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Marina Marchisio

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A $54-(114-)$ Dimensional Family of Smooth Unirational Quartic $3-(4-)$ Folds.

Marina Marchisio

Sunto. – *Costruiamo una famiglia di dimensione* 54 (114) *di ipersuperficie quartiche lisce unirazionali di dimensione* 3 (4).

Summary. – *We build a* $54-(114-)$ *dimensional family of smooth unirational quartic* $3-(4-)$ *folds.*

The question of deciding whether the generic quartic hypersurface of dimension 3 or 4 (quartic 3–fold or 4–fold) is unirational or not is one of the most important open rationality problem. In [18] we outlined the construction of a 54–(114–) dimensional family of smooth unirational quartic 3– (4–) folds. Since up to now these are the biggest known families of such quartic hypersurfaces, we thought it could be useful to give here the full proof of their existence.

1. – Main known results on the unirationality of the quartic hypersurfaces X_4 in $\mathbb{P}^n$.

Let X_4 be a smooth quartic hypersurface defined over a field K.

DEFINITION 1.1. – *i)* $X_4 \subseteq \mathbb{P}^n$ *is called unirational if there exists a rational generically surjective (i.e. dominant) map*

$$\varphi : \mathbb{P}^{n-1} - -- > X_4,$$

ii) $X_4 \subseteq \mathbb{P}^n$ *is called rational if there exists a birational map*

$$\varphi : \mathbb{P}^{n-1} - \overset{\simeq}{-} - > X_4.$$

If $n = 2$ X_4 is a smooth plane quartic of genus 3 hence it is not rational. By Lüroth Theorem $X_4 \subseteq \mathbb{P}^2$ is not unirational.

If $n = 3$ X_4 is a smooth quartic surface in $\mathbb{P}^3$. X_4 is not rational and if K is an algebrically closed field and $K(\mathbb{P}^2)/K(X_4)$ is a separable extension then X_4, by the rationality criterion of Castelnuovo, is not unirational. The two hypotesis on K are essential; in fact there exist several counterexamples, for istance the one given by Shioda in [37] of $X_4 \subseteq \mathbb{P}^3(K)$, where K is an algebraic closed field of characteristic $p \neq 0$ and $p \equiv 3 \pmod 4$, which is a K3 (hence not rational) unirational surface.

For $n \geq 4$ the two notions of unirationality and rationality don't coincide, moreover there are no criterions of unirationality or rationality. For these reasons the answer to the problem *"Is X_4 in $\mathbb{P}^n$ unirational?"* is not so obvious.

If $n \geq 7$ U. Morin in the 1936 in [22] proved the following

THEOREM 1.1. – *The generic quartic hypersurface defined over any field K in $\mathbb{P}^n$ with $n \geq 7$ is unirational.*

Always Morin in 1940 in [23] proved the following

THEOREM 1.2. – *Given a hypersurface $X_d \subseteq \mathbb{P}^n$, there exists a constant $c(d)$ such that the generic $X_d \subseteq \mathbb{P}^n$ is unirational if $n - 1 > c(d)$.*

J. P. Murre in 1979 in [25] discussed and wrote in modern language the proof of the previous result. In 1980 in [4] C. Ciliberto, for K algebraic closed field of characteristic zero, gave a new proof of the previous theorem finding as particular case the Theorem 1.1.

Always the Theorem 1.2 was extended to the complete intersecions by A. Predonzan in [28] in 1949 and this extension was discussed and generalized in modern language by L. Ramero in [33] and by K. Paranjape and V. Srinivas in [26].

For $n = 6$ U. Morin in 1952 proved in [24] the following

THEOREM 1.3. – *The generic quartic hypersurface in $\mathbb{P}^6$ defined over any field K is unirational.*

In 1998 A. Conte and J. P. Murre in [6] gave a new and much simpler proof of the unirationality of the quartic fivefold using in an essential way a theorem of B. Segre of the 1954 not available to Morin.

For $n = 4$ and $n = 5$ the problem of the unirationality of the generic quartic hypersurface is still open, i.e. it is still unknown if the generic X_4 of dimension 3 and 4 is unirational or not. This is considered one of the most interesting and difficult open problem in this kind of questions.

Nevertheless B. Segre in 1960 in [36] gave an example of a particular smooth quartic hypersurface which is unirational. It has equation

$$x_0^4 + x_0x_4^3 + x_1^4 - 6x_1^2x_2^2 + x_2^4 + x_3^4 + x_3^3x_4 = 0,$$

where $(x_0 : x_1 : x_2 : x_3 : x_4)$ are the homogeneous coordinates in $\mathbb{P}^4$.

In 1971 V. A. Iskovskikh and Yu. I. Manin in [15] proved the following

THEOREM 1.4. – *The quartic hypersurface* X_4 *in* $\mathbb{P}^4$ *is not rational.*

They showed that the group of the birational automorphisms, which is a birational invariant, of the X_4 is finite. Since Bir($\mathbb{P}^3$) is the Cremona group and it is infinite it follows the non-rationality of the quartic threefold.

By the previous example given by B. Segre V. A. Iskovskikh and Yu. I. Manin gave a negative answer in dimension three to the Lüroth problem formulated by Lüroth in 1861 and which asks *"Is an unirational variety necessarly rational?"*.

In 1972 other counterexamples to the Lüroth problem in dimension three were given by H. Clemens and Ph. A. Griffiths in [5] proving the non-rationality of the cubic hypersurface in $\mathbb{P}^4$ and by M. Artin and D. Mumford in [2] building unirational varieties with torsion in $H_3(\mathbb{Z})$ different from zero hence not rational.

In 1996 J. Harris, B. Mazur and R. Pandharipande in [13] proved the following

THEOREM 1.5. – *Every hypersurface* X_d *of degree* d *in* $\mathbb{P}^n$ *is unirational if the codimension of the singular locus Sing* X_d *is sufficiently big with respect to* d *and* n.

Recently some progresses were made. First of all in 1997 A. V. Pukhlikov, in [32], extending the tecniques of proof of Iskovskikh and Manin, proved that the generic Fano hypersurface X_M in $\mathbb{P}^M$ with $M \geq 4$ is not rational.

In 1998 in [14] J. Harris and Yu. Tschinkel studied the rational points over the quartics and in particular proved the following

THEOREM 1.6. – *Let* $X_4 \subseteq \mathbb{P}^n$ *be a quartic smooth hypersurface defined over* K. *If* $n \geq 4$, *then for any finite extension* K' *of* K *the set* $X_4(K')$ *of the* K'*-rational points of* X_4 *is dense in the Zariski topology.*

2. – Examples of smooth quartic unirational hypersurfaces in $\mathbb{P}^4$ and $\mathbb{P}^5$.

To find examples of smooth quartic hypersurfaces which are unirational we extend the tecniques of Conte-Murre used in [6] to prove the unirationality of the quartic fivefold.

2.1 – *Existence of a rational surface*

In [6] Conte and Murre proved that the generic $X_4 \subseteq \mathbb{P}^n$ with $n \geq 5$, contains a rational surface. It is possible to prove that also the generic quartic of dimension 3 in $\mathbb{P}^4$ contains a rational surface, more precisely we prove the following

PROPOSITION 2.1. – *Every* $X_4 \subseteq \mathbb{P}^4(K)$ *contains a rational surafce* S^0 *and moreover if* $P^* \in X_4$ *is a fixed point of* X_4 *we can take* S^0 *going through it.*

See [17] for the proof.

2.2 – *Construction of the quadric bundle and unirationality of the* X_4 *in* $\mathbb{P}^4$ *and* $\mathbb{P}^5$

Consider $X_4 \subseteq \mathbb{P}^{m+1}$, with $m \geq 3$ and S^0 the rational surface contained in X_4. Fix $R \in S^0$ and H^0 a hyperplane in $\mathbb{P}^{m+1}$ and take the tangent cone $C_R(X_4)$ to X_4 in R. Let $Q_R = C_R(X_4) \cap H^0$ be the quadric hypersurface of dimension $m-2$ obtained intersecting $C_R(X_4)$ with H^0. Consider the quadric bundle

$$\pi : X^+ \longrightarrow S^0$$

with $X^+ = \{(R, P')/R \in S^0, P' \in Q_R\}$ and $\pi((R, P')) = R$. X^+ is an irreducible variety defined over K_0 of dimension m. If $m = 4$ Q_R is a quadric in $\mathbb{P}^3$ while if $m = 3$ Q_R is a conic in $\mathbb{P}^2$ and hence $X^+ \longrightarrow S^0$ is a conic bundle.

If $m \geq 5$ the existence of the rational surface S^0 in X_4 is sufficient to prove, applying in an essential way the Segre's theorem, that the previous quadric bundle has a rational section and hence to prove the unirationality of the $X_4 \subseteq \mathbb{P}^{m+1}$.

If $m = 3, 4$ the existence of the rational surface S^0 is not, alone, sufficient to conclude that the quadric bundle admits a rational section because it is not possible to apply the Segre's theorem. We note that if all conic bundles over a rational surface were unirational then the previous construction would imply automatically the unirationality of the quartic $X_4 \subseteq \mathbb{P}^4$. Unfortunately this problem, also if it is not known if there exist examples of conic bundles which are not unirational, is very far to be solved and it is, togheter the one of the unirationality or not of the quartic hypersurface in $\mathbb{P}^4$ and $\mathbb{P}^5$, the most important open problem in these rationality questions. It seems that the answer to this problem is negative and a possible counterexample should be given by the hypersurface of degree n, X_n, in $\mathbb{P}^n$ containing a line of molteplicity $n-2$ for $n \geq 5$.

Nevertheless there are special rational surfaces S^0 such that the previous quadric bundle admits a rational section. We will study one of these special cases, in particular when S^0 is a surfaces with separable asymptotics.

Let F_n be a surface in $\mathbb{P}^3(K)$, irreducible, of order $n \geq 3$, which is not a developable ruled surface. Let x be a generic point of F_n and Π_x the tangent plane to F_n in x. The intersection $F_n \cap \Pi_x$ is a curve with a double point in x and the lines $l \subseteq \Pi_x$ such that $x \in l$ are the tangent lines to F_n in x with

$$\mathrm{molt}_x(l \cap F_n) \geq 2.$$

If Q_x is the polar quadric to F_n in x, the intersection $Q_x \cap \Pi_x$ is a conic C_x with a double point in x. Then in $K(x)$ or in its quadratic extension

$$C_x = l'_x + l''_x$$

where l'_x and l''_x are two tangent lines to F_n in x, more precisely the two aymptotics lines. Let S be the congruence, defined over K, generated by the asymptotic lines to F_n moving x, contained in the Grassmannian $\mathbb{G}(1,3)$ of the lines in $\mathbb{P}^3$.

DEFINITION 2.1. – *F_n has separable asymptotics if $S = S' + S''$, that is if S consists of two irreducible components over an algebraic extension of K.*

We have that

$$\mathrm{molt}_x(l'^{,''}_x \cap F_n) \geq 3$$

that is $l'^{,''}_x \cap F_n = 3x + p_1 + ... + p_a$.

REMARK 2.1. – *a*) Every (non developable) ruled F_n has separable asymptotics. Moreover $\dim S' = 1, \dim S'' = 2$ and $S' = \{$generatrices of the ruled surface$\}$. If F_n is a cone or a developable surface $S' = S''$.

b) If $n = 2$, F_2 is a quadric and hence it has always separable asymptotics and $\dim S' = \dim S'' = 1$.

c) If F_n has separable asymptotics then every surface in $\mathbb{P}^3$, obtained starting from F_n by a nondegenerate projectivity of $\mathbb{P}^3$, has separable asymptotics.

It is possible to prove the following

THEOREM 2.1. – *Let $S^0 \subseteq X_4 \subseteq \mathbb{P}^4, \mathbb{P}^5$ be a surface with separable asymptotics and let*

$$\pi : X^+ \longrightarrow S^0$$

be the quadric bundle constructed above. Then the quadric bundle admits a rational section.

PROOF. – Let $R \in S^0$ be a non singular point of S^0, then the tangent plane to S^0 in R intersects its quadric tangent cone $C_R(X_4)$ (i.e. the tangent cone in R to the quartic hypersurafce $X_4 \cap T_R(X_4)$ where $T_R(X_4)$ is the tangent space to X_4 in R) in two generatrices which correspond to the two points in X^+ ((R, P') and (R, P'')). Let S be the closure of the geometric locus of these couples of points moving R on S^0. S is the congruence generated by the asymptotic lines moving R on S^0 and hence, by hypotesis, consists of two components S' and S'', over an algebraic extension of K, such that each determines a rational section of π. ■

In the next section we will study the monoidal quartic surfaces with separable asymptotics.

3. – Monoidal quartic surfaces with separable asymptotics and families of smooth quartic unirational hypersurfaces in $\mathbb{P}^4$ and $\mathbb{P}^5$.

3.1 – *Monoidal quartic surfaces with separable asymptotics*

We consider the monoidal quartic surfaces F_4 that is the algebraic surfaces which have a triple point as unique singularity. If we suppose that the triple point T has homogeneous coordinates $T = (0 : 0 : 0 : 1)$, the equation of F_4 is

$$x_3 \cdot \beta(x_0, x_1, x_2) - \alpha(x_0, x_1, x_2) = 0$$

where $\alpha(x_0, x_1, x_2)$ and $\beta(x_0, x_1, x_2)$ are homogeneous polynomials, without common factors, of degree four and three respectively.

Among all monoidal quartic surfaces $F_4 \subseteq \mathbb{P}^3$ it is possible to characterize the ones with separable asymptotics, to which we are interested for the reasons explained before, in several ways for istance by the following

PROPOSITION 3.1. – *Let F_4 be as above then F_4 has separable asymptotics if and only if for the generic point x of F_4 on K*

$$h(x) = l(x)^2$$

where $h(x)$ is the equation of the Hessian surface of F_4 and $l(x)$ is an element of $K_1(x)$ with K_1 quadratic extension of K (eventually equal to K).

See [29] or [36] for the proof.

We note that the example of the particular quartic smooth hypersurface which is unirational given by B. Segre in [36] in 1960 contains the monoidal quartic surface with separable asymptotics of equation

$$x_1^4 - 6x_1^2x_2^2 + x_2^4 + x_3^4 + x_3^3x_4 = 0.$$

A. Predonzan in [30] gave a complete projective classification of all monoidal quartic surfaces with separable asymptotics by the following

THEOREM 3.1. – *A monoidal surface F_4 of order 4, not ruled, in a projective space $\mathbb{P}^3(K)$, with K algebraic closed field of characteristic zero, has separable asymptotics if and only if is one of the following different projective types of canonical equation*

$$I) \quad X_0X_1X_2X_3 + X_0^4 + X_1^4 + X_2^4 - 2(X_0^2X_1^2 + X_1^2X_2^2 + X_2^2X_0^2) = 0$$

$$II) \quad X_0X_1X_2X_3 + (X_1^2 - X_2^2)^2 = 0$$

$$III) \quad X_1^2X_2X_3 + X_0^2X_1^2 + X_2^4 = 0$$

$$IV) \quad X_1^2X_2X_3 + X_0^4 = 0$$

$$V) \quad X_0^3X_3 + X_1X_2(X_1^2 - X_2^2) = 0$$

$$VI) \quad X_0^3X_3 + X_1^2X_2^2 = 0.$$

In the proof of the theorem, A. Predonzan, didn't make explicitly most of the very complicated and painful computations which led him to the result stated. We checked them by using the modern symbolic computation systems Maple [16] and CoCoA [3].

3.2 – *Tethraedral surfaces and families of smooth quartic unirational hypersurfaces in $\mathbb{P}^4$ and $\mathbb{P}^5$*

The most general monoidal quartic surface S in $\mathbb{P}^3$ with separable asymptotics is the one of equation

$$X_0X_1X_2X_3 + X_0^4 + X_1^4 + X_2^4 - 2(X_0^2X_1^2 + X_1^2X_2^2 + X_2^2X_0^2) = 0$$

and it is called tethraedral surface.

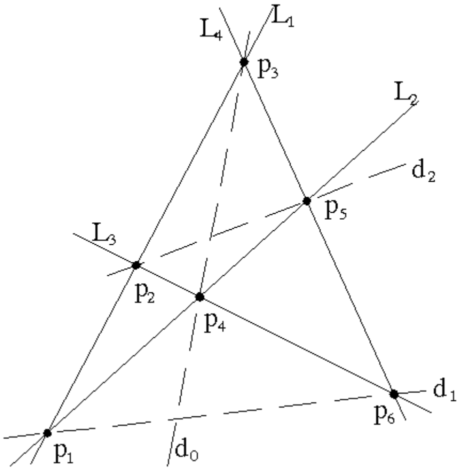
We determined in a rigorous way the equation of this surface by the modern tools of symbolic computation Maple and CoCoA in the following way. The equation of the generic monoidal quartic surface, as we saw before, is

$$x_3 \cdot \beta(x_0, x_1, x_2) - \alpha(x_0, x_1, x_2) = 0.$$

$\beta(x_0, x_1, x_2) = 0$ is the equation of the cubic tangent cone to S in the triple point. It is possible to prove that if this surface has separable asymptotics it consists of irreducible linear components i.e. it breaks into planes. The most general case is the one in which the cubic tangent cone consists of three planes not going through a same line. Hence we can suppose that the equation of the tangent cone to the tethraedral surface S in T is

$$x_0x_1x_2 = 0.$$

Moreover, by other characterizations of the surfaces with separable asymptotics, it is possible to prove, see [30], that S has 6 double points p_i which are the vertices of a complete plane quadrilateral and the diagonal trilateral of which is the intersection of the plane containing the quadrilateral with the cubic tangent cone. If we take $x_3 = 0$ as the equation of the plane containing the quadrilateral we can consider the following picture



d_0, d_1, d_2 are the lines of equations $x_0 = 0, x_1 = 0, x_2 = 0$ and p_i are the 6 double points of coordinates, supposed $a_0, a_1, a_2 \neq 0$,

$$p_1\left(\frac{\sqrt{a_2}}{\sqrt{a_0}} : 0 : 1\right), \quad p_2\left(-\frac{\sqrt{a_1}}{\sqrt{a_0}} : 1 : 0\right), \quad p_3\left(0 : \frac{\sqrt{a_2}}{\sqrt{a_1}} : 1\right),$$

$$p_4\left(0 : -\frac{\sqrt{a_2}}{\sqrt{a_1}} : 1\right), \quad p_5\left(\frac{\sqrt{a_1}}{\sqrt{a_0}} : 1 : 0\right), \quad p_6\left(-\frac{\sqrt{a_2}}{\sqrt{a_0}} : 0 : 1\right).$$

Hence we obtain for L_1, L_2, L_3, L_4 respectively the equations:

$$\begin{aligned}
\sqrt{a_0}x_0 + \sqrt{a_1}x_1 - \sqrt{a_2}x_2 &= 0\\
\sqrt{a_0}x_0 - \sqrt{a_1}x_1 - \sqrt{a_2}x_2 &= 0\\
\sqrt{a_0}x_0 + \sqrt{a_1}x_1 + \sqrt{a_2}x_2 &= 0\\
\sqrt{a_0}x_0 - \sqrt{a_1}x_1 + \sqrt{a_2}x_2 &= 0,
\end{aligned}$$

and we can put

$$a(x_0, x_1, x_2) = L_1 \cdot L_2 \cdot L_3 \cdot L_4$$

i.e.

$$a(x_0, x_1, x_2) = a_0^2x_0^4 + a_1^2x_1^4 + a_2^2x_2^4 - 2(a_0a_1x_0^2x_1^2 + a_1a_2x_1^2x_2^2 + a_0a_2x_0^2x_2^2).$$

The equation of S becomes

$$x_0x_1x_2x_3 + a_0^2x_0^4 + a_1^2x_1^4 + a_2^2x_2^4 - 2(a_0a_1x_0^2x_1^2 + a_1a_2x_1^2x_2^2 + a_0a_2x_0^2x_2^2) = 0$$

and by the coordinate change

$$\begin{cases} X_0 &= \sqrt{a_0}x_0 \\ X_1 &= \sqrt{a_1}x_1 \\ X_2 &= \sqrt{a_2}x_2 \\ X_3 &= \dfrac{x_3}{\sqrt{a_0a_1a_2}} \end{cases}$$

it assumes the canonical form I).

In order to find families of quartic smooth hypersurfaces in $\mathbb{P}^4$ and $\mathbb{P}^5$ which are unirational we want to determine the dimension of the algebraic systems of all quartic hypersurfaces $X_4 \subseteq \mathbb{P}^r(K)$ which contain a tethraedral surface (and hence they are unirational over a finite extension K_1 of K as we saw before).

The dimension of the family of all tethraedral surfaces S in $\mathbb{P}^3$ is 15. Denoting by $\mathbb{P}^r = \mathbb{P}^r(K)$, we can put

$$\Phi = \{S \text{ tethraedral } / \exists\, \mathbb{P}^3 \subset \mathbb{P}^r, \text{ with } \mathbb{P}^3 \supseteq S\},$$

$$\Sigma = \{X = X_4 \subseteq \mathbb{P}^r / \exists\, S \in \Phi, \text{ with } S \subseteq X\},$$

and we consider the incidence corrispondence:

$$I = \{(S,X) \in \Phi \times \Sigma / S \subseteq X\}$$

with the relative projections:

$$\begin{tikzcd} I \arrow[r, "\sigma"] \arrow[d, "\varphi"'] & \Sigma \\ \Phi & \end{tikzcd}$$

We note that Φ is obviously irreducible over K and that:

$$\dim \Phi = \dim \mathbb{G}(3,r) + 15 = 4(r-3) + 15 = 4r + 3. \tag{1}$$

In order to compute the dimension of the generic fiber of Φ, we fix $S \in \Phi$ and we suppose that, in homogeneous coordinates $(x_0 : \ldots : x_r)$ of $\mathbb{P}^r$, S is contained in the tridimensional linear space $\{x_4 = \ldots = x_r = 0\}$ and that, in it, it has equation:

$$S: \;\; \varphi_4(x_0, \ldots, x_3) = 0. \tag{2}$$

Then, the generic $X \in \Sigma$ which contains S will have equation of the type:

$$\sum_{i=4}^{r} x_i f_i(x_0, \ldots, x_r) + \lambda\varphi_4(x_0, \ldots, x_3) = 0, \tag{3}$$

where the f_i are general homogeneous polynomials of degree 3 and $\lambda \in K^*$, so that for $\varphi^{-1}(S) = \{X \in \Sigma / X \supseteq S\}$ we have:

$$\dim \varphi^{-1}(S) = \binom{r+4}{4} - \binom{4+3}{3} = \binom{r+4}{4} - 35 \tag{4}$$

We remember that, if we fix $X_s^0 \subseteq \mathbb{P}^k$ and put

$$\Psi = \{X_n \subseteq \mathbb{P}^r / X_n \supseteq X_s^0\},$$

we have:

$$\dim \Psi = \binom{n+r}{r} - \binom{n+k}{k} + \binom{n+k-s}{k} - 1.$$

$\varphi^{-1}(S)$ is hence a linear system of costant dimension when $S \in \Phi$ varies, and hence I and Σ are algebraic systems irreducible over the base field K.

Moreover since $\Sigma \subseteq |\mathcal{O}_{\mathbb{P}^r}(4)|$, we can put:

$$\dim \Sigma = \binom{r+4}{4} - 1 - \varepsilon, \quad \varepsilon \geq 0. \tag{5}$$

We have hence, for $S \in \Phi$ and $X \in \Sigma$ generic:

$$\dim I = \dim \Phi + \dim \varphi^{-1}(S) = \dim \Sigma + \dim \sigma^{-1}(X), \tag{6}$$

i.e.:

$$\dim I = 4r + 3 + \binom{r+4}{4} - 35 = \binom{r+4}{4} - 1 - \varepsilon + \dim \sigma^{-1}(X) \tag{7}$$

from which it follows:

$$\dim \sigma^{-1}(X) = 4r - 31 + \varepsilon. \tag{8}$$

LEMMA 3.1. – *If the generic quartic hypersurface $X = X_4 \subseteq \mathbb{P}^r$ contains a tethraedral surface S, then $r \geq 8$.*

PROOF. – We note that the generic X contains a $S \Longleftrightarrow \Sigma = |\mathcal{O}_{\mathbb{P}^r}(4)| \Longleftrightarrow \varepsilon = 0$. In our hypotesis hence it will be:

$$\dim \sigma^{-1}(X) = 4r - 31 \geq 0 \Longrightarrow r \geq 8.$$

In order to prove that it is true also the viceversa of the Lemma 3.1, we suppose hence that $r \geq 8$ but, by contradiction, that $\varepsilon > 0$. We put, for semplicity:

$$\left\{\begin{array}{lcl} a & = & \dim \varPhi \\ b & = & \dim \varphi^{-1}(S) \\ c & = & \dim \varSigma \\ d & = & \dim \sigma^{-1}(X) \end{array}\right. \tag{9}$$

and we remember that we have:

$$a + b = c + d. \tag{10}$$

Now we fix a $S_0 \in \varPhi$ and we put:

$$\varSigma' = \{X \in \varSigma / S_0 \subseteq X\} \simeq \varphi^{-1}(S_0); \tag{11}$$

$$I' = \{(S, Y) \in I / Y \in \varSigma'\} = \sigma^{-1}(\varSigma'). \tag{12}$$

Then let $\overline{I}$ be an irreducible maximal component of I' and $\varPhi' = \varphi(\overline{I})$.

We obtain the following commutative diagramme

$$\begin{tikzcd}
 & \overline{I} \arrow[d] \arrow[rrd, "\overline{\sigma}"] \arrow[ldd, "\overline{\varphi}"'] & & \\
 & I \arrow[r, "\sigma"] \arrow[d, "\varphi"'] & \Sigma \arrow[r, phantom, "\supseteq"] & \Sigma' \\
\Phi' \arrow[r, phantom, "\subseteq"] & \Phi & &
\end{tikzcd}$$

where $\overline{\varphi} = \varphi|\overline{I}$ and $\overline{\sigma} = \sigma|\overline{I}$.

We note that we have

$$\begin{aligned} \dim \overline{I} &= \dim I' = \dim \varSigma' + \dim \overline{\sigma}^{-1}(X') = \\ &= \dim \varphi^{-1}(S_0) + \dim \sigma^{-1}(X') = b + d \end{aligned} \tag{13}$$

with $X' \in \varSigma'$ generic.

On the other hand, if we put:

$$\overline{a} = \dim \varPhi', \;\; \overline{b} = \dim \overline{\varphi}^{-1}(S'), \;\; (S' \in \varPhi' \text{ generic}), \tag{14}$$

it will be:

$$\dim \overline{I} = \overline{a} + \overline{b}, \tag{15}$$

and hence:

$$\overline{a} + \overline{b} = b + d. \tag{16}$$

We want to prove that (16) is impossible, by analyzing the various cases that can appear.

For this aim, we indicate by L_0 the $\mathbb{P}^3$ in which is contained S_0 and by L' the $\mathbb{P}^3$ in which is contained the generic $S' \in \Phi'$ and let $L = L_0 \cap L'$.

(I) $L = \emptyset$.

Since $\Phi' \subseteq \Phi$, we can put:

$$\overline{a} = a - \overline{\varepsilon}, \qquad \text{with } \overline{\varepsilon} \geq 0. \tag{17}$$

Moreover, for the hypotesis $L = \emptyset$, it will be:

$$\overline{b} = b - 35 + 1 = b - 34. \tag{18}$$

Substituting in (16) we obtain:

$$a + b - 34 - \overline{\varepsilon} = b + d, \tag{19}$$

i.e., substituting the values of a and of d given by (1) and (8):

$$4r + 3 - 34 - \overline{\varepsilon} = 4r - 31 + \varepsilon. \tag{20}$$

and hence:

$$\varepsilon + \overline{\varepsilon} = 0, \tag{21}$$

which is impossible, being for hypotesis $\varepsilon > 0$, $\overline{\varepsilon} \geq 0$.

(II) $L \neq \emptyset, \dim L = h \geq 0, L \not\subset S_0$.

In this case we shall have, since all the $S' \in \Phi'$ contained in the same L' go through $L \cap S_0$:

$$\overline{a} = (3-h)(r-3) + (h+1)(3-h) + 15 - \binom{4+h}{h} + 1 - \overline{\varepsilon}, \quad \text{with } \overline{\varepsilon} \geq 0 \tag{22}$$

$$\overline{b} = b - 35 + \binom{4+h}{h} \tag{23}$$

from which, substituting in the (16) we obtain:

$$\begin{aligned} &(3-h)(r-3) + (h+1)(3-h) + 15 - \binom{4+h}{h} + \\ &+ 1 - \overline{\varepsilon} + b - 35 + \binom{4+h}{h} = \\ &= b + 4r - 31 + \varepsilon \end{aligned} \tag{24}$$

and hence:

$$0 < \varepsilon + \bar{\varepsilon} = \tag{25}$$

$$= 3r - hr - 9 + 3h + 3h - h^2 + 3 - h + 15 + 1 - 35 - 4r + 31 =$$

$$= r(3 - h - 4) + 5h - h^2 + 6 = r(-1 - h) - (6 - h)(-1 - h)$$

$$\Longrightarrow r < 6 - h \text{ (remember that } h \geq 0!\text{)},$$

which is impossible being for hypotesis $r \geq 8, h \geq 0$.

(III) $L \neq \emptyset, \dim L = h \geq 0, L \subseteq S_0 \cap S'$.

This case can be studied in the same way as the previous one and one finds again the limitation (25).

(IV) $L \neq \emptyset, \dim L = h \geq 0, L \nsubseteq S', L \subseteq S_0$.

Here, by the same argumentations of the case (II), we have that $\bar{b}$ is still given by (23), while $\bar{a}$ will be given by:

$$\bar{a} = (3 - h)(r - 3) + \delta + 15 - \bar{\varepsilon}, \quad \bar{\varepsilon} \geq 0 \tag{26}$$

where δ is the dimension of the family of the linear spaces of dimension h which lye in S_0 and it must be $0 \leq \delta < (h + 1)(3 - h)$, being this the dimension of the family of all the L of dimension h of $\mathbb{P}^3$. Since, moreover, the unique values possible for h are $0, 1$, we shall have $\delta < 3$ or 4 respectively if $h = 0$ or 1. Substituting as usual in the (16) and doing the easy calculations we obtain:

$$(h + 1)(r - 3) - \binom{h + 4}{h} - \delta + 1 < 0. \tag{27}$$

If hence $h = 0$ the (27) becomes:

$$r - 3 - \delta < 0 \Longrightarrow r < \delta + 3 = 6 \tag{28}$$

against the hypotesis $r \geq 8$.

If instead $h = 1$, the (27) becomes:

$$2r - 6 - 5 - \delta + 1 \Longrightarrow r < \frac{\delta + 10}{2} = 7, \tag{29}$$

still against the hypotesis $r \geq 8$.

Hence we can conclude:

THEOREM 3.2. – $\Sigma = |\mathcal{O}_{\mathbb{P}^r}(4)| \Longleftrightarrow r \geq 8$.

If hence $r \leq 7, \Sigma \underset{\neq}{\subset} |\mathcal{O}_{\mathbb{P}^r}(4)|$ and, since the generic $X \in |\mathcal{O}_{\mathbb{P}^r}(4)|$ doesn't contain any $S \in \Phi$ and the generic X which contains a $S \in \Phi$, contains no more then one of them, it will be $\dim \sigma^{-1}(X) = 0$, i.e. for the (8):

$$\varepsilon = 31 - 4r \tag{30)}$$

and hence, substituting in the (5):

$$\dim \Sigma = \binom{r+4}{4} + 4r - 32. \tag{31}$$

Hence we can conclude with the following:

THEOREM 3.3. – *Let Σ be as defined above. If $r \geq 8, \Sigma = |\mathcal{O}_{\mathbb{P}^r}(4)|$. If $r \leq 7$, Σ is an irreducible algebraic system over K whose dimension is given by* (31).

We know, by Theorem 2.1, that all $X_4 \in \Sigma$ are unirational over a simple finite extension $K_1 \supset K$. If $r \geq 6$, the prevoius theorem doesn't add anything new because the generic X_4 is, in this case, unirational.

But if $r = 4,\ 5$ the (31) says that $\dim \Sigma = 54,\ 114$ (while $\dim |\mathcal{O}_{\mathbb{P}^r}(4)| = 69{,}125$). In this way we constructed two families of quartic hypersurfaces in $\mathbb{P}^4$, $\mathbb{P}^5$ which are unirational of dimension respectively 54 and 114.

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Università degli Studi di Torino, Dipartimento di Matematica
Via Carlo Alberto, 10 - 10123 Torino, Italy
e-mail: marina.marchisio@unito.it

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