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On Some Properties of Explicit Toric Degenerations.

M. MARCHISIO - V. PERDUCA

Sunto. – *Nella presente nota si studiano delle degenerazioni semi-stabili di varietà toriche determinate da certe partizioni dei loro politopi associati. In un caso particolare vengono date le loro equazioni attraverso un'analisi combinatorica. I dettagli, le dimostrazioni e ulteriori esempi si trovano nel preprint* [7] *e verranno pubblicati altrove. In un successivo articolo* [4] *verrà discussa una interpretazione geometrica.*

Summary. – *We study semi-stable degenerations of toric varieties determined by certain partitions of their moment polytopes. We investigate in a particular case their defining equations via a combinatorial analysis. Details, proofs and further examples are contained in the preprint* [7] *and will be published elsewhere. In a sequel paper* [4] *we will discuss a geometric interpretation.*

1. – Background.

1.1 – *Polytopes and semi-stable partitions.*

In his paper [6], Hu provides a toric construction for semi-stable degenerations of toric varieties. We study the uniqueness of this construction for a toric variety X in the particular case of a semi-stable partition of its moment polytope in two subpolytopes. Adapting a theorem by Strumfels on toric ideals (Lemma 4.1 in [10] and Section 2 in [9]) to particular open polytopes, we investigate the equations of the degeneration of X as embedded variety.

Let $M \simeq \mathbb{Z}^n$ be a lattice and N its dual. We consider polytopes $\Delta \subset M$ which describe smooth algebraic varieties X_Δ; Δ determines the normal fan $\Sigma_{X_\Delta} \subset N$. Recall that convex polytopes Δ determine a toric manifold X_Δ together with an ample line bundle $\mathcal{L}_\Delta$: $(X_\Delta, \mathcal{L}_\Delta)$. If the polytope is non singular, then $\mathcal{L}_\Delta$ is very ample, we then have an embedding $X_\Delta \hookrightarrow \mathbb{P}^\ell$, for some ℓ [8].

Now fix a (compact) polytope Δ and suppose $\Delta \cap M = \{\boldsymbol{m}_0, \dots, \boldsymbol{m}_\ell\}$. Take $x_0, \dots, x_l$ as homogeneous coordinates in $\mathbb{P}^\ell$. We can define $X = X_\Delta$ as the closure in $\mathbb{P}^\ell$ of the image of the map

$$(1) \qquad \begin{aligned} \varphi : (\mathbb{C}^*)^n &\to \mathbb{P}^\ell \\ \boldsymbol{t} &\mapsto [\boldsymbol{t}^{\boldsymbol{m}_0}, \dots, \boldsymbol{t}^{\boldsymbol{m}_\ell}], \end{aligned}$$

where $\boldsymbol{t} = (t_1, \dots, t_n) \in (\mathbb{C}^*)^n$ and given $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathbb{Z}^n$ we use the notation $\boldsymbol{t}^{\boldsymbol{u}} = t_1^{u_1} \cdot \dots \cdot t_n^{u_n}$.

We assume that there exists a suitable finite partition Γ of Δ in subpolytopes $\{\Delta_j\}_{j=1}^k$. We will assume that the toric varieties X_{Δ_j} corresponding to each Δ_j are also smooth, moreover following [1, 6] we ask Γ to be *semi-stable*. In fact:

THEOREM 1.1. – [1, 6] *If $\{\Delta_j\}_{j=1}^k$ is a semi-stable partition of Δ, then there exists a semi-stable degeneration of X, $f : \tilde{X} \to \mathbb{C}$ with central fiber $f^{-1}(0) = \cup_{j=1}^k X_{\Delta_j}$; the central fiber is completely described by the polytope partition $\{\Delta_j\}_{j=1}^k$.*

$\tilde{X}$ is constructed by a *lift* of Δ (see below). Theorem 2.8 in [6] claims that $\tilde{X}$ is unique: we study the uniqueness of $\tilde{X}$ for semi-stable partitions of Δ in two subpolytopes Δ_1, Δ_2, and we describe its defining equations. In particular, in Section 2 of [6], Hu shows that the ordering (arbitrarily fixed) $\{\Delta_1, \dots, \Delta_k\}$ of the polytopes in Γ determines a piecewise affine function on the partition $F : \Delta \to \mathbb{R}$, which takes rational values on the points in the lattice M. F can be chosen to be concave and it is called *lifting function*. He therefore calls the open polytope

$$\tilde{\Delta}_F = \{(m, \tilde{m}) \in M \times \mathbb{Z} \text{ such that } m \in \Delta \text{ and } \tilde{m} \geq F(m)\}$$

an *open lifting* (here simply *lift*) of Δ with respect to Γ. There are many possible lifts of Δ with respect to Γ; if Γ consists of two subpolytopes, then two lifts exist. By construction there exists a morphism $f : \tilde{X}_F := X_{\tilde{\Delta}_F} \to \mathbb{C}$ which realizes a semi-stable degeneration of X. As before we have embeddings $X \hookrightarrow \mathbb{P}^\ell$ and $\tilde{X}_F \hookrightarrow \mathbb{P}^\ell \times \mathbb{C}$. In particular we can define $\tilde{X}_F$ as the closure in $\mathbb{P}^\ell \times \mathbb{C}$ of the image of the map:

$$\psi = \psi_F : (\mathbb{C}^*)^n \times \mathbb{C} \to \mathbb{P}^\ell \times \mathbb{C} \qquad (2)$$

$$(\boldsymbol{t}, \lambda) \mapsto ([\lambda^{F(\boldsymbol{m}_0)} \boldsymbol{t}^{\boldsymbol{m}_0}, \lambda^{F(\boldsymbol{m}_1)} \boldsymbol{t}^{\boldsymbol{m}_1}, \dots, \lambda^{F(\boldsymbol{m}_\ell)} \boldsymbol{t}^{\boldsymbol{m}_\ell}], \lambda).$$

Theorem 2.8 in [6] claims that the image of ψ_F, and hence $\tilde{X}_F$, is independent of the lifting function F.

We explicitly study this statement for semi-stable partitions of Δ in two subpolytopes Δ_1, Δ_2 in order to prove that the two non-compact toric varieties defined by the open polytopes $\tilde{\Delta}_F$ and $\tilde{\Delta}_G$ associated to the two possible lifting functions F, G, have the same toric ideals. To do this we adapt a Strumfels's theorem on toric ideals (Lemma 4.1 in [10] and Section 2 in [9]) to this non-compact context.

1.2 – *Toric ideals.*

In [9] Sottile describes the ideal I of the compact toric variety X (*toric ideal*) defined as the image of a map (1), following Strumfels's book [10].

Take $x_0, \dots, x_l$ as homogeneous coordinates in $\mathbb{P}^\ell$. With the notation of the previous section, suppose $\boldsymbol{m}_j = (m_{1j}, \dots, m_{nj})$, $j = 0, \dots, \ell$ and consider the $(n+1) \times (\ell+1)$ matrix

$$\mathcal{A}^+ = \begin{pmatrix} 1 & 1 & \dots & 1 \\ m_{10} & m_{11} & \dots & m_{1\ell} \\ \vdots & \vdots & & \vdots \\ m_{n0} & m_{n1} & \dots & m_{n\ell} \end{pmatrix}.$$

Observe that if $\boldsymbol{u} \in \mathbb{Z}^{\ell+1}$, then we may write $\boldsymbol{u}$ uniquely as $\boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^-$, where $\boldsymbol{u}^+, \boldsymbol{u}^- \in \mathbb{N}^{\ell+1}$, but $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ have no non-zero components in common. For instance, if $\boldsymbol{u} = (1, -2, 1, 0)$, then $\boldsymbol{u}^+ = (1, 0, 1, 0)$ and $\boldsymbol{u}^- = (0, 2, 0, 0)$ (Sottile's notation).

We therefore have:

THEOREM 1.2. – ([9], Corollary 2.3)

$$I = \langle \boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-} | \boldsymbol{u} \in \ker(\mathcal{A}^+) \textit{ and } \boldsymbol{u} \in \mathbb{Z}^{\ell+1} \rangle.$$

There are no simple formulas for a finite set of generators of a general toric ideal, but algorithms for this computation are implemented in the computer algebra system Macaulay 2 [5].

1.3 – *An example: semi-stable degeneration of the twisted cubic.*

To illustrate the previous section, we describe the semi-stable degeneration of the twisted cubic $X \subset \mathbb{P}^3$ determined by a subdivision of its moment polytope in two subpolytopes; the two varieties associated to the two possible lifts have the same defining equations.

The twisted cubic $X \subset \mathbb{P}^3$ can be defined as $\mathbb{P}^1$ embedded in $\mathbb{P}^3$ by cubics, that is, as the toric curve $(X_\Delta, \mathcal{L}_\Delta) = (\mathbb{P}^1, \mathcal{O}(3))$, where Δ is the polytope below

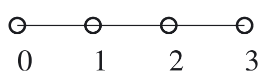

FIGURE 1. – The moment polytope Δ of the twisted cubic $X \subset \mathbb{P}^3$.

Here $M = \mathbb{Z}$, $\Delta \cap M = \{\boldsymbol{m}_j = j, j = 0, \dots, 3\}$, X is the closure of the image of

$$\varphi : \mathbb{C}^* \to \mathbb{P}^3$$
$$t \mapsto [1, t, t^2, t^3],$$

and we have

$$\mathcal{A}^+ = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

The toric ideal of X is

$$I = \langle x_0x_2 - x_1^2, x_1x_3 - x_2^2, x_0x_3 - x_1x_2 \rangle.$$

Now consider the semi-stable partition $\{\Delta_1, \Delta_2\}$ of Δ, where $\Delta_1 = [0, 1] \subset \mathbb{R}$ and $\Delta_2 = [1, 3] \subset \mathbb{R}$. This partition gives the semi-stable degeneration of X to the union of two curves $X_1 \cup X_2$, where $X_1 = (\mathbb{P}^1, \mathcal{O}(1))$ and $X_2 = (\mathbb{P}^1, \mathcal{O}(2))$.

The two possible lifting functions are

$$F(j) = \begin{cases} 1 & j = 0 \\ 0 & j \neq 0 \end{cases}, \quad G(j) = \begin{cases} 0 & j = 0, 1 \\ j - 1 & j = 2, 3 \end{cases}.$$

Figure 2 shows the two lifts Δ_F and Δ_G.

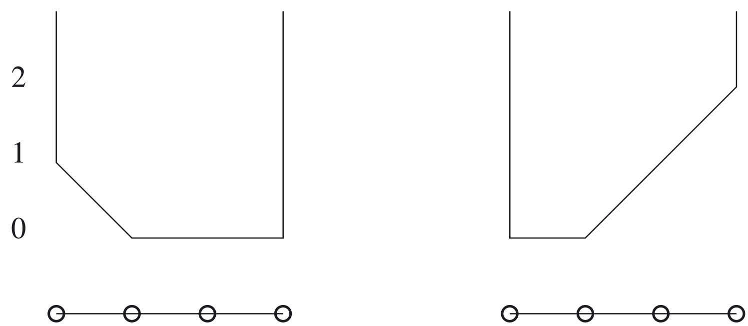

FIGURE 2. – Δ_F and Δ_G.

Using the notation in (2), in local coordinates the embeddings of $\tilde{X}_F$ and $\tilde{X}_G$ in $\mathbb{P}^3 \times \mathbb{C}$ are $([\lambda, t, t^2, t^3], \lambda)$ and $([1, t, \lambda t^2, \lambda^2 t^3], \lambda)$. We therefore observe that $\tilde{X}_F$ and $\tilde{X}_G$ have different parametric equations, nevertheless it is easy to see that both of them are defined in $\mathbb{P}^3 \times \mathbb{C}$ by the equations

$$x_0x_2 - \eta x_1^2 = 0, x_1x_3 - x_2^2 = 0, x_0x_3 - \eta x_1x_2 = 0,$$

where η is the non-homogeneous coordinate in $\mathbb{C}$ (one can do this computation by hand or he can use computer algebra systems which implement elimination theory algorithms).

2. – Main results.

The following theorem is a generalisation to our specific non-compact situation of Theorem 1.2. We use the notion of the previous sections.

Let I_F be the ideal of all polynomials in the coordinates $x_0, \dots, x_\ell, \eta$ homogeneous in $x_0, \dots, x_\ell$ and vanishing on $\tilde{X}_F$. In analogy with the compact case we use the notation

$$\boldsymbol{z}^{\boldsymbol{u}} = x_0^{u_0} \dots x_\ell^{u_\ell} \eta^{u_{\ell+1}},$$

with $\boldsymbol{u} = (u_0, \dots, u_\ell, u_{\ell+1}) \in \mathbb{Z}^{\ell+2}$.

Consider the $(n+2) \times (\ell+2)$ matrix

$$\mathcal{B}^+ = \mathcal{B}_F^+ = \begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ m_{10} & m_{11} & \dots & m_{1\ell} & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ m_{n0} & m_{n1} & \dots & m_{n\ell} & 0 \\ F(\boldsymbol{m}_0) & F(\boldsymbol{m}_1) & \dots & F(\boldsymbol{m}_\ell) & 1 \end{pmatrix}.$$

THEOREM 2.1. – $I_F = \langle \boldsymbol{z}^{\boldsymbol{u}^+} - \boldsymbol{z}^{\boldsymbol{u}^-} | \boldsymbol{u} \in \ker(\mathcal{B}^+)$ *and* $\boldsymbol{u} \in \mathbb{Z}^{\ell+2} \rangle$.

Now let G be the second lift, then we can consider the matrix $\mathcal{B}_G^+$ and characterize the toric ideal I_G of $\tilde{X}_G$ as above. In general $\tilde{X}_G$ will have a different parametrisation from the one of $\tilde{X}_F$, moreover the normal fans are different.

Our main result is

THEOREM 2.2. – *$\tilde{X}_F$ and $\tilde{X}_G$ have the same equations in $\mathbb{P}^\ell \times \mathbb{C}$, i.e. $I_F = I_G$.*

If X is the twisted cubic, we have

$$\mathcal{B}_F^+ = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \mathcal{B}_G^+ = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}.$$

Observe that

$$\mathcal{B}_F^+ = E \cdot \mathcal{B}_G^+,$$

where E is the 3×3 elementary matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \in SL_3(\mathbb{Z}),$$

and hence $\ker \mathcal{B}_F^+ = \ker \mathcal{B}_G^+$.

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