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Sunto. – *Definiamo relazioni canoniche discretizzate associate ad automorfismi di ordine finito di gruppi abeliani discreti. Questa è una generalizzazione di automorfismi di ordine finito di algebre di rotazione. Si dimostrano anche proprietà di particolari operatori di Schrödinger che derivano da queste relazioni.*

Summary. – *We define discretized canonical commutation relations associated to finite order automorphisms of discrete abelian groups. This generalizes the situation for rotation algebras and their finite order automorphisms. We also consider the almost Schrödinger operator associated to the given commutation relations.*

Introduction.

Arveson [1], [2] introduced the idea of discretized canonical commutation relations that certain pivotal operators satisfy, in considering the problem of a discrete version for the Hamiltonian of a one-dimensional quantum system.

This led naturally to the non-commutative spheres of Bratteli, Elliott, Evans and Kishimoto [3], [4] (the fixed point subalegbra of the "flip" automorphism of the rotation algebra) and in turn the appearance of the associated almost Schrödinger and almost Mathieu operators which have been extensively studied [6], [8], [17], [18], [19]. These commutation relations could be viewed as associated to a bicharacter on $\mathbb{Z}^2$, together with an order two automorphism given by $x \mapsto -x$ (the "flip"). This viewpoint naturally suggests an extension to consider any finite order automorphism of a discrete abelian group together with a bicharacter and the study of the "discrete models" thus obtained. The main idea, as with the earlier situation, is to exploit the commutation relations that certain pivotal operators satisfy. In constructing these discrete models we are again led to the rotation algebra and more generally, higher dimensional non-commutative tori [20], [22], in particular certain fixed point subalgebras thereof. For more precise details see Theorems 3.5 and 3.6. This type of characterization allows us to deduce information on the norm of certain almost Scrödinger operarators (c.f. Theorems 5.2 and 5.7). We also derive properties of the associated eigenvalues systems (c.f. Theorems 6.2 and 6.3).

In more detail the contents of this paper are as follows. In Section 1 we define for a family of operators the discretized canonical commutation relations associated to a finite order automorphism of a discrete abelian group together with an invariant bicharacter. In Section 2 we give the two main examples, Examples 2.2 and 2.4, we consider throughout the paper involving the groups $\mathbb{Z}^2$ and $\mathbb{Z}^n$. In Section 3 we detail our two main examples and their link with fixed point subalgebras of non-commutative tori. In Section 4, we use a theorem of Rieffel, to show our examples are representative of the general situation. Finally in Sections 5 and 6 we define and consider almost Schrödinger operators and study in detail some of their norm bounds and associated eigenvalue equations.

1. – Discretized Canonical Commutation Relations.

Let G be a discrete abelian group and let ω be a bicharacter of G, i.e., ω is a function $\omega : G \times G \to \mathbb{T}$ satisfying,

$$\begin{aligned} &\omega(x,-x) = \omega(-x,x) = 1 && \forall x \in G,\\ &\omega(x,0) = \omega(0,x) = 1 && \forall x \in G,\\ &\omega(x,x) = 1 && \forall x \in G. \end{aligned}$$

Let F be a finite order automorphism (of order $|F|$) of G. Then a bicharacter ω of G is called an F-invariant bicharacter if ω satisfies the additional property:

$$\omega(x,y) = \omega(Fx,Fy) \quad \forall x,y \in G.$$

DEFINITION 1.1. – *Given a discrete abelian group G, a finite automorphism F of G, and an F-invariant bicharacter of G, we say that a uniformly bounded family of operators $\{D_x\}_{x\in G} \subseteq \mathcal{B}(H)$ satisfies the Discretized Canonical Commutation Relations associated to F if,*

(1) $D_0 = |F|I$,
(2) $D_x^* = D_{-x}, \quad \forall x \in G$,
(3) $D_xD_y = \sum_{j=1}^{|F|} \omega(x,F^jy)D_{x+F^jy}, \quad \forall x,y \in G$,

where F^jy denotes the j-th power of the automorphism F applied to y.

REMARK 1.2. – The assumption $D_0 = |F|I$ is very natural since if we let $y = 0$ in the product formula (3) we obtain $D_xD_0 = |F|D_x$. Note also if we let $x = 0$ instead we obtain $|F|D_y = \sum_{j=1}^{|F|} D_{F^jy}$ or $(|F|-1)D_y = \sum_{j=1}^{|F|-1} D_{F^jy}$.

REMARK 1.3. – If the order of F is even, say $2n$, and $F^n = -I$, using the product formula (3), it is easy to show $D_0 D_x = D_0 D_{-x}$. Consequently $D_x = D_{-x} = D_x^*$ and the operators D_x are self adjoint.

PROPOSITION 1.4. – $\|D_x\| \leq |F| \; \forall x \in G$.

PROOF. – Let M be the l.u.b. for $\{\|D_x\|\}$, $x \in G$. Then, by (2) and (3),

$$D_x D_x^* = D_x D_{-x} = \sum_{j=1}^{|F|} \omega(x, -F^j x) D_{x-F^j x}.$$

Thus,

$$\|D_x\|^2 = \|D_x D_x^*\| \leq M|F|,$$

hence $M \leq |F|$. ∎

Note that Proposition 1.4 implies the existence of concrete representations for the universal C^*-algebra generated by a family of operators satisfying the Discretized Canonical Commutation Relations.

2. – Main Examples.

In this Section we will present examples of families of operators satisfying the discretized canonical commutation relations for $G = \mathbb{Z}^2$, and $G = \mathbb{Z}^n$, $n \geq 3$.

We will first define a bicharacter of $G = \mathbb{Z}^2$.

LEMMA 2.1. – *Let $G = \mathbb{Z}^2$ and $x = \begin{pmatrix} m \\ n \end{pmatrix}$, $y = \begin{pmatrix} k \\ \ell \end{pmatrix}$ be elements of G. Define $\omega(x, y) = \rho^{\frac{(nk-m\ell)}{2}}$, where $\rho = e^{2\pi i\theta}$, with $\theta \in [0, 1)$. Then,*

(4) $\omega(x, -x) = \omega(-x, x) = 1$, $\forall x \in \mathbb{Z}^2$,
(5) $\omega(0, x) = \omega(x, 0) = 1$, $\forall x \in \mathbb{Z}^2$,
(6) $\omega(x, y) = \omega(Ax, Ay)$, $\forall x, y \in \mathbb{Z}^2$, $\forall A \in SL(2, \mathbb{Z})$.

PROOF. – Straightforward computation. ∎

The non-identity, finite order elements of $SL(2, \mathbb{Z})$ (up to conjugacy class) are given by the matrices,

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \pm\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Consequently we will consider the following automorphisms of $\mathbb{Z}^2$,

$$\Sigma(y) = \begin{pmatrix} -k \\ -\ell \end{pmatrix} \quad \text{(order 2)}, \qquad Z(y) = \begin{pmatrix} -(k+\ell) \\ k \end{pmatrix} \quad \text{(order 3)},$$

$$T(y) = \begin{pmatrix} \ell \\ -k \end{pmatrix} \quad \text{(order 4)}, \qquad E(y) = \begin{pmatrix} -\ell \\ (k+\ell) \end{pmatrix} \quad \text{(order 6)}.$$

Note that $\Sigma = -I$ was first considered by Arveson in [1]. Now $-Z = E^5$, $-T = T^3$ and $-E = Z^2$ so $-Z$, $-T$, and $-E$ are included. Also $T^2 = -I$, $T^3 = -T$, $E^2 = Z$, and $E^3 = -I$ so explicitly writing the product rule for the discretized canonical commutation relations for the four automorphisms above, we have:

$$\begin{aligned}
&\Sigma : D_x D_y = \omega(x,-y)D_{x-y} + \omega(x,y)D_{x+y} \\
&T : D_x D_y = \omega(x,Ty)D_{x+Ty} + \omega(x,-y)D_{x-y} + \omega(x,-Ty)D_{x-Ty} + \omega(x,y)D_{x+y} \\
&Z : D_x D_y = \omega(x,Zy)D_{x+Zy} + \omega(x,Z^2y)D_{x+Z^2y} + \omega(x,y)D_{x+y} \\
&E : D_x D_y = \omega(x,Ey)D_{x+Ey} + \omega(x,Zy)D_{x+Zy} + \omega(x,-y)D_{x-y} \\
&\qquad\quad + \omega(x,-Ey)D_{x-Ey} + \omega(x,-Zy)D_{x-Zy} + \omega(x,y)D_{x+y}
\end{aligned}$$

where D_x is self adjoint for Σ, T and E from Remark 1.3.

EXAMPLE 2.2. – *Let $\mathcal{A}_\theta$ be the rotation algebra, that is the universal C^*-algebra generated by two unitaries operators U and V satisfying $VU = \rho UV$, with $\rho = e^{2\pi i\theta}$, $\theta \in [0,1)$. If δ is a finite order automorphism of $\mathcal{A}_\theta$ of order $r \in \mathbb{N}$, define*

$$\delta(m,n) = \rho^{mn/2} \sum_{j=0}^{r-1} \delta^j(U^m V^n).$$

Introduce the following automorphisms of $\mathcal{A}_\theta$:

$$\begin{aligned}
\sigma(U) = U^{-1},\ \sigma(V) = V^{-1} \quad &\text{(order 2)}, \\
\tau(U) = V,\ \tau(V) = U^{-1} \quad &\text{(order 4)}, \\
\zeta(U) = e^{-\pi i\theta}U^{-1}V,\ \zeta(V) = U^{-1} \quad &\text{(order 3)}, \\
\eta(U) = V,\ \eta(V) = e^{-\pi i\theta}U^{-1}V \quad &\text{(order 6)}.
\end{aligned}$$

As noticed in [1], *the choice of the family of operators $D_x = \sigma(m,n)$ $(m,n \in \mathbb{Z})$ in the rotation algebra, $\mathcal{A}_\theta$, (notation as above for $\delta = \sigma$) gives a family of self-adjoint operators satisfying the discretized canonical commutation relations associated to Σ. The choices $D_x = \tau(m,n)$, $D_x = \zeta(m,n)$ and $D_x = \eta(m,n)$ (notation as above for $\delta = \tau, \zeta$, and η respectively) give families of operators, also in $\mathcal{A}_\theta$, satisfying the discretized canonical commutation relations associated to T, Z and E respectively.*

Now we will give an example of a family of operators satsfying the discretized canonical commutation relations for the flip automorphism S of $\mathbb{Z}^n$. Firstly, we will define an S-invariant bicharacter of $\mathbb{Z}^n$.

LEMMA 2.3. – *Let $G = \mathbb{Z}^n$ and $x = (t_1, \dots, t_n)$, $y = (j_1, \dots, j_n)$ be elements of G. Let S denote the order two automorphism of $\mathbb{Z}^n$ given by $x \to -x$ $\forall x \in \mathbb{Z}^n$. (When $n = 2$, $S = \Sigma$.) Define*

$$\omega(x, y) = e\left(\sum_{s<\ell} \frac{\theta_{s,\ell}}{2}(t_\ell j_s - t_s j_\ell)\right),$$

where $\Theta = \{\theta_{k,j}\}_{1 \le k < j \le n} \in M_n(\mathbb{R})$ is skew-symmetric, and $e(t) = e^{2\pi i t}$. Then,

(7) $\omega(x, -x) = \omega(-x, x) = 1$ $\forall x \in \mathbb{Z}^n$,
(8) $\omega(0, x) = \omega(x, 0) = 1$ $\forall x \in \mathbb{Z}^n$,
(9) $\omega(x, y) = \omega(Sx, Sy)$ $\forall x, y \in \mathbb{Z}^n$.

PROOF. – Clear from the definitions of ω and S. ■

EXAMPLE 2.4. – *Let $\mathcal{A}_\Theta$ denote the non-commutative torus, that is, the universal C^*-algebra generated by unitary operators U_k, $k = 1, \dots, n$ subject to the relations $U_k U_\ell = e^{2\pi i \theta_{\ell,k}} U_\ell U_k$ for $1 \le k < \ell \le n$.*

Define the operators $[x]$, $x = (t_1, \dots, t_n) \in \mathbb{Z}^n$, in $\mathcal{A}_\Theta$ by,

$$[x] = e\left(\sum_{s<\ell} \frac{\theta_{s,\ell}}{2} t_s t_\ell\right)\left(U_1^{t_1} \dots U_n^{t_n} + U_1^{-t_1} \dots U_n^{-t_n}\right).$$

Clearly $[-x] = [x]$ and $[0] = 2I$. Using the relations among the U_ℓ's,

$$\begin{aligned} [x]^* &= e\left(-\sum_{s<\ell} \frac{\theta_{s,\ell}}{2} t_s t_\ell\right)\left(U_n^{-t_n} \dots U_1^{-t_1} + U_n^{t_n} \dots U_1^{t_1}\right) \\ &= e\left(\sum_{s<\ell} \frac{\theta_{s,\ell}}{2} t_s t_\ell\right)\left(U_1^{t_1} \dots U_n^{t_n} + U_1^{-t_1} \dots U_n^{-t_n}\right) \\ &= [x] = [-x], \end{aligned}$$

and, if $y = (j_1, \dots, j_n)$,

$$[x][y] = \omega(x, y)[x + y] + \omega(x, -y)[x - y].$$

Thus $\{[x]\}_{x \in \mathbb{Z}^n}$ is a family that satisfies the discretized canonical commutation relations associated to S.

Moreover we can define the "antisymmetric" self-adjoint operators $\{x\}$,

$x \in \mathbb{Z}^n$, by

$$\{x\} = ie\left(\sum_{s<\ell} \frac{\theta_{s,\ell}}{2} t_s t_\ell\right)\left[U_1^{t_1} \dots U_n^{t_n} - U_1^{-t_1} \dots U_n^{-t_n}\right],$$

and the following multiplication relation holds

$$[x]\{y\} = \omega(x,y)\{x+y\} - \omega(x,-y)\{x-y\}.$$

3. – Embeddings in Non-Commutative Tori.

In this section we will construct a concrete C^*-algebra that faithfully realizes the discretized canonical commutation relations. Define,

$$\ell^1(G,\omega) \underset{\text{def}}{=} \begin{cases} \text{Banach space of all absolutely summable complex} \\ \text{functions on } G \text{ with multiplication and involution} \\ (f * g)(x) = \sum_{y \in G} \omega(y,x) f(y) g(x-y),\ f^*(x) = \overline{f(-x)} \} \end{cases}$$

and,

$$\mathcal{D} \underset{\text{def}}{=} \{f \in \ell^1(G,\omega) \mid f(Fx) = f(x)\ \forall x \in G\}, \quad \mathcal{D} \subseteq \ell^1(G,\omega).$$

LEMMA 3.1. – *$\mathcal{D}$ is a $*$-subalgebra of $\ell^1(G,\omega)$.*

PROOF. – Straightforward computation. ■

Since by Proposition 1.4, the $*$-subalgebra $\mathcal{D}$ admits $*$-representations on a Hilbert space, we define

DEFINITION 3.2. –

$$C^*(\mathcal{D}) \underset{\text{def}}{=} \text{enveloping } C^*\text{-algebra of } \mathcal{D}$$

that is the universal C^*-algebra generated by $\{D_x\}_{x \in G}$ subject to the discretized canonical commutation relations associated to F.

REMARK 3.3. – With G and ω as in Example 2.2, $\mathcal{D}$ is mapped into the rotation algebra $\mathcal{A}_\theta$. This is since $\mathcal{D} \subseteq \ell^1(\mathbb{Z}^2,\omega)$. In fact, $\ell^1(\mathbb{Z}^2,\omega)$ is the universal Banach *-algebra generated by unitary operators $\{W_x \mid x \in \mathbb{Z}^2\}$ satisfying:

$$W_x W_y = \omega(x,y) W_{x+y}.$$

Clearly $U = W_{(1,0)}$ and $V = W_{(0,1)}$ generate this algebra and it is easy to show that $VU = e^{2\pi i\theta}UV$. Thus the completion $C^*(\ell^1(\mathbb{Z}^2,\omega))$ of $\ell^1(\mathbb{Z}^2,\omega)$ is isomorphic to $\mathcal{A}_\theta$. Hence we obtain a homomorphism $\gamma : \ell^1(\mathbb{Z}^2,\omega) \to \mathcal{A}_\theta$, and also a homomorphism $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathcal{A}_\theta$, or $\gamma_{\mathcal{D}} : C^*(\mathcal{D}) \to \mathcal{A}_\theta$ by passing to completion.

REMARK 3.4. – Using a similar argument to that above when G and ω are as in Example 2.4 then $\mathcal{D}$ is mapped into the non-commutative torus $\mathcal{A}_\Theta$ with $\gamma_{\mathcal{D}} : C^*(\mathcal{D}) \to \mathcal{A}_\Theta$ a homomorphism.

The proofs of the following two theorems are straightforward. (c.f. [1] for the order two case)

THEOREM 3.5. – *Let θ be irrational and let $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathcal{A}_\theta$ be the morphism defined by:*

$$\gamma_{\mathcal{D}}(d_x) = \sum_{j=1}^{|F|} W_{F^j x}, \text{ where } F = \Sigma, T, Z, E, \quad x \in \mathbb{Z}^2$$
$$\text{with } \mathcal{A}_\theta = C^*(W_x \mid W_x W_y = \omega(x,y) W_{x+y}, x, y \in \mathbb{Z}^2)$$
$$= C^*(U, V \mid VU = e^{2\pi i\theta} UV).$$

Then the natural homomorphism $\gamma_{\mathcal{D}} : C^*(\mathcal{D}) \to \mathcal{A}_\theta$ induces an isomorphism of C^*-algebras: $C^*(\mathcal{D}) \cong \mathcal{A}_\theta^F$ ($F = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ for $F = \Sigma, T, Z, E$ respectively), where $\mathcal{A}_\theta^F$ denotes the fixed point subalgebra of $\mathcal{A}_\theta$ determined by the action of $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ on $\mathcal{A}_\theta$ given by

$$\begin{cases} U = W_{(1,0)} \overset{F}{\mapsto} e^{\pi iac\theta} U^a V^c, \\ V = W_{(0,1)} \overset{F}{\mapsto} e^{\pi ibd\theta} U^b V^d. \end{cases}$$

(See [3], [4], [5], [9], [10], [11], [12], [13] *and* [14] *for more information on these fixed point subalgebras.)*

THEOREM 3.6. – *Let Θ be totally irrational and let $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathcal{A}_\Theta$ be the morphism defined by:*

$$\gamma_{\mathcal{D}}(d_x) = W_x + W_{Sx}, \text{ where}$$
$$\mathcal{A}_\Theta = C^*(W_x \mid W_x W_y = \omega(x,y) W_{x+y}, x \in \mathbb{Z}^n)$$
$$= C^*(U_1, \ldots, U_n \mid U_j U_i = e^{2\pi i\theta_{i,j}} U_i U_j, i < j).$$

Then the natural homomorphism $\gamma_{\mathcal{D}} : C^*(\mathcal{D}) \to \mathcal{A}_\Theta$ induces an isomorphism of C^*-algebras: $C^*(\mathcal{D}) \cong \mathcal{A}_\Theta^S$, where $\mathcal{A}_\Theta^S$ denotes the fixed point subalgebra of $\mathcal{A}_\Theta$

determined by the action of S on $\mathcal{A}_\Theta$ given by $S(U_i) = U_i^{-1}$, $i = 1, \ldots, n$. (See [7], [15], [16] *for more details on this fixed point subalgebra.)*

4. – The General Case.

Here we will explain how to reduce the case of general abelian groups to one of our main examples 2.2 and 2.4.

First, we will look at 2.2 and 2.4 from a slightly different viewpoint.

With $G = \mathbb{Z}^2$, let $M = \mathbb{R}^2$, and then construct $\widetilde{G} = M \times \widehat{M} = \mathbb{R}^2 \times \mathbb{R}^2$, where $\widehat{M}$ is the dual of M, with canonical Heisenberg bicharacter $\beta(\begin{pmatrix} m \\ s \end{pmatrix}, \begin{pmatrix} n \\ t \end{pmatrix}) = e^{2\pi imt}$. The skew bicharacter B on $\widetilde{G}$ defined by $B(x,y) = \beta(x,y)\overline{\beta}(y,x)$ corresponds, when restricted to $\mathbb{Z}^2$, to $\overline{\omega}$, with ω as in Section 2. The rotation algebra $\mathcal{A}_\theta$ is characterized by the relation $u_x u_y = \beta(x,y) u_{x+y}$.

More in general, when G is any discrete abelian group and α is any (continuous) bicharacter on G, by a result of Rieffel, we can find an appropriate locally compact abelian group $\widetilde{G}$ in which G embeds as a closed subgroup and α is the restriction of the Heisenberg cocycle on $\widetilde{G}$, as the situation is for the case $G = \mathbb{Z}^2$. More precisely

DEFINITION 4.1 [21]. – *Let M be any locally compact abelian group, $\widehat{M}$ its dual group and let $N = M \times \widehat{M}$. Then on N we define the canonical Heisenberg cocycle by*

$$H((m,s),(n,t)) = \langle m,t \rangle,$$

where $\langle , \rangle$ denotes the duality between M and $\widehat{M}$.

Furthermore N has a canonical square-integrable H-representation Π on $L^2(M)$, the Heisenberg representation, defined by

$$(\Pi_{m,s} f)(n) = < n, s > f(n+m).$$

The commutation relations among the operators of Π are given by

$$\Pi_x \Pi_y = H(x,y)\Pi_{x+y} = H(x,y)\overline{H(y,x)}\Pi_y \Pi_x.$$

The skew-symmetric character R, defined on N by

$$R(x,y) = H(x,y)\overline{H(y,x)}$$

is such that

$$\Pi_x \Pi_y = R(x,y)\Pi_y \Pi_x.$$

PROPOSITION 4.2 [21]. – *Let G be any discrete abelian group, α a cocycle on G, $\widehat{G}$ the dual group of G. Let $\widetilde{G} = G \times \widehat{G}$. G is embedded in $\widetilde{G}$ by $\psi : G \to \widetilde{G}$,*

$\psi(x) = (x, \phi(x))$, where $\phi : G \to \widehat{G}$ is defined by $\langle x, \phi(y)\rangle = a(x, y)$. On $\widetilde{G}$ we have the canonical bicharacter β defined by

$$\beta((m, s), (n, t)) = \langle m, t\rangle,$$

where $\langle , \rangle$ denotes the duality between M and $\widehat{M}$. Then

$$\beta(\psi(x), \psi(y)) = \langle x, \phi(y)\rangle = a(x, y),$$

so a is the restriction to G of the canonical Heisenberg cocycle on $\widetilde{G}$.

5. – The almost Schrödinger operator: bounds for its norm.

DEFINITION 5.1. – *Let $G \cong \mathbb{Z}^n \oplus \mathbb{Z}_{r_1} \oplus \ldots \oplus \mathbb{Z}_{r_k}$, and denote by $\{e_j\}_{j=1,\ldots,n}$ the canonical basis of $\mathbb{Z}^n$, and by b_j the canonical basis of $\mathbb{Z}_{r_j}$, $j = 1, \ldots, k$. Then we will call the operator*

$$H(a_1, \ldots, a_n; \beta_1, \ldots, \beta_k) = \sum_{j=1}^{n} a_j D_{e_j} + \sum_{j=1}^{k} \beta_j D_{b_j}, \quad a_j, \beta_j \in \mathbb{R}$$

the almost Schrödinger operator associated to the Discretized Canonical Commutation Relations associated to F.

Considering the case $G = \mathbb{Z}^3$, $F = S$, set $h = H(1, 1, 1; 0)$; we have

$$h = H(1, 1, 1; 0) = U_1 + U_1^* + U_2 + U_2^* + U_3 + U_3^* = [e_1] + [e_2] + [e_3].$$

We are interested in determining bounds for the norm of h. Our aim will be the generalization to dimension three of the formulas in [8]. A long, but straightforward, computation using the discretized canonical commutation relations shows that,

$$h^2 = 6I + \sum_{i=1}^{3} [2e_i] + \sum_{i<j} (\rho_{i,j} + \rho_{i,j}^{-1})\{[e_i + e_j] + [e_i - e_j]\}$$

and,

$$\begin{aligned} h^3 = 9h &+ \sum_{i=1}^{3} [3e_i] \\ &+ \sum_{i<j} (\rho_{i,j}^{-2} + 1 + \rho_{i,j}^{2})\{[2e_i + e_j] + [2e_i - e_j] + [e_i + 2e_j] + [e_i - 2e_j] + [e_i] + [e_j]\} \\ &+ \{P(1,2,3) - (\rho_{1,2}\rho_{1,3}^{-1}\rho_{2,3} + \rho_{1,2}^{-1}\rho_{1,3}\rho_{2,3}^{-1})\}[e_1 + e_2 + e_3] \\ &+ \{P(1,2,3) - (\rho_{1,2}\rho_{1,3}\rho_{2,3}^{-1} + \rho_{1,2}^{-1}\rho_{1,3}^{-1}\rho_{2,3})\}[e_1 + e_2 - e_3] \\ &+ \{P(1,2,3) - (\rho_{1,2}\rho_{1,3}^{-1}\rho_{2,3}^{-1} + \rho_{1,2}^{-1}\rho_{1,3}\rho_{2,3})\}[e_1 - e_2 - e_3] \\ &+ \{P(1,2,3) - (\rho_{1,2}\rho_{1,3}\rho_{2,3} + \rho_{1,2}^{-1}\rho_{1,3}^{-1}\rho_{2,3}^{-1})\}[e_1 - e_2 + e_3] \end{aligned}$$

where $P(1, 2, 3) = (\rho_{1,2} + \rho_{1,2}^{-1})(\rho_{1,3} + \rho_{1,3}^{-1})(\rho_{2,3} + \rho_{2,3}^{-1})$ and $\rho_{i,j} = e^{\pi i \theta_{i,j}}$. Hence, THEOREM 5.2. – *In $\mathcal{A}_\Theta$, for $\Theta = \theta\eth_3$, where $\eth_3$ is the skew-symmetric* (3×3) *matrix with 1's in the upper diagonals, and $\theta \in \mathbb{R}$, $\rho = e^{\pi i \theta}$, we have,*

$$\|h^2\| \leq 12 + 12|\rho + \rho^{-1}|,$$
$$\|h^3 - 9h\| \leq 6 + 36|\rho^{-2} + 1 + \rho^2| + 6\{|2(\rho + \rho^{-1}) + (\rho^3 + \rho^{-3})| + |\rho + \rho^{-1}|\}.$$

COROLLARY 5.3. – *If $a \in \mathbb{R}$ is such that,*

$$|a| > 2\sqrt{3}\sqrt{1 + |\rho + \rho^{-1}|} \text{ or}$$
$$|a^3 - 9a| > 6 + 36|\rho^{-2} + 1 + \rho^2| + 6\{|2(\rho + \rho^{-1}) + (\rho^3 + \rho^{-3})| + |\rho + \rho^{-1}|\},$$

then a is not in the spectrum $\sigma(h)$ of h in $\mathcal{A}_\Theta$, for $\Theta = \theta\eth_3$.

COROLLARY 5.4. – *The norm of h in $\mathcal{A}_\Theta$, $\Theta = \theta\eth_3$, is always less than or equal to $2\sqrt{3}\sqrt{1 + |\rho + \rho^{-1}|}$. If $\rho = e^{\pi i/3}$ then the norm of h is less than or equal to $3^{1/3} + 3^{2/3}$.*

REMARK 5.5. – By continuity arguments it is clear that the estimates above are also going to hold for triples $(\rho_{1,2}, \rho_{1,3}, \rho_{2,3})$ "close" to (ρ, ρ, ρ).

REMARK 5.6. – For $n = 2$ we can bound the size of $\|h\|$ (at least near $\rho = e^{\pi i/3}$) by using the estimate [8]

$$\|h^3 - 6h\| \leq 4 + 8|\rho^{-2} + 1 + \rho^2|.$$

In general, for $G = \mathbb{Z}^n$, $F = S$ and $h = H(1, \ldots, 1; 0)$, using similar computations we obtain

$$h^2 = 2nI + \sum_{i=1}^{n} [2e_i] + \sum_{i<j} (\rho_{i,j} + \rho_{i,j}^{-1})\{[e_i + e_j] + [e_i - e_j]\}$$

and,

$$\begin{aligned} h^3 &= 3nh + \sum_{i=1}^{n} [3e_i] \\ &+ \sum_{i<j} (\rho_{i,j}^{-2} + 1 + \rho_{i,j}^2)\{[2e_i + e_j] + [2e_i - e_j] + [e_i + 2e_j] + [e_i - 2e_j] + [e_i] + [e_j]\} \\ &+ \sum_{i<j<k} \Big[\{P(i,j,k) - (\rho_{i,j}\rho_{i,k}^{-1}\rho_{j,k} + \rho_{i,j}^{-1}\rho_{i,k}\rho_{j,k}^{-1})\}[e_i + e_j + e_k]. \\ &+ \{P(i,j,k) - (\rho_{i,j}\rho_{i,k}\rho_{j,k}^{-1} + \rho_{i,j}^{-1}\rho_{i,k}^{-1}\rho_{j,k})\}[e_i + e_j - e_k] \\ &+ \{P(i,j,k) - (\rho_{i,j}\rho_{i,k}^{-1}\rho_{j,k}^{-1} + \rho_{i,j}^{-1}\rho_{i,k}\rho_{j,k})\}[e_i - e_j - e_k] \\ &+ \{P(i,j,k) - (\rho_{i,j}\rho_{i,k}\rho_{j,k} + \rho_{i,j}^{-1}\rho_{i,k}^{-1}\rho_{j,k}^{-1})\}[e_i - e_j + e_k]\Big] \end{aligned}$$

where $P(i,j,k) = (\rho_{i,j} + \rho_{i,j}^{-1})(\rho_{i,k} + \rho_{i,k}^{-1})(\rho_{j,k} + \rho_{j,k}^{-1})$. Hence, THEOREM 5.7. – *In $\mathcal{A}_\Theta$, for $\Theta = \theta\eth_n$, where $\eth_n$ is the skew-symmetric $(n \times n)$ matrix with 1's in the upper diagonals, and $\theta \in \mathbb{R}$, $\rho = e^{\pi i\theta}$, we have,*

$$\begin{aligned} \|h^2\| &\le 4n + 2n(n-1)|\rho + \rho^{-1}|, \\ \|h^3 - (3n)h\| &\le 2n + 6n(n-1)|\rho^{-2} + 1 + \rho^2| \\ &\quad + n(n-1)(n-2)\big\{|2(\rho+\rho^{-1}) + (\rho^3 + \rho^{-3})| + |\rho + \rho^{-1}|\big\}. \end{aligned}$$

COROLLARY 5.8. – *If $a \in \mathbb{R}$ is such that*

$$\begin{aligned} |a| &> \sqrt{4n + 2n(n-1)|\rho + \rho^{-1}|} \text{ or} \\ |a^3 - 3na| &> 2n + 6n(n-1)|\rho^{-2} + 1 + \rho^2| \\ &\quad + n(n-1)(n-2)\big\{|2(\rho+\rho^{-1}) + (\rho^3 + \rho^{-3})| + |\rho + \rho^{-1}|\big\}, \end{aligned}$$

then a is not in the spectrum $\sigma(h)$ of h in $\mathcal{A}_\Theta$, for $\Theta = \theta\eth_n$.

COROLLARY 5.9. – *The norm of h in $\mathcal{A}_\Theta$, $\Theta = \theta\eth_n$, is always less than or equal to $\sqrt{4n + 2n(n-1)|\rho + \rho^{-1}|}$. If $\rho = e^{\pi i/3}$ then the norm of h is less than or equal to the positive solution of $h^3 - 3nh = 2n + n(n-1)(n-2)$*

REMARK 5.10. – By continuity arguments it is clear that the above estimates hold for Θ's "close" to $\Theta = \theta\eth_n$.

6. – The almost Schrödinger operator: eigenvalue equations.

In this section we generalize to higher dimensions some of the computations of Riedel [17], [18] on the spectrum of the almost Schrödinger operator $H(1,1,1;0)$ $(G = \mathbb{Z}^3)$.

DEFINITION 6.1 [17]. – *A state ϕ on $\mathcal{A}_\Theta$ is an eigenstate for the almost Schrödinger operator $h = H(a_1, \ldots, a_n; 0)$ if*

$$\phi(ha) = \chi\phi(a) \quad \forall a \in \mathcal{A}_\Theta.$$

For $n = 3$ the above eigenvalue equation, on the linear span of the elements $a = [t], t \in \mathbb{Z}^3$ (See Example 2.2) is equivalent to the system of equations $S(x)$ given below. (Where $S_t = \phi([t])$, $\omega(e_j, t) = e^{i\gamma_j}$ with $e_j, j = 1,2,3$, the canonical base of $\mathbb{Z}^3$.)

$$S(x) : \begin{cases} a_1 \cos(\gamma_1)[S_{x+e_1} + S_{x-e_1}] + a_2 \cos(\gamma_2)[S_{x+e_2} + S_{x-e_2}] + \\ \qquad a_3 \cos(\gamma_3)[S_{x+e_3} + S_{x-e_3}] = \chi S_x \\ \qquad S_x = S_{-x} \\ a_1 \sin(\gamma_1)[S_{x+e_1} - S_{x-e_1}] + a_2 \sin(\gamma_2)[S_{x+e_2} - S_{x-e_2}] + \\ \qquad a_3 \sin(\gamma_3)[S_{x+e_3} - S_{x-e_3}] = 0. \end{cases}$$

A system analogous to the one given above (call it $T(x)$) represents the solutions of the eigenvalue equation belonging to the linear span of the elements $\{t\}$, $t \in \mathbb{Z}^3$ (see Example 2.2 also). The two systems of equations $S(x)$ and $T(x)$ can be rewritten in the more compact form given below (Set $Z_t = \phi([t]) + \phi(\{t\})$, $A_j = a_j \cos(\gamma_j)$, $B_j = a_j \sin(\gamma_j)$, $j = 1, 2, 3$.)

$$Z(x) : \begin{cases} \sum_{i=1}^{3} A_j(Z_{x+e_j} + Z_{x-e_j}) = \chi Z_x \\ \sum_{i=1}^{3} B_j(Z_{x+e_j} - Z_{x-e_j}) = 0 \end{cases}$$

$Z(x)$ is a linear system of two equations in seven unknowns. To construct solutions of $Z(x)$ we will consider the following three systems below.

$$W(1) : \begin{cases} Z(x) \\ Z(x+e_1) \end{cases}, \quad W(2) : \begin{cases} Z(x) \\ Z(x+e_2) \end{cases}, \quad W(3) : \begin{cases} Z(x) \\ Z(x+e_3) \end{cases}.$$

The system $W(k)$, $k = 1, 2, 3$ is a system of four equations in the unknowns Z_x, $Z_{x\pm e_j}$, $Z_{x+e_k\pm e_j}$, $j = 1, 2, 3$. By eliminating Z_{x-e_k} and Z_{x+2e_k}, we obtain the following system (call it $W_1(k)$).

$$2A_kB_kZ_{x+e_k} + \sum_{j\neq k}(A_jB_k + A_kB_j)Z_{x+e_j} + \sum_{j\neq k}(A_jB_k - A_kB_j)Z_{x-e_j} = \chi B_k Z_x$$
$$2A_kB_kZ_x + \sum_{j\neq k}(A_jB_k - A_kB_j)Z_{x+e_k+e_j} + \sum_{j\neq k}(A_jB_k + A_kB_j)Z_{x+e_k-e_j} = \chi B_k Z_{x+e_k}$$

Note that $W_1(k)$ is a system of two equations in the unknowns Z_x, Z_{x+e_k}, Z_{x+e_j}, Z_{x-e_j}, $Z_{x+e_k+e_j}$, $Z_{x+e_k-e_j}$, $j = 1, 2, 3$, $j \neq k$. If we represent Z_t, $t \in \mathbb{Z}^3$, using coordinates in three space, the points representing the ten unknowns form two rectangles intersecting along a segment parallel to the k-axis.

THEOREM 6.2. – *Given 14 initial points, we can fill out the space by using equations $W_1(k)$, $k = 1, 2, 3$. However, there are only 7 degrees of freedom, as we see from evaluating $S(x)$ for $x = 0$.*

Computations analogous to the ones performed above also show:

THEOREM 6.3. – *Given a finite set of initial points, we can fill out the space by using equations $W_1(k)$, $k = 1, \ldots, n$. However, there are only $2n + 1$ degrees of freedom, as we see from evaluating $S(x)$ for $x = 0$.*

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