1$. Hence

$$\sum_{k \leq nt} Q_{n,k}^{(a)}(x) \leq 1 - \sum_{k \geq [nt]+1} p_{n,k}(x) = \sum_{k \leq nt} p_{n,k}(x) \leq \sum_{k \leq nt} \frac{(k/n - x)^2}{(t-x)^2} p_{n,k}(x).$$

This means that

$$\sum_{k \leq nt} Q_{n,k}^{(a)}(x) \leq \frac{1}{(t-x)^2} \mu_{n,2}(x). \tag{10}$$

If $t > x,\ t, x \in I$ we can write

$$\sum_{k \geq nt} Q_{n,k}^{(a)}(x) = \sum_{k \geq nt} (q_{n,k}^a(x) - q_{n,k+1}^a(x))$$
$$= \left(\sum_{k \geq nt} p_{n,k}(x) \right)^a \leq \left(\sum_{k \geq nt} \frac{|k/n - x|^{2/a}}{|t-x|^{2/a}} p_{n,k}(x) \right)^a.$$

Hence

$$\sum_{k \geq nt} Q_{n,k}^{(a)}(x) \leq \frac{1}{(t-x)^2} (\mu_{n,2/a}(x))^a. \tag{11}$$

Coming back to the estimate of $|L_n^{(a)} g_x(x)|$ given by (9) let us write

$$\sum_{k \in A_x(a,b)} g_x\left(\frac{k}{n}\right) Q_{n,k}^{(a)}(x)$$
$$= \sum_{\frac{k}{n} \in I_x(-a)} g_x\left(\frac{k}{n}\right) Q_{n,k}^{(a)}(x) + \sum_{\frac{k}{n} \in I_x(b)} g_x\left(\frac{k}{n}\right) Q_{n,k}^{(a)}(x) = \sum\nolimits_1 + \sum\nolimits_2,$$

where $I_x(h) = [x+h, x] \cap I$ if $h < 0$, $I_x(h) = [x, x+h] \cap I$ if $h > 0$. Arguing similarly to the proof of Lemma in [1] (see also proof of Lemma 2 in [9]) and using inequalities (10), (11) we obtain

$$\left|\sum\nolimits_1\right| \leq \left(1 + \frac{8n}{a^2} \mu_{n,2}(x)\right) \left\{ \sum_{i=1}^{m-1} \frac{1}{i^3} v_i\big(g_x; I_x(-ia/\sqrt{n})\big) + \frac{1}{m^2} v_m(g_x; I_x(-a)) \right\},$$
$$\left|\sum\nolimits_2\right| \leq \left(1 + \frac{8n}{b^2} \left(\mu_{n,2/a}(x)\right)^a\right) \left\{ \sum_{i=1}^{m-1} \frac{1}{i^3} v_i\big(g_x; I_x(ib/\sqrt{n})\big) + \frac{1}{m^2} v_m(g_x; I_x(b)) \right\}.$$

Further, applying the Hölder inequality we observe that

$$\mu_{n,2}(x) \leq (\mu_{n,2/a}(x))^a \quad \text{if} \quad 0 < a < 1.$$

Next using the obvious inequality

$$v_i(g_x; I_x(-a)) + v_i(g_x; I_x(b)) \leq 2 v_i(g_x; Y_x(a,b)),$$

we easily get the estimate for $\Big|\sum_{k\in A_x(a,b)} g_x\left(\frac{k}{n}\right)Q^{(a)}_{n,k}(x)\Big|$.

If at least one of the points $x-a, x+b$ belongs to Int I then inequalities (10), (11) yield

$$\left|\sum_{k\in D_x(a,b)} g_x\left(\frac{k}{n}\right)Q^{(a)}_{n,k}(x)\right|$$

$$=\left|\sum_{\frac{k}{n}<x-a} g_x\left(\frac{k}{n}\right)Q^{(a)}_{n,k}(x)+\sum_{\frac{k}{n}>x+b} g_x\left(\frac{k}{n}\right)Q^{(a)}_{n,k}(x)\right|$$

$$\leq v_1(g_x;(-\infty,x)\cap I)\sum_{\frac{k}{n}<x-a} Q^{(a)}_{n,k}(x)+v_1(g_x;(x,\infty)\cap I)\sum_{\frac{k}{n}>x+b} Q^{(a)}_{n,k}(x)$$

$$\leq \frac{1}{c^2}(\mu_{n,2/a}(x))^a v_1(g_x;I),$$

where $c=\min\{a,b\}$.

Now, it is enough to apply identities (3) and (9) and the proof is complete. ■

Let $p\geq 1$. Denote by $BV_p(I)$ the class of all functions of bounded p-th power variation on the interval I. If $g\in BV_p(Y)$, then for every integer k,

$$V_k(g;Y)\leq k^{1-1/p}V_p(g;Y), \tag{12}$$

where $V_p(g;Y)$ denotes the total p-th power variation of g on Y, defined as $\sup\Big(\sum_i |g(t_i)-g(\tau_i)|^p\Big)^{1/p}$ over all finite systems of non-overlapping intervals $(\tau_i,t_i)\subset Y$.

Note that for many known operators there exist a non-negative function ψ_a and a positive integer $n(a)$ such that

$$(\mu_{n,2/a}(x))^a\leq \psi_a(x)n^{-1}\quad\text{for all}\quad x\in I, n\geq n(a). \tag{13}$$

The inequality (11), Theorem 1 and some calculation (cf. [9], the proof of Theorem 2) lead to

THEOREM 2. – *Let $f\in BV_p(I)$, $p\geq 1$, and let condition* (13) *hold. Then for every $x\in$ Int I at which* (5) *is satisfied and for every $n\geq\max\{4,n_0(x),n(a)\}$ we have*

$$|L^{(a)}_n f(x)-2^{-a}f(x+)-(1-2^{-a})f(x-)|$$

$$\leq\frac{16(1+4(a^{-2}+b^{-2})\psi_a(x))}{(\sqrt{n})^{1+1/p}}\sum_{k=1}^{n}\frac{1}{(\sqrt{k})^{1-1/p}}V_p(g_x;Y_x(a/\sqrt{k},b/\sqrt{k}))$$

$$+\frac{1}{n}\vartheta_x(a,b)\psi_a(x)V_p(g_x;I)+|\varDelta^{(a)}_n(f;x)|,$$

where $Y_x(h,\eta), n_0(x), \vartheta_x(a,b)$ *are as in Theorem* 1 *and* $|\Delta_n^{(a)}(f;x)|$ *is estimated as in the Lemma.*

REMARK 1. *Similar results for function* f *of bounded* Φ*-variation in the Young sense on* I *can be obtained, too.* (cf. [7, Corollary 1]).

REMARK 2. *In view of the continuity of the function* g_x *at* x*, the right-sides of the inequalities given in Theorems* 1 *and* 2 *tend to* 0 *as* $n \to \infty$ (see [9, Remark 1]).

3. – Examples.

Now, we present an application of our results to some operators of the form (1).

1) Let $L_n^{(a)}f \equiv B_n^{(a)}f$ be the Bernstein-Bézier operators of $f \in M(I)$ defined by (1) and (2), in which $I = [0,1], J_n = N_0, p_{n,j}(x) = \binom{n}{j}x^j(1-x)^{n-j}, 0 < a < 1$.

In this case $\sigma^2(x) = x(1-x), \beta(x) = x(1-x)(2x^2-2x+1)$ and conditions (5) hold for all $x \in (0,1)$. In view of our Lemma,

$$|\Delta_n^{(a)}(f;x)| \le \frac{5}{2\sqrt{nx(1-x)}}(|f(x+)-f(x-)| + e_n(x)|f(x)-f(x-)|)$$

for $n \ge n_0(x), n_0(x) \le 16/x(1-x)$. As is shown in [10] (p. 337), for all $n \ge 1$ there holds the following inequality

$$(\mu_{n,2/a}(x))^a \equiv \left(\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^{2/a} p_{n,k}(x)\right)^a \le A_a(x(1-x))^a n^{-1},$$

where A_a is a positive constant depending only on a. This means that condition (13) is satisfied with $\psi_a(x) = A_a(x(1-x))^a$ and $n(a) = 1$.

For example, choosing $a = x$ and $b = 1-x$ in our Theorem 2 and observing that $\vartheta_x(x, 1-x) = 0$ we easily get

COROLLARY 1. – *If* $f \in BV_p([0,1]), p \ge 1$ *and if* $0 < a < 1$*, then*

$$\begin{aligned} &|B_n^{(a)}f(x) - 2^{-a}f(x+) - (1-2^{-a})f(x-)| \\ &\le \frac{B_a}{x(1-x)^{2-a}(\sqrt{n})^{1+1/p}}\sum_{k=1}^{n}\frac{1}{(\sqrt{k})^{1-1/p}}V_p(g_x; Y_x(x/\sqrt{k},(1-x)/\sqrt{k})) \\ &+\frac{5}{2\sqrt{nx(1-x)}}(|f(x+)-f(x-)| + e_n(x)|f(x)-f(x-)|) \end{aligned}$$

for all $x \in (0,1), n \ge 16/x(1-x)$*, where* B_a *is a positive constant depending only on* a*.*

In case $p = 1$ this Corollary gives the result of Zeng [10, Theorem 1].

Also, more general result for $f \in M(I)$ can be formulated by applying Theorem 1.

2) Next, let us consider the modification of Baskakov operators $U_n^{(a)} f$ given by (1) and (2) in which $p_{n,j}(x) = \binom{n+j-1}{j} x^j (1+x)^{-n-j}$ for $x \in I = [0, \infty), j \in J_n = N_0$. Theorems 1 and 2 apply with $\sigma^2(x) = x(1+x), \beta(x) = \sum_{j=0}^{\infty} |j - x|^3 p_{1,j}(x) \le 3x(1+x)^2$ and

$$|\Delta_n^{(a)}(f;x)| \le \frac{6\sqrt{1+x}}{\sqrt{nx}} (|f(x+) - f(x-)| + e_n(x)|f(x) - f(x-)|)$$

for $x > 0$, $n \ge n_0(x), n_0(x) \le 144(1+x)/x$. In order to verify condition (13), we will estimate the function $(\mu_{n,2/a}(x))^a$. Write $l = 2/a$ and denote by $[l]$ the greatest integer not exceeding l. As in [10, Lemma 6] choose the numbers $p = \frac{2[l]}{2[l]+2-l}$, $p' = \frac{2[l]}{l-2}$, $r = \frac{2}{p}$, $s = \frac{2\upsilon}{p'}$, $\upsilon = [l] + 1$.

Clearly, $l > 2, p > 1, p' > 1, 1/p + 1/p' = 1$ an $l = r + s$. Applying the Hölder inequality we obtain

$$\begin{aligned}(\mu_{n,2/a}(x))^a &= \left(\sum_{k=0}^{\infty} \left| \frac{k}{n} - x \right|^{2/a} p_{n,k}(x) \right)^a \\ &\le \left(\sum_{k=0}^{\infty} \left| \frac{k}{n} - x \right|^{rp} p_{n,k}(x) \right)^{a/p} \left(\sum_{k=0}^{\infty} \left| \frac{k}{n} - x \right|^{sp'} p_{n,k}(x) \right)^{a/p'} \\ &= \left(\frac{x(1+x)}{n} \right)^{a/p} \left(\frac{1}{n^{2\upsilon}} T_{n,2\upsilon}(x) \right)^{a/p'},\end{aligned}$$

where $T_{n,2\upsilon}(x) = \sum_{k=0}^{\infty} (k - nx)^{2\upsilon} p_{n,k}(x)$. As it is known [6, Corollary 3.7], $T_{n,2\upsilon}(x) = \sum_{j=1}^{\upsilon} c_{j,\upsilon}(n)(x(1+x))^j n^j$, where $c_{j,\upsilon}(n)$ denote real numbers independent of x and bounded uniformly in n. Thus

$$\begin{aligned}(\mu_{n,2/a}(x))^a &\le \left(\frac{x(1+x)}{n} \right)^{a/p} \left(\frac{1}{n^{2\upsilon}} \sum_{j=1}^{\upsilon} |c_{j,\upsilon}(n)| (x(1+x))^j n^j \right)^{a/p'} \\ &\le c(\upsilon, a)(x(1+x))^{a/p} n^{-a(1/p+\upsilon/p')} \left(\sum_{j=1}^{\upsilon} (x(1+x))^j \right)^{a/p'} \\ &= c(\upsilon, a)(x(1+x))^{a/p} \left(\sum_{j=1}^{\upsilon} (x(1+x))^j \right)^{a/p'} n^{-1},\end{aligned}$$

where $c(\upsilon, a) = \left(\sup_{n \in N} \max_{1 \le j \le \upsilon} |c_{j,\upsilon}(n)| \right)^{a/p'}$.

This means that condition (13) is satisfied for all $n \in N$ with the function

$$\psi_a(x) = \lambda(a)\left(\sum_{j=1}^{[2/a]+1} (x(1+x))^j\right)^a, \tag{14}$$

where $\lambda(a)$ is a positive constant depending only on a.

Using the above estimate and choosing $a = b = 1$ $(\vartheta_x(1,1) = 1)$ in Theorem 2, we easily get the following

Corollary 2. – *If $f \in BV_p(I)$, where $I = [0,\infty), p \geq 1$, and if $0 < a < 1$, then for all $x > 0$ and $n \geq 144(1+x)/x$,*

$$\begin{aligned} &|U_n^{(a)} f(x) - 2^{-a} f(x+) - (1-2^{-a}) f(x-)| \\ &\leq \frac{16(1+8\psi_a(x))}{(\sqrt{n})^{1+1/p}} \sum_{k=1}^{n} \frac{1}{(\sqrt{k})^{1-1/p}} V_p(g_x; Y_x(1/\sqrt{k}, 1/\sqrt{k})) \\ &+ \frac{1}{n}\psi_a(x) V_p(g_x; I) + \frac{6\sqrt{1+x}}{\sqrt{nx}}(|f(x+) - f(x-)| + e_n(x)|f(x) - f(x-)|), \end{aligned}$$

where $\psi_a(x)$ is given by (14).

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Faculty of Mathematics and Computer Science, Adam Mickiewicz University
Umultowska 87, 61-614 Poznań, Poland
E-mail: ganiol@amu.edu.pl

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