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Adding or Removing an Element from a Pseudo-Symmetric Numerical Semigroup.

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Sunto. – *Se S è un semigruppo numerico* pseudo-simmetrico*, se g è il suo numero di Frobenius e se x è un generatore minimo di S allora anche $S \cup \{g\}$, $S \setminus \{x\}$ e $S \cup \{\frac{1}{2}g, g\}$ sono semigruppi numerici. In questo lavoro ci proponiamo di studiare tali costruzioni.*

Summary. – *If S is a* pseudo-symmetric *numerical semigroup, g is its Frobenius number and x is a minimal generator of S, then $S \cup \{g\}$, $S \setminus \{x\}$ and $S \cup \{\frac{1}{2}g, g\}$ are also numerical semigroups. In this paper we study these constructions.*

Introduction.

A *numerical semigroup* is a subset S of $\mathbb{N}$ (the set of nonnegative integers) closed under addition, $0 \in S$ and generates $\mathbb{Z}$ as a group (as usual $\mathbb{Z}$ denotes the set of integers). It is well known (see for instance [2] or [13]) that the set $\mathrm{H}(S) = \mathbb{N} \setminus S$ is finite. The elements of $\mathrm{H}(S)$ are called *gaps* of S and the cardinality of $\mathrm{H}(S)$ is an important invariant of S called gender of S in [8] or degree of singularity in [2]. The largest integer not belonging to S is the *Frobenius number* of S, denoted by $\mathrm{g}(S)$. Its study has been the subject of many papers (see for instance [3, 4, 10, 17, 18]). Given a numerical semigroup S, set

$$\mathrm{Pg}(S) = \{x \in \mathbb{Z} \setminus S | x + s \in S \text{ for all } s \in S \setminus \{0\}\}.$$

The elements of $\mathrm{Pg}(S)$ are called *pseudo-Frobenius numbers* of S (see [14]). The cardinality of $\mathrm{Pg}(S)$ is another invariant widely studied of S (see for instance [2, 5]) and it is known as the *type* of S.

Given a nonempty subset A of $\mathbb{N}$, denote by $\langle A \rangle$ the submonoid of $\mathbb{N}$ generated by A. A subset A of a numerical semigroup is a *system of generators* of S if $S = \langle A \rangle$ and we say that A is a *minimal* system of generators if in addition no proper subset of A generates S. It is well known (see for instance [13]) that every numerical semigroup S admits a unique minimal system of generators, which is finite and equals $S \setminus \{0\} \setminus (S \setminus \{0\} + S \setminus \{0\})$, that is, the set of elements of

$S \setminus \{0\}$ that cannot be expressed as a sum of two nonzero elements of S. This fact motivates the introduction of two other invariants of the numerical semigroup S. If $\{n_1 < \cdots < n_p\}$ is a minimal system of generators of S, then n_1 is the *multiplicity* of S, denoted by $\mathrm{m}(S)$, and p is the *embedding dimension* of S, denoted by $\mathrm{e}(S)$. The elements of $\{n_1, \ldots, n_p\}$ are called *minimal generators* of S. These concepts have their Algebraic Geometric interpretation through what is known as the semigroup ring associated to S (see for instance [6, 7]), and it is from this connection that they are named in this way.

A numerical semigroup is *irreducible* if it cannot be expressed as an intersection of two numerical semigroups that contain it properly. In [16] it is shown that S is irreducible if and only if it is maximal (with respect to set inclusion) in the set of numerical semigroups with Frobenius number $\mathrm{g}(S)$. From [2] and [5], this implies that the class of irreducible numerical semigroups with even (respectively odd) Frobenius number coincides with the class of pseudo-symmetric (respectively symmetric) numerical semigroups. These two classes of numerical semigroups have special interest in Ring Theory since their associated semigroup rings are Kunz (see [2]) and Gorenstein (see [9]), respectively.

If S is a numerical semigroup, $S \neq \mathbb{N}$ (observe that if $S = \mathbb{N}$, then $\mathrm{g}(\mathbb{N}) = -1$), and x is a minimal generator of S, then $S \cup \{\mathrm{g}(S)\}$ and $S \setminus \{x\}$ are also numerical semigroups. In [12] the author studies those numerical semigroups obtained in this way when S is a symmetric numerical semigroup. Here we focus our attention in this construction for S a pseudo-symmetric semigroup. Moreover, in this setting, $\mathrm{Pg}(S) = \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$ and thus $S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$ is also a numerical semigroup. This new construction is also studied in this work. If S is a pseudo-symmetric semigroup and x is a minimal generator of S, we will say that $S \setminus \{x\}$ is unitary reduction of a pseudo-symmetric semigroup, URPSY-semigroup for short; $S \cup \{\mathrm{g}(S)\}$ is a unitary extension of a pseudo-symmetric semigroup, UEPSY-semigroup for short; and $S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$ an extension by the pseudo-Frobenius numbers of S, a PEPSY-semigroup for short.

The contents of this paper are organized as follows. In Section 1 we study UEPSY-semigroups. We give a characterization in Theorem 10 for this kind of semigroups in terms of their number of gaps and a minimal generator. Corollary 11 provides us with a formula for the Frobenius number of a UEPSY-semigroup S in terms of its minimal generators, and Theorem 18 describes the set $\mathrm{Pg}(S)$. Section 2 is devoted to the URPSY-semigroups. In Theorem 24 these semigroups are characterized in terms of their number of gaps. Theorem 28 shows that the type of a URPSY-semigroup is 2, 3 or 4, and describes each of these situations. In Corollary 29 we prove that for every integer $e \geq 4$ there exists a numerical semigroup S with $\mathrm{e}(S) = e$ and $\mathrm{t}(S) = 4$, giving in this way continuity to the results obtained in [11, 15, 12] where it is shown that for every integer $e \geq 2$ there exists numerical semigroups with type t equal to 1, 2 and 3 (except for the cases $(e, t) \in \{(2, 2), (3, 3)\}$). Finally in Section 3 we deal with PEPSY-semigroups. Theorem 33 characterizes them in terms of their number of gaps, a minimal generator and an element which must be of unique expression. As a consequence of this result in Corollary 34 we give an upper bound for the Frobenius number of these semigroups in terms of their minimal generators. We end this section with Theorem 38 that states that in this class of numerical semigroups the type equals the embedding dimension minus one.

1. – UEPSY-semigroups.

A numerical semigroup S is *pseudo-symmetric* if $\mathrm{g}(S)$ is even and for all $x \in \mathbb{Z} \setminus S$ we have that either $x = \frac{1}{2}\mathrm{g}(S)$ or $\mathrm{g}(S) - x \in S$. A numerical semigroup S is a *UEPSY-semigroup* if there exists a pseudo-symmetric numerical semigroup S' such that $S' \subset S$ and $\#(S \setminus S') = 1$, that is, S is obtained from a pseudo-symmetric semigroup by adjoining one element to it. This element must be its Frobenius number as we see next.

PROPOSITION 1. – *Let S be a numerical semigroup. Then S is a UEPSY-semigroup if and only if there exists a pseudo-symmetric numerical semigroup S' such that $S = S' \cup \{\mathrm{g}(S')\}$.*

PROOF. – *Necessity.* Since S is UEPSY-semigroup, there exists S' a pseudo-symmetric numerical semigroup such that $S' \subset S$ and $\#(S \setminus S') = 1$, that is, $S = S' \cup \{h\}$ for $h \in S \setminus S'$. As $h \notin S'$ and S' is a pseudo-symmetric numerical semigroup, it follows that either $h = \frac{1}{2}\mathrm{g}(S')$ or $\mathrm{g}(S') - h \in S'$. Note that $S' \cup \{\frac{1}{2}\mathrm{g}(S')\}$ is not a numerical semigroup, because $\frac{1}{2}\mathrm{g}(S') + \frac{1}{2}\mathrm{g}(S') = \mathrm{g}(S') \notin S' \cup \{\frac{1}{2}\mathrm{g}(S')\}$. Hence $\mathrm{g}(S') - h \in S'$ and thus $\{h, \mathrm{g}(S') - h\} \subset S$, which leads to $\mathrm{g}(S') = h + (\mathrm{g}(S') - h) \in S$. Using now that $S = S' \cup \{h\}$, $\mathrm{g}(S') \notin S'$ and $\mathrm{g}(S') \in S$, we conclude that $h = \mathrm{g}(S')$.

Sufficiency. Follows from the definition. ■

The following result can be found in [2] and [5].

LEMMA 2. – *A numerical semigroup S is pseudo-symmetric if and only if $\mathrm{g}(S)$ is even and S is maximal (with respect to set inclusion) in the set of all numerical semigroups with Frobenius number $\mathrm{g}(S)$.*

In the sequel we use the symbol $\rightarrow$ in a set $X = \{x_1, \ldots, x_n, \rightarrow\} \subseteq \mathbb{N}$ to indicate that $x_n + k \in X$ for all $k \in \mathbb{N}$.

LEMMA 3. – *Let S be a numerical semigroup with $\mathrm{g}(S) = \mathrm{m}(S) - 1$. Then S is* pseudo-symmetric *if and only if $S = \{0, 3, \rightarrow\}$.*

PROOF. – *Necessity.* If $\mathrm{g}(S) = \mathrm{m}(S) - 1$, then $S = \{0, \mathrm{m}(S), \rightarrow\}$. If $\mathrm{m}(S) \geq 4$, then $1 \notin S$, $1 \neq \frac{1}{2}\mathrm{g}(S)$ and $\mathrm{g}(S) - 1 = \mathrm{m}(S) - 2 \notin S$. Hence S is not pseudo-symmetric. We conclude by observing that neither n nor $\{0, 2, \rightarrow\}$ are pseudo-symmetric numerical semigroups (their frobenius numbers are -1 and 1, respectively).

Sufficiency. Trivial. ■

From [5] or [2] we can deduce the following result.

LEMMA 4. – *A numerical semigroup is pseudo-symmetric if and only if* $\#\mathrm{H}(S) = \frac{\mathrm{g}(S)+2}{2}$.

LEMMA 5. – *Let S be a numerical semigroup such that* $\mathrm{m}(S) < \mathrm{g}(S) < 2\mathrm{m}(S)$. *Then S is pseudo-symmetric if and only if*

$$S = \{0, \mathrm{m}(S), \mathrm{m}(S) + 1, \ldots, 2\mathrm{m}(S) - 3, 2\mathrm{m}(S) - 1, \rightarrow\}.$$

PROOF. – *Necessity.* Let $i \in \{1, \ldots, \mathrm{m}(S) - 1\}$ be such that $\mathrm{g}(S) = \mathrm{m}(S) + i$. By Lemma 2, we deduce that $S = \{0, \mathrm{m}(S), \ldots, \mathrm{m}(S) + i - 1, \mathrm{m}(S) + i + 1, \rightarrow\}$. Hence $\#\mathrm{H}(S) = \mathrm{m}(S)$, and by Lemma 4, $\mathrm{m}(S) = \frac{\mathrm{m}(S)+i+2}{2}$, whence $i = \mathrm{m}(S) - 2$.

Sufficiency. Follows trivially from Lemma 4. ■

PROPOSITION 6. – *A numerical semigroup S is a* UEPSY-semigroup *if and only if one of the following conditions holds:*

(1) $S = \{0, \mathrm{m}(S), \rightarrow\}$ *and* $\mathrm{m}(S) \geq 2$,

(2) *there exists a* pseudo-symmetric *numerical semigroup S' such that* $\mathrm{g}(S') > 2\mathrm{m}(S')$ *and* $S = S' \cup \{\mathrm{g}(S')\}$.

PROOF. – Assume that S is a UEPSY-semigroup. By Proposition 1, there exists a pseudo-symmetric numerical semigroup S' such that $S = S' \cup \{\mathrm{g}(S')\}$. If $\mathrm{g}(S') > 2\mathrm{m}(S')$, then (2) holds. Otherwise, either $\mathrm{g}(S') = \mathrm{m}(S') - 1$ or $\mathrm{m}(S') < \mathrm{g}(S') < 2\mathrm{m}(S')$. The rest of the proof follows from Lemmas 3 and 5.

If condition (2) holds, then S clearly is a UEPSY-semigroup. If $S = \{0, \mathrm{m}(S), \rightarrow\}$ with $\mathrm{m}(S) \geq 3$, then $S = S' \cup \{\mathrm{g}(S')\}$, where $S' = \{0, \mathrm{m}(S), \ldots, 2\mathrm{m}(S) - 3, 2\mathrm{m}(S) - 1, \rightarrow\}$, and by Lemma 5 we know that S' is a pseudo-symmetric numerical semigroup, whence S is a UEPSY-semigroup. If $S = \{0, 2 \rightarrow\}$, then $S = S' \cup \{\mathrm{g}(S')\}$ with $S' = \{0, 3, \rightarrow\}$, which in view of Lemma 3 is a pseudo-symmetric numerical semigroup. ■

The semigroups of the form $\{0, m, \rightarrow\}$ will appear again later in this paper. We call them *interval semigroups*. For a semigroup of this kind it is straightforward to compute its Frobenius number, type, multiplicity, embedding dimension and number of gaps. Proposition 6 allows us to focus in the sequel on the study of numerical semigroups of the form $S' \cup \{g(S')\}$ with S' a pseudo-symmetric numerical semigroup such that $g(S') > 2m(S')$. Our next goal is Theorem 10; before proving it we need some previous results.

LEMMA 7. – *If S is a pseudo-symmetric numerical semigroup with* $g(S) > 2m(S)$, *then* $g(S \cup \{g(S)\}) = g(S) - m(S)$.

PROOF. – The integer $g(S) - m(S)$ cannot be in $S \cup \{g(S)\}$, since otherwise $\{g(S) - m(S), m(S)\} \subset S$ and this yields $g(S) \in S$. Let us prove that $g(S) - m(S) + i \in S \cup \{g(S)\}$ for all $i \in \mathbb{N} \setminus \{0\}$. If $g(S) - m(S) + i \notin S$, then as S is pseudo-symmetric this implies that either $g(S) - m(S) + i = \frac{1}{2}g(S)$ or $g(S) - (g(S) - m(S) + i) \in S$. Since $g(S) > 2m(S)$, we have that $g(S) - m(S) + i \neq \frac{1}{2}g(S)$, and thus $m(S) - i \in S$. But $m(S)$ is the least positive integer of S and $i \neq 0$, whence $m(S) = i$, or in other words, $g(S) - m(S) + i = g(S) \in S \cup \{g(S)\}$. Therefore $g(S) - m(S) + i \in S \cup \{g(S)\}$ for all $i \in \mathbb{N} \setminus \{0\}$ and as $g(S) - m(S) \notin S \cup \{g(S)\}$, this means that $g(S) - m(S) = g(S \cup \{g(S)\})$. ■

The proof of the following two lemmas is trivial.

LEMMA 8. – *Let S be a numerical semigroup with* $g(S) > m(S)$. *Then* $m(S \cup \{g(S)\}) = m(S)$.

LEMMA 9. – *Let S be a numerical semigroup and let $x \in S$. Then $S \setminus \{x\}$ is a numerical semigroup if and only if x is a minimal generator of S.*

THEOREM 10. – *Let S be a numerical semigroup that is not an interval semigroup. The following conditions are equivalent:*

(1) *S is a UEPSY-semigroup,*
(2) $\#H(S) = \frac{g(S)+m(S)}{2}$ *and* $g(S) + m(S)$ *is a minimal generator of S.*

PROOF. – (1) *implies* (2). By Proposition 6, there exists a pseudo-symmetric numerical semigroup S' such that $g(S') > 2m(S')$ and $S = S' \cup \{g(S')\}$. Lemmas 7 and 8 assert that $g(S) = g(S') - m(S)$. Hence from Lemma 4 we deduce that $\#H(S) = \#H(S') - 1 = \frac{g(S')+2}{2} - 1 = \frac{g(S)+m(S)}{2}$. Finally, note that $g(S') = g(S) + m(S)$ is a minimal generator of $S = S' \cup \{g(S')\}$.

(2) *implies* (1). If $g(S) + m(S)$ is a minimal generator of S, by Lemma 9 we obtain that $S' = S \setminus \{g(S) + m(S)\}$ is a numerical semigroup. Observe that $g(S') = g(S) + m(S)$. In this way, $\#H(S') = \#H(S) + 1 = \frac{g(S')+2}{2}$, which by Lemma 4 implies that S' is a pseudo-symmetric numerical semigroup. As $S = S' \cup \{g(S')\}$ we conclude that S is a UEPSY-semigroup. ■

Given a numerical semigroup S and an element $n \in S \setminus \{0\}$. The *Apéry set* of S with respect to n (see [1]) is defined by

$$\mathrm{Ap}(S,n) = \{s \in S | s - n \notin S\}.$$

It is well known that $\#\mathrm{Ap}(S,n) = n$, every minimal generator x of S, $x \neq n$, belongs to $\mathrm{Ap}(S,n)$, and the maximum of $\mathrm{Ap}(S,n)$ equals $\mathrm{g}(S) + n$. As a consequence of Theorem 10 and these remarks we obtain the following.

COROLLARY 11. – *Let S be a UEPSY-semigroup and let $\{n_1 < \cdots < n_p\}$ be its minimal system of generators. Then* $\mathrm{g}(S) = n_p - n_1$.

Recall that for a numerical semigroup S, $\mathrm{Pg}(S) = \{x \in \mathbb{Z} \setminus S | x + s \in S \text{ for all } s \in S \setminus \{0\}\}$ and that its cardinality is the type of S, $\mathrm{t}(S)$. Our next goal is to prove Theorem 18. We first need several lemmas.

LEMMA 12. – [2, 5] *Let S be a numerical semigroup. Then S is pseudo-symmetric if and only if* $\mathrm{Pg}(S) = \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$.

The next result can be found in [12].

LEMMA 13. – *Let S be a numerical semigroup and $\{n_1 < \cdots < n_p\}$ be its minimal system of generators. If $\mathrm{g}(S) > n_1$, then*

$$(\mathrm{Pg}(S) \setminus \{\mathrm{g}(S)\}) \cup \{\mathrm{g}(S) - n_1\} \subseteq \mathrm{Pg}(S \cup \{\mathrm{g}(S)\}) \subseteq (\mathrm{Pg}(S) \setminus \{\mathrm{g}(S)\}) \cup \{\mathrm{g}(S) - n_1, \ldots, \mathrm{g}(S) - n_p\}.$$

LEMMA 14. – *Let S be a pseudo-symmetric numerical semigroup with minimal system of generators $\{n_1, \ldots, n_p\}$ and let $i \in \{1, \ldots, p\}$. Then* $\mathrm{g}(S) - n_i \in \mathrm{Pg}(S \cup \{\mathrm{g}(S)\})$ *if and only if* $\mathrm{g}(S) > n_i$ *and* $n_i - \frac{1}{2}\mathrm{g}(S) \notin S$.

PROOF. – *Necessity.* If $n_i > \mathrm{g}(S)$, then $\mathrm{g}(S) - n_i < 0$ and thus $\mathrm{g}(S) - n_i \notin \mathrm{Pg}(S \cup \{\mathrm{g}(S)\})$. Hence $\mathrm{g}(S) > n_i$. If $n_i - \frac{1}{2}\mathrm{g}(S) \in S$, then $n_i - \frac{1}{2}\mathrm{g}(S) \in (S \cup \{\mathrm{g}(S)\}) \setminus \{0\}$ and as $\mathrm{g}(s) - n_i \in \mathrm{Pg}(S \cup \{\mathrm{g}(S)\})$ this leads to $\mathrm{g}(S) - n_i + (n_i - \frac{1}{2}\mathrm{g}(S)) = \frac{1}{2}\mathrm{g}(S) \in S \cup \{\mathrm{g}(S)\}$, which is impossible.

Sufficiency. Clearly $\mathrm{g}(S) - n_i \notin S \cup \{\mathrm{g}(S)\}$. We claim that $\mathrm{g}(S) - n_i + s \in S \cup \{\mathrm{g}(S)\}$ for all $s \in (S \cup \{\mathrm{g}(S)\}) \setminus \{0\}$. If $s = \mathrm{g}(S)$, then the claim follows easily. Assume that $\mathrm{g}(S) - n_i + s \notin S \cup \{\mathrm{g}(S)\}$ for some $s \in S$, $s \neq 0$. Then $\mathrm{g}(S) - n_i + s \neq \mathrm{g}(S)$ and $\mathrm{g}(S) - n_i + s \notin S$. This implies that $s \neq n_i$ and as S is pseudo-symmetric, we have that either $\mathrm{g}(S) - n_i + s = \frac{1}{2}\mathrm{g}(S)$ or $\mathrm{g}(S) - (\mathrm{g}(S) - n_i + s) = n_i - s \in S$. Since $\mathrm{n}_i - \frac{1}{2}\mathrm{g}(S) \notin S$, we get that $\mathrm{g}(S) - n_i + s \neq \frac{1}{2}\mathrm{g}(S)$. Hence $n_i - s \in S$. But as n_i is a minimal generator of S, this forces s to be n_i, in contradiction with $s \neq n_i$. ■

The following result appears in [16].

LEMMA 15. – *Let S be a numerical semigroup and let n be a nonzero element of S. Then S is pseudo-symmetric if and only if*

$$\mathrm{Ap}(S,n)=\{0=w(1)<w(2)<\cdots<w(n-1)=\mathrm{g}(S)+n\}\cup\left\{\frac{1}{2}\mathrm{g}(S)+n\right\}$$

and $w(i)+w(n-i)=w(n-1)$ for all $i\in\{1,\ldots,n-1\}$.

LEMMA 16. – *Let S be a pseudo-symmetric numerical semigroup such that $\mathrm{g}(S)>2\mathrm{m}(S)$ and $\mathrm{m}(S)\geq 4$. Then every minimal generator of S is less than $\mathrm{g}(S)$.*

PROOF. – By hypothesis, $\mathrm{m}(S)<\mathrm{g}(S)$. From the remark made before Corollary 11, we know that every minimal generator of S not equal to $\mathrm{m}(S)$ is in $\mathrm{Ap}(S,\mathrm{m}(S))$. Note also that under the standing hypothesis $\frac{1}{2}\mathrm{g}(S)+\mathrm{m}(S)<\mathrm{g}(S)$. Let x be a minimal generator of S such that $x\notin\{\mathrm{m}(S),\frac{1}{2}\mathrm{g}(S)+\mathrm{m}(S)\}$. Using Lemma 15 we have that $x\in\{w(2)<\cdots<w(\mathrm{m}(S)-1)=\mathrm{g}(S)+\mathrm{m}(S)\}$. Moreover, $x\neq\mathrm{g}(S)+\mathrm{m}(S)$ since otherwise by Lemma 15 we would have that $x=\mathrm{g}(S)+\mathrm{m}(S)=w(2)+w(\mathrm{m}(S)-2)$, contradicting that x is a minimal generator of S. Hence $x=w(i)$ for some $i\in\{2,\ldots,\mathrm{m}(S)-2\}$ and by Lemma 15, $x+w(\mathrm{m}(S)-i)=\mathrm{g}(S)+\mathrm{m}(S)$. As $w(\mathrm{m}(S)-i)\in S\setminus\{0\}$, we get $w(\mathrm{m}(S)-i)\geq\mathrm{m}(S)$ and thus $x<\mathrm{g}(S)$. ■

LEMMA 17. – *Let S be a pseudo-symmetric numerical semigroup with minimal system of generators $\{n_1<\cdots<n_p\}$, $\mathrm{g}(S)>2\mathrm{m}(S)$ and $\mathrm{m}(S)\geq 4$. Then*

$$\{n_1,\ldots,n_p,\mathrm{g}(S)\}$$

is a minimal system of generators of $S\cup\{\mathrm{g}(S)\}$ and thus $\mathrm{e}(S\cup\{\mathrm{g}(S)\})=\mathrm{e}(S)+1$.

PROOF. – Clearly $\{n_1,\ldots,n_p,\mathrm{g}(S)\}$ generates $S\cup\{\mathrm{g}(S)\}$. The minimality of the set $\{n_1,\ldots,n_p,\mathrm{g}(S)\}$ follows from the fact that $\{n_1,\ldots,n_p\}$ is a minimal system of generators of S and that $n_i<\mathrm{g}(S)$ for all $i\in\{1,\ldots,p\}$ (see Lemma 16). ■

We are ready to prove the announced result describing the set of pseudo-Frobenius numbers of a UEPSY-semigroup.

THEOREM 18. – *Let S be a UEPSY-semigroup such that $\mathrm{m}(S)\geq 4$ and S is not an interval semigroup. Let $\{n_1<\cdots<n_p\}$ be a minimal system of generators of S. Then*

$$\mathrm{Pg}(S)=\left\{n_p-n_i|n_i\neq n_p\text{ and }n_i-\frac{1}{2}n_p\notin S\right\}\cup\left\{\frac{1}{2}n_p\right\}.$$

PROOF. – By Proposition 6, we know that there exists a pseudo-symmetric numerical semigroup S' such that $\mathrm{g}(S') > 2\mathrm{m}(S')$ and $S = S' \cup \{\mathrm{g}(S')\}$. Corollary 11 states that $n_p = \mathrm{g}(S) + n_1$ and from Lemmas 7, and 8 we obtain $n_1 = \mathrm{m}(S) = \mathrm{m}(S')$ and $\mathrm{g}(S) = \mathrm{g}(S') - n_1$, whence $n_p = \mathrm{g}(S')$. Using now Lemma 17 we deduce that $\{n_1, \ldots, n_{p-1}\}$ is a minimal system of generators of S'. By Lemma 12 we know that $\mathrm{Pg}(S') = \{\frac{1}{2}n_p, n_p\}$ and by Lemma 13, we have that $\mathrm{Pg}(S) = \mathrm{Pg}(S' \cup \{\mathrm{g}(S')\}) \subseteq \{n_p - n_1, \ldots, n_p - n_{p-1}\} \cup \{\frac{1}{2}n_p\}$ and that $\frac{1}{2}n_p \in \mathrm{Pg}(S)$. The proof concludes by using Lemma 14. ■

COROLLARY 19. – *Let S be a* UEPSY-semigroup *such that* $\mathrm{m}(S) \geq 4$ *and S is not an interval semigroup. Then* $2 \leq \mathrm{t}(S) \leq \mathrm{e}(S)$.

PROOF. – It suffices to apply Theorem 18, observing that if $\{n_1 < \cdots < n_p\}$ is a minimal system of generators of S, then $\{\frac{1}{2}n_p, n_p - n_1\} \subseteq \mathrm{Pg}(S)$. ■

To conclude this section we focus our attention on those UEPSY-semigroup s that are not interval semigroups and with $\mathrm{m}(S) \leq 3$. If S is such a numerical semigroup, by Proposition 6, we know that there exists a pseudo-symmetric numerical semigroup S' such that $\mathrm{g}(S') > 2\mathrm{m}(S')$ and $S = S' \cup \{\mathrm{g}(S')\}$. It follows that in this situation $\mathrm{m}(S) = \mathrm{m}(S')$. Therefore our study is equivalent to the study of pseudo-symmetric numerical semigroups with multiplicity less than or equal to 3. Clearly, there are no pseudo-symmetric numerical semigroups with multiplicity 1 or 2, and thus there are no UEPSY-semigroup with multiplicity 1 or 2 other than interval semigroups. The following result can be deduced from [16].

LEMMA 20. – *Let S be a numerical semigroup. Then S is a pseudo-symmetric numerical semigroup with* $\mathrm{m}(S) = 3$ *if and only if* $S = \langle 3, x+3, 2x+3 \rangle$ *with x a positive integer not divisible by* 3.

As a consequence of this we obtain the following result.

PROPOSITION 21. – *Let S be a numerical semigroup that is not an interval semigroup. Then S is a UEPSY-semigroup with* $\mathrm{m}(S) \leq 3$ *if and only if* $S = \langle 3, x+3, 2x \rangle$ *with x a positive integer greater than or equal to* 4 *not divisible by* 3.

2. – URPSY-semigroup.

A numerical semigroup S is a URPSY-semigroup if there exists a pseudo-symmetric numerical semigroup S' such that $S \subseteq S'$ and $\#(S' \setminus S) = 1$. In this setting, $S = S' \setminus \{x\}$ for some $x \in S'$, which in view of Lemma 9 must be a minimal generator of S'. We summarize this in the next proposition.

PROPOSITION 22. – *Let S be a numerical semigroup. Then S is a URPSY-semigroup if and only if there exists a* pseudo-symmetric *numerical semigroup S' and a minimal generator x of S' such that $S = S' \setminus \{x\}$.*

PROPOSITION 23. – *Let S be a numerical semigroup. Then S is a URPSY-semigroup if and only if one of the following conditions holds:*

(1) $S \in \{\langle 4,5,6,7\rangle, \langle 3,5,7\rangle, \langle 4,5,11\rangle\}$,

(2) $S = \langle 3, x+3\rangle$ *with x a positive integer not divisible by* 3,

(3) $S = \langle m, m+1, \ldots, 2m-3\rangle$ *with m a positive integer greater than or equal to* 5,

(4) *there exists a pseudo-symmetric numerical semigroup S' and a minimal generator $x < \mathrm{g}(S')$ of S' such that $S = S' \setminus \{x\}$.*

PROOF. – *Necessity.* By Proposition 22 there exists a pseudo-symmetric numerical semigroup S' and a minimal generator x of S' such that $S = S' \setminus \{x\}$. If $x < \mathrm{g}(S')$, then (4) holds. Assume that $x > \mathrm{g}(S')$. From Lemma 16 we get that either $\mathrm{g}(S') < 2\mathrm{m}(S')$ or $\mathrm{m}(S') = 3$ (there are no pseudo-symmetric numerical semigroups with multiplicity 1 or 2). Using now Lemmas 3, 5 and 20, we have that S' is of one of the following forms:

- $S' = \langle 3, x+3, 2x+3\rangle$ with x a positive integer not divisible by 3,
- $S' = \{0, 3, \rightarrow\}$,
- $S' = \langle m, m+1, \ldots, 2m-3, 2m-1\rangle$ with m an integer greater than or equal to 4.

Considering the semigroups $S' \setminus \{x\}$ with x a minimal generator greater than $\mathrm{g}(S')$, where S' ranges in any of the above cases, yields the desired result.

Sufficiency. The reader can check that any of the semigroups fulfilling conditions (1) to (4) is a URPSY-semigroup. ■

Since the semigroups given in (1)-(3) of the preceding proposition are well typified, we focus in the sequel in the study of those URPSY-semigroup fulfilling condition (4) of Proposition 23.

THEOREM 24. – *Let S be a numerical semigroup not fulfilling any of the conditions* (1)-(3) *of Proposition* 23. *Then S is an URPSY-semigroup if and only if* $\#\mathrm{H}(S) = \frac{\mathrm{g}(S)+4}{2}$.

PROOF. – *Necessity.* By Proposition 23, there exists a pseudo-symmetric numerical semigroup S' and a minimal generator x of S' such that $x < \mathrm{g}(S')$ and $S = S' \setminus \{x\}$. From this it follows that $\mathrm{g}(S) = \mathrm{g}(S')$, and by Lemma 4, $\#\mathrm{H}(S) = \#\mathrm{H}(S') + 1 = \frac{\mathrm{g}(S')+2}{2} + 1 = \frac{\mathrm{g}(S)+4}{2}$.

Sufficiency. By Lemma 4 we know that S is not a pseudo-symmetric numerical semigroup. Hence there exists $h = \max\{h \in \mathbb{Z} \setminus S | \mathrm{g}(S) - h \notin S, h \neq \frac{1}{2}\mathrm{g}(S)\}$ (here max stands for maximum). Clearly $h > \frac{1}{2}\mathrm{g}(S)$ and thus $2h \in S$. Moreover, the maximality of h warranties that $h + s \in S$ for all $s \in S \setminus \{0\}$. Therefore $S' = S \cup \{h\}$ is a numerical semigroup with $\mathrm{g}(S') = \mathrm{g}(S)$. Moreover, $\#\mathrm{H}(S') = \#\mathrm{H}(S) - 1 = \frac{\mathrm{g}(S')+2}{2}$, which by Lemma 4 implies that S' is pseudo-symmetric, whence S is a URPSY-semigroup. ∎

Theorem 28 describes the type of a URPSY-semigroup. Before proving it we need a couple of technical lemmas.

LEMMA 25. – *Let S be a pseudo-symmetric numerical semigroup and x be one of its minimal generators. If $x < \mathrm{g}(S)$, then*

$$\{\mathrm{g}(S), x\} \subseteq \mathrm{Pg}(S \setminus \{x\}) \subseteq \{\mathrm{g}(S), x, \frac{1}{2}\mathrm{g}(S), \mathrm{g}(S) - x\}.$$

PROOF. – Clearly $\{\mathrm{g}(S), x\} \subseteq \mathrm{Pg}(S \setminus \{x\})$. Let $g' \in \mathrm{Pg}(S \setminus \{x\}) \setminus \{\mathrm{g}(S), x, \frac{1}{2}\mathrm{g}(S)\}$. Then $g' + x$ cannot be in S, since otherwise $g' + s \in S$ for all $s \in S \setminus \{0\}$ and this would lead to $g' \in \mathrm{Pg}(S)$, which by Lemma 12 equals $\{\mathrm{g}(S), \frac{1}{2}\mathrm{g}(S)\}$. It is easy to see that $g' + x + s \in S$ for all $s \in S \setminus \{0\}$ and thus $g' + x \in \mathrm{Pg}(S) = \{\mathrm{g}(S), \frac{1}{2}\mathrm{g}(S)\}$, whence either $g' = \mathrm{g}(S) - x$ or $g' = \frac{1}{2}\mathrm{g}(S) - x$. As $\frac{1}{2}\mathrm{g}(S) \in \mathrm{Pg}(S)$, $\frac{1}{2}\mathrm{g}(S) + x \in S \setminus \{x\}$, and since $\frac{1}{2}\mathrm{g}(S) - x + (\frac{1}{2}\mathrm{g}(S) + x) = \mathrm{g}(S) \notin S \setminus \{x\}$, we have that $\frac{1}{2}\mathrm{g}(S) - x \notin \mathrm{Pg}(S \setminus \{x\})$. Hence $g' = \mathrm{g}(S) - x$.

LEMMA 26. – *Let S be a pseudo-symmetric numerical semigroup and let x be one of its minimal generators with $x < \mathrm{g}(S)$. Then $\frac{1}{2}\mathrm{g}(S) \in \mathrm{Pg}(S \setminus \{x\})$ if and only if $x - \frac{1}{2}\mathrm{g}(S) \notin S$.*

PROOF. – If $x - \frac{1}{2}\mathrm{g}(S) \in S$, then $x - \frac{1}{2}\mathrm{g}(S) \in S \setminus \{0, x\}$. As $\frac{1}{2}\mathrm{g}(S) + (x - \frac{1}{2}\mathrm{g}(S)) = x \notin S \setminus \{x\}$, we have that $\frac{1}{2}\mathrm{g}(S) \notin \mathrm{Pg}(S \setminus \{x\})$.

Assume now that $\frac{1}{2}\mathrm{g}(S) \notin \mathrm{Pg}(S \setminus \{x\})$. Since $\frac{1}{2}\mathrm{g}(S) \in \mathrm{Pg}(S)$ (see Lemma 12), we deduce that there exists $s \in S \setminus \{0, x\}$ such that $\frac{1}{2}\mathrm{g}(S) + s = x$, whence $x - \frac{1}{2}\mathrm{g}(S) = s \in S$. ∎

LEMMA 27. – *Let S be a pseudo-symmetric numerical semigroup and let x be a minimal generator of S such that $x < \mathrm{g}(S)$. Then $\mathrm{g}(S) - x \in \mathrm{Pg}(S \setminus \{x\})$ if and only if $2x - \mathrm{g}(S) \notin S$.*

PROOF. – *Necessity.* As $\mathrm{g}(S) - x \in \mathrm{Pg}(S \setminus \{x\})$, we have that $\mathrm{g}(S) - x + s \neq x$ for all $s \in S \setminus \{0, x\}$. Hence $2x - \mathrm{g}(S) \notin S \setminus \{0, x\}$ and thus $2x - \mathrm{g}(S) \notin S$.

Sufficiency. Clearly $\mathrm{g}(S) - x \notin S \setminus \{x\}$. If $\mathrm{g}(S) - x \notin \mathrm{Pg}(S \setminus \{x\})$, then there exists $s \in S \setminus \{0, x\}$ such that $\mathrm{g}(S) - x + s \notin S \setminus \{x\}$. Hence either $\mathrm{g}(S) -$

$x+s=x$ or $\mathrm{g}(S)-x+s\notin S$. This first condition yields $2x-\mathrm{g}(S)\in S$, which is impossible. Therefore $\mathrm{g}(S)-x+s\notin S$ and using that S is a pseudo-symmetric numerical semigroup, this leads to either $\mathrm{g}(s)-x+s=\frac{1}{2}\mathrm{g}(S)$ or $\mathrm{g}(S)-(\mathrm{g}(S)-x+s)=x-s\in S$. Again, the first condition cannot hold, since $\mathrm{g}(S)-x+s=\frac{1}{2}\mathrm{g}(S)$ implies $x-\frac{1}{2}\mathrm{g}(S)\in S$ and thus $2x-\mathrm{g}(S)\in S$, which is impossible. Hence $x-s\in S$, and as x is a minimal generator and $s\in S\setminus\{0,x\}$, this yields a contradiction. ∎

THEOREM 28. – *Let S be a pseudo-symmetric numerical semigroup and let x be one of its minimal generators with* $x<\mathrm{g}(S)$*. Then* $\mathrm{t}(S\setminus\{x\})\in\{2,3,4\}$*. Moreover*

(1) $t(S\setminus\{x\})=2$ *if and only if* $x-\frac{1}{2}\mathrm{g}(S)\in S$*; in this setting* $\mathrm{Pg}(S\setminus\{x\})=\{\mathrm{g}(S),x\}$,

(2) $t(S\setminus\{x\})=3$ *if and only if* $2x-\mathrm{g}(S)\in S$ *and* $x-\frac{1}{2}\mathrm{g}(S)\notin S$*; in this setting* $\mathrm{Pg}(S\setminus\{x\})=\{\mathrm{g}(S),x,\frac{1}{2}\mathrm{g}(S)\}$,

(3) $t(S\setminus\{x\})=4$ *if and only if* $2x-\mathrm{g}(S)\notin S$*; in this setting* $\mathrm{Pg}(S\setminus\{x\})=\{\mathrm{g}(S),x,\frac{1}{2}\mathrm{g}(S),\mathrm{g}(S)-x\}$.

PROOF. – The proof follows easily from Lemmas 25, 26 and 27. Observe that if $x-\frac{1}{2}\mathrm{g}(S)\in S$, then $2x-\mathrm{g}(S)=2(x-\frac{1}{2}\mathrm{g}(S))\in S$. ∎

From [11] it can be deduced that for every integer $e\geq 2$ there exists a numerical semigroup S with $\mathrm{e}(S)=e$ and $\mathrm{t}(S)=1$. In [12] we prove that

(1) for all $e\geq 3$, there exists a numerical semigroup S such that $\mathrm{e}(S)=e$ and $\mathrm{t}(S)=2$ (this result also follows from [15]),

(2) for all $e\geq 4$, there exists a numerical semigroup S such that $\mathrm{e}(S)=e$ and $\mathrm{t}(S)=3$.

As an application of the results presented so far in this section, we prove that for all integer $e\geq 4$ there exists a numerical semigroup S with $\mathrm{e}(S)=e$ and $\mathrm{t}(S)=4$.

It is well known (see [5]) that if S is a numerical semigroup with $\mathrm{e}(S)=2$ (respectively $\mathrm{e}(S)=3$), then $\mathrm{t}(S)=1$ (respectively $\mathrm{t}(S)\in\{1,2\}$), whence there are no numerical semigroups with type equal to 4 and embedding dimension 2 or 3.

COROLLARY 29. – *For every integer* $e\geq 4$*, there exists a numerical semigroup S with* $\mathrm{e}(S)=e$ *and* $\mathrm{t}(S)=4$.

PROOF. – For $e=4$, the numerical semigroup $S=\langle 8,20,21,23\rangle$ has embedding dimension and type equal to 4 (the reader will not find any difficulties in proving this by using that $\mathrm{Ap}(S,8)=\{0,41,42,43,20,21,46,23\}$ and [5, Proposition 7]). Now assume that $e\geq 5$ and let $S=\{0,e,e+$

$1, \dots, 2e-3, 2e-1, \rightarrow\}$. By Lemma 5, we know that S is a pseudo-symmetric numerical semigroup. Moreover, $\mathrm{g}(S) = 2e-2$, e is a minimal generator of S less than $\mathrm{g}(S)$, and $2e - \mathrm{g}(S) = 2 \notin S$. Hence by Theorem 28, $\mathrm{t}(S \setminus \{e\}) = 4$. Finally, it is easy to prove that $\{e+1, \dots, 2e-3, 2e-1, 2e, 2e+1\}$ is a minimal system of generators of $S \setminus \{e\}$ and thus $\mathrm{e}(S \setminus \{e\}) = e$. ∎

3. – PEPSY-semigroup.

Let S be a pseudo-symmetric numerical semigroup. By Lemma 12, we know that $\mathrm{Pg}(S) = \{\mathrm{g}(S), \frac{1}{2}\mathrm{g}(S)\}$. From this it follows easily that $S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$ is a numerical semigroup. This fact allows us to give the following definition. A numerical semigroup S is a *PEPSY-semigroup* if there exists a pseudo-symmetric numerical semigroup S' such that $S = S' \cup \{\mathrm{g}(S'), \frac{1}{2}\mathrm{g}(S')\}$.

PROPOSITION 30. – *A numerical semigroup S is a PEPSY-semigroup if and only if one of the following conditions holds:*

(1) $S = \{0, m, \rightarrow\}$ *with* $m \geq 1$,

(2) *there exists a pseudo-symmetric numerical semigroup S' with* $\mathrm{g}(S') > 2\mathrm{m}(S')$ *and* $S = S' \cup \{\mathrm{g}(S'), \frac{1}{2}\mathrm{g}(S')\}$.

PROOF. – *Necessity.* Let S' be a pseudo-symmetric numerical semigroup such that $S = S' \cup \{\mathrm{g}(S'), \frac{1}{2}\mathrm{g}(S')\}$. If $\mathrm{g}(S') > 2\mathrm{m}(S')$, then (2) is fulfilled by S. If to the contrary $\mathrm{g}(S') < 2\mathrm{m}(S')$, then by Lemmas 3 and 5, we know that $S' = \{0, 3, \rightarrow\}$ or $S' = \{0, m, \dots, 2m-3, 2m-1, \rightarrow\}$ for some $m \geq 3$. The reader can check that in any of these two cases $S' \cup \{\mathrm{g}(S'), \frac{1}{2}\mathrm{g}(S')\}$ is of the form $\{0, x, \rightarrow\}$ for some $x \geq 1$.

Sufficiency. Trivial. ∎

In view of this last proposition, every PEPSY-semigroup that is not an interval semigroup comes from a pseudo-symmetric numerical semigroup S' with $\mathrm{g}(S') > 2\mathrm{m}(S')$. Theorem 33 is the analogue of Theorem 10 for PEPSY-semigroups. Before proving it, we need to introduce some results and concepts.

LEMMA 31. – *Let S be a pseudo-symmetric numerical semigroup. Then*

$$\mathrm{g}\left(S \cup \left\{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\right\}\right) = \mathrm{g}(S) - \mathrm{m}(S).$$

PROOF. – The proof is similar to the proof of Lemma 7. In this case we do not need $\mathrm{g}(S)$ to be greater than $2\mathrm{m}(S)$, since we can allow $\mathrm{g}(S) - \mathrm{m}(S) + i$ to be $\frac{1}{2}\mathrm{g}(S)$. ∎

The following result has an immediate proof.

LEMMA 32. – *Let S be a pseudo-symmetric numerical semigroup with* $\mathrm{g}(S) > 2\mathrm{m}(S)$. *Then*

$$\mathrm{m}\left(S \cup \left\{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\right\}\right) = \mathrm{m}(S).$$

Let S be a numerical semigroup and let $\{n_1, \ldots, n_p\}$ be its minimal system of generators. Given $s \in S$ we know that there exists $(a_1, \ldots, a_p) \in \mathbb{N}^p$ such that $s = a_1 n_1 + \cdots + a_p n_p$. An element $s \in S$ has *unique expression* if the corresponding $(a_1, \ldots, a_p) \in \mathbb{N}^p$ is unique.

THEOREM 33. – *Let S be a numerical semigroup that is not an interval semigroup. The following conditions are equivalent:*

(1) *S is a* PEPSY-semigroup,

(2) $\#\mathrm{H}(S) = \frac{\mathrm{g}(S)+\mathrm{m}(S)-2}{2}$, $\frac{\mathrm{g}(S)+\mathrm{m}(S)}{2}$ *is a minimal generator of S and* $\mathrm{g}(S) + \mathrm{m}(S)$ *is an element of S of unique expression.*

PROOF. – (1) *implies* (2). By Proposition 30, we know that there exists a pseudo-symmetric numerical semigroup S' with $\mathrm{g}(S') > 2\mathrm{m}(S')$ and such that $S = S' \cup \{\frac{1}{2}\mathrm{g}(S'), \mathrm{g}(S')\}$. By Lemmas 31 and 32, we know that $\mathrm{g}(S) = \mathrm{g}(S') - \mathrm{m}(S')$ and $\mathrm{m}(S) = \mathrm{m}(S')$. Using now Lemma 4, we obtain $\#\mathrm{H}(S) = \#\mathrm{H}(S') - 2 = \frac{\mathrm{g}(S)+\mathrm{m}(S)-2}{2}$. Clearly, if $\{n_1, \ldots, n_p\}$ is a minimal system of generators of S', then there exists $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, p\}$ such that $\{n_{i_1}, \ldots, n_{i_r}, \frac{1}{2}\mathrm{g}(S')\}$ is a minimal system of generators of S, and thus $\frac{\mathrm{g}(S)+\mathrm{m}(S)}{2} = \frac{1}{2}\mathrm{g}(S')$ is a minimal generator of S. We prove that $\mathrm{g}(S) + \mathrm{m}(S) = 0n_{i_1} + \cdots + 0n_{i_r} + 2\frac{1}{2}\mathrm{g}(S')$ is the unique expression of $\mathrm{g}(S) + \mathrm{m}(S)$ in S. Assume to the contrary there exists $(a_1, \ldots, a_r, a_{r+1}) \in \mathbb{N}^{r+1}$ such that $\mathrm{g}(S) + \mathrm{m}(S) = a_1 n_{i_1} + \cdots + a_r n_{i_r} + a_{r+1}\frac{1}{2}\mathrm{g}(S')$ and $(a_1, \ldots, a_{r+1}) \neq (0, \ldots, 0, 2)$. Since $\mathrm{g}(S) + \mathrm{m}(S) = \mathrm{g}(S') \notin S'$ and $\{n_{i_1}, \ldots, n_{i_r}\} \subset S'$, we have that $a_{r+1} \neq 0$ and as $(a_1, \ldots, a_{r+1}) \neq (0, \ldots, 0, 2)$, we deduce that $a_{r+1} = 1$. But then we obtain $\frac{1}{2}\mathrm{g}(S') = \mathrm{g}(S) + \mathrm{m}(S) - \frac{1}{2}\mathrm{g}(S') = a_1 n_{i_1} + \cdots + a_r n_{i_r} \in S'$, which is impossible.

(2) *implies* (1). As $\frac{\mathrm{g}(S)+\mathrm{m}(S)}{2}$ is a minimal generator of S and $\mathrm{g}(S) + \mathrm{m}(S)$ has unique expression (which is $\mathrm{g}(S) + \mathrm{m}(S) = 2\frac{\mathrm{g}(S)+\mathrm{m}(S)}{2}$), we have that $S' = S \setminus \{\frac{\mathrm{g}(S)+\mathrm{m}(S)}{2}, \mathrm{g}(S) + \mathrm{m}(S)\}$ is a numerical semigroup. The rest of the proof follows by counting the number of gaps of S' and applying Lemma 4. ■

As a consequence of Theorem 33 and the remark preceding Corollary 11 we have the following result.

COROLLARY 34. – *Let S be a PEPSY-semigroup that is not an interval semigroup and let $\{n_1 < \cdots < n_p\}$ be its minimal system of generators. Then* $\mathrm{g}(S) \leq 2n_p - n_1$. *Moreover,* $\mathrm{g}(S) = 2n_i - n_1$ *for some $i \in \{2, \ldots, p\}$ such that $2n_i > n_p$ and $2n_i$ has unique expression in S.*

We finish this section with Theorem 38. For its proof we still need some lemmas.

LEMMA 35. – *Let S be a pseudo-symmetric numerical semigroup with minimal system of generators $\{n_1, \ldots, n_p\}$. Then*

$$\mathrm{Pg}\left(S \cup \left\{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\right\}\right) \subseteq \{\mathrm{g}(S) - n_1, \ldots, \mathrm{g}(S) - n_p\}.$$

PROOF. – Let $g' \in \mathrm{Pg}(S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\})$. Assume that $\frac{1}{2}\mathrm{g}(S) - g' \in S$. Then there exists $s \in S \setminus \{0\}$ such that $g' = \frac{1}{2}\mathrm{g}(S) - s$ and as $g' \in \mathrm{Pg}(S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\})$, we have that $\mathrm{g}(S) - s = g' + \frac{1}{2}\mathrm{g}(S) \in S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$. Since $\mathrm{g}(S) - s \notin S$ and $\mathrm{g}(S) - s \neq \mathrm{g}(S)$, this leads to $\mathrm{g}(S) - s = \frac{1}{2}\mathrm{g}(S)$ and thus $\mathrm{g}(S) = 2s \in S$, in contradiction with the definition of $\mathrm{g}(S)$. Therefore $\frac{1}{2}\mathrm{g}(S) - g' \notin S$. From the definition of g', we have that $g' \notin \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$, which by Lemma 12 implies that $g' \notin \mathrm{Pg}(S)$. Hence $g' \in \mathrm{Pg}(S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\} \setminus \mathrm{Pg}(S)$, that is, there exists $y \in S \setminus \{0\}$ such that $g' + y \in \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$. We already know that $g' + y$ cannot be equal to $\frac{1}{2}\mathrm{g}(S)$ and consequently $g' + y = \mathrm{g}(S)$. As $y \in S \setminus \{0\}$, there exists $i \in \{1, \ldots, p\}$ such that $y - n_i = z \in S$. Besides, $g' + n_i \in S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$ (recall that $g' \in \mathrm{Pg}(S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}))$ and we already know that $g' + n_i \neq \frac{1}{2}\mathrm{g}(S)$ and that $g' + n_i \notin S$, since otherwise $g' + n_i + z = g' + y$ would be in S. Hence $g' + n_i$ must be equal to $\mathrm{g}(S)$, forcing y to be n_i. We conclude that $g' = \mathrm{g}(S) - n_i$.

LEMMA 36. – *Let S be a pseudo-symmetric numerical semigroup with minimal system of generators $\{n_1, \ldots, n_p\}$. Then $\mathrm{g}(S) - n_i \in \mathrm{Pg}(S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\})$ if and only if $\mathrm{g}(S) > n_i$ and $n_i - \frac{1}{2}\mathrm{g}(S) \notin S$.*

PROOF. – *Necessity.* As in the proof of Lemma 14, n_i must be less than $\mathrm{g}(S)$. Moreover, if $n_i - \frac{1}{2}\mathrm{g}(S) \in S$, then $\mathrm{g}(S) - (n_i - \frac{1}{2}\mathrm{g}(S)) \notin S$, and thus $\mathrm{g}(S) - n_i + \frac{1}{2}\mathrm{g}(S) \notin S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$, which implies that $\mathrm{g}(S) - n_i \notin \mathrm{Pg}(S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\})$.

Sufficiency. The proof is analogue to the *sufficiency* part of Lemma 14. The only case the reader must check is that $\mathrm{g}(S) - n_i + s \in S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$ for $s = \frac{1}{2}\mathrm{g}(S)$. ■

LEMMA 37. – *Let S be a pseudo-symmetric numerical semigroup such that $\mathrm{g}(S) > 2\mathrm{m}(S)$ and let $\{n_1 < \cdots < n_p\}$ be its minimal system of generators. Let*

$$\{n_{i_1}, \ldots, n_{i_r}\} = \left\{n_i | n_i - \frac{1}{2}\mathrm{g}(S) \notin S\right\}.$$

Then $\{n_{i_1}, \ldots, n_{i_r}, \frac{1}{2}\mathrm{g}(S)\}$ is the minimal system of generators of $S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$.

PROOF. – Clearly $\{n_1, \ldots, n_p, \frac{1}{2}\mathrm{g}(S)\}$ generates $S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$. If $n_i - \frac{1}{2}\mathrm{g}(S) = s \in S$, then $s \neq 0$ and thus $n_i = s + \frac{1}{2}\mathrm{g}(S)$ is not a minimal generator of $S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$. From this it can be deduced that $\{n_{i_1}, \ldots, n_{i_r}, \frac{1}{2}\mathrm{g}(S)\}$ generates $S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$. As $\frac{1}{2}\mathrm{g}(S) \notin S$, we have that $\frac{1}{2}\mathrm{g}(S) \notin \langle n_{i_1}, \ldots, n_{i_r}\rangle$, which means that $\frac{1}{2}\mathrm{g}(S)$ is a minimal generator of $S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$. Now take $j \in \{1, \ldots, r\}$. We prove that n_{i_j} cannot be expressed as a sum of two nonzero elements of $S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$. This in particular would imply that n_{i_j} is a minimal generator of $S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$. Assume to the contrary that $n_{i_j} = x + y$ with $x, y \in (S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}) \setminus \{0\}$. Since n_{i_j} is a minimal generator of S, we have that x and y cannot be both in S. Hence either $x \in \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$ or $y \in \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$. Suppose without loss of generality that $x \in \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$.

- If $x = \frac{1}{2}\mathrm{g}(S)$, then $n_{i_j} - \frac{1}{2}\mathrm{g}(S) = y$ and as we know that $n_{i_j} - \frac{1}{2}\mathrm{g}(S) \notin S$, this forces y to be in $\{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$. The case $y = \frac{1}{2}\mathrm{g}(S)$ leads to $n_{i_j} = \mathrm{g}(S)$, which is impossible. Hence $y = \mathrm{g}(S)$ and thus $n_{i_j} = \mathrm{g}(S) + \frac{1}{2}\mathrm{g}(S) > \mathrm{g}(S) + \mathrm{m}(S)$ and this implies that $n_{i_j} - \mathrm{m}(S) > \mathrm{g}(S)$. But this yields $n_{i_j} - \mathrm{m}(S) \in S$, which is also impossible since n_{i_j} is a minimal generator of S.
- If $x = \mathrm{g}(S)$, then $n_{i_j} = \mathrm{g}(S) + y$. Since $y \in (S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}) \setminus \{0\}$ and both $\frac{1}{2}\mathrm{g}(S)$ and $\mathrm{g}(S)$ are greater than $\mathrm{m}(S)$, we have that $y \geq \mathrm{m}(S)$. As above, y cannot be greater than $\mathrm{m}(S)$ and consequently $y = \mathrm{m}(S)$, that is, $n_{i_j} = \mathrm{g}(S) + \mathrm{m}(S)$. Hence $n_{i_j} - \frac{1}{2}\mathrm{g}(S) = \frac{1}{2}\mathrm{g}(S) + \mathrm{m}(S)$ and since S is pseudo-symmetric, $\frac{1}{2}\mathrm{g}(S) \in \mathrm{Pg}(S)$, we have that $\frac{1}{2}\mathrm{g}(S) + \mathrm{m}(S) \in S$, whence $n_{i_j} - \frac{1}{2}\mathrm{g}(S) \in S$, contradicting the choice of n_{i_j}.

In any case we get a contradiction and therefore n_{i_j} is a minimal generator of $S \cup \{\frac{1}{2}\mathrm{g}(S), \mathrm{g}(S)\}$. ■

THEOREM 38. – *Let S be a PEPSY-semigroup that is not an interval semigroup. Then* $\mathrm{t}(S) = \mathrm{e}(S) - 1$.

PROOF. – By Proposition 30, we know that there exists a pseudo-symmetric numerical semigroup S' such that $S = S' \cup \{\frac{1}{2}\mathrm{g}(S'), \mathrm{g}(S')\}$ and $\mathrm{g}(S') > 2m(S')$.

- For $\mathrm{m}(S') \geq 4$, by Lemma 16, we know that all the minimal generators of S' are less than $\mathrm{g}(S')$. The result now follows from Lemmas 35, 36 and 37.
- For $\mathrm{m}(S') \leq 3$, Lemma 20 ensures that $S' = \langle 3, x+3, 2x+3\rangle$ for some positive integer x that is not a multiple of 3. Then $\mathrm{g}(S') = 2x$, $S = S' \cup \{x, 2x\} = \langle 3, x\rangle$, and $\mathrm{t}(S) = 1 = \mathrm{e}(S) - 1$. ■

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