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Normal Forms and Long Time Existence for Semi-Linear Klein-Gordon Equations.

JEAN-MARC DELORT (*)

Sunto. – *Presentiamo in questo testo due risultati di esistenza di lungo periodo per soluzioni di equazioni non lineari di Klein-Gordon, ottenuti mediante metodi di forme normali. In particolare indichiamo come questi metodi permettono di ottenere soluzioni quasi globali per tale equazione sulle sfere, a dispetto del fatto che tali soluzioni non tendono a zero quando il tempo tende ad infinito.*

Summary. – *We present in this text two results of long time existence for solutions of nonlinear Klein-Gordon equations, obtained through normal forms methods. In particular, we indicate how these methods allow one to obtain almost global solutions for that equation on spheres, despite the fact that such solutions do not go to zero when time goes to infinity.*

0. – Introduction.

Our aim in this talk is to present results – some old, some new – concerning long time existence for a particular class of nonlinear evolution partial differential equations. We shall emphasize how normal forms methods can be useful to tackle such problems. Our goal is to describe these questions in the less technical possible way, to keep this text accessible to a general audience.

One of the basic questions in evolution problems for a nonlinear partial differential equation is the one of the largest time interval over which a solution with given initial data exists. Like in the similar problem for ordinary differential equations, two different types of problems arise. One of them concerns equations having a first integral which, in favorable cases, can be used to globalize a local solution. Such an approach allows one to handle Cauchy data of any size, but only quite special nonlinearities, coming most frequently from physics, give rise to

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conserved quantities providing first integrals. The second type of questions deals with long time existence problems for small Cauchy data, which can be studied for quite general nonlinearities. We shall discuss here the latter, for long range nonlinear perturbations of the Klein-Gordon equation. We refer to the book of Hörmander [13], and references therein, for discussions of these topics in the related case of nonlinear wave equations.

The first works concerning global solutions for nonlinear Klein-Gordon equations with small, smooth, quickly decaying Cauchy data, are due to Klainerman [14] and Shatah [19]. These authors treated cases in which the nonlinearity was playing the role of a "short range" perturbation of the linear part of the problem i.e. a perturbation that does not change too much the behaviour of solutions. More precisely, solutions of the linear Klein-Gordon equation on $\mathbb{R}^d$ with $C_0^\infty(\mathbb{R}^d)$ initial data decay uniformly like $t^{-d/2}$ when $t \to +\infty$. Short range nonlinear perturbations are by definition nonlinearities $V(u)u$ such that the nonlinear potential, computed on a linear solution, satisfies $\|V(u(t,\cdot))\|_{L^\infty} = O(t^{-\kappa})$ when $t \to +\infty$ with $\kappa > 1$. For instance, for a quadratic nonlinearity in d space dimension, $V(u) = u$, and the nonlinearity is short range when $\kappa = \dfrac{d}{2} > 1$.

The first case of a "long range" perturbation of the linear Klein-Gordon equation was studied by Ozawa, Tsutaya and Tsutsumi [18] for a quadratic nonlinearity in two space dimension. Using a normal forms method, due initially to Shatah [19], they were able to reduce that long range perturbation to a short range one. The first result we present here, proved in [6], concerns the case of a cubic nonlinearity in one space dimension. This is again a problem falling in the long range category, with the extra difficulty that it cannot be reduced to a short range case by normal forms. Nevertheless, normal forms methods remain useful, and allow one to prove a global existence result. Section 3 of this text describes, on an example, the main ideas of the proof of this theorem.

We then switch to long time existence problems for nonlinear Klein-Gordon equations on a special class of compact manifolds. We fall then in the category of "very long range" perturbations of linear equations, as solutions of the corresponding linear problem *do not* go to zero when $t \to +\infty$. The main theorem we state, obtained in collaboration with D. Bambusi, B. Grébert and J. Szeftel, asserts that a semi-linear Klein-Gordon equation on the *sphere*, with smooth data of size $\varepsilon > 0$, has when $\varepsilon \to 0$ an almost global solution, i.e. a solution defined on an interval of length ε^{-N} for any $N \in \mathbb{N}$ under a *non-resonance condition* on the linear part of the operator. This condition says that the natural parameter appearing in the Klein-Gordon operator should avoid an exceptional subset of zero measure, and can be though of as a sort of Diophantine condition imposed on that parameter.

The proof of the result given in [2] combines an Hamiltonian approach, and in particular Birkhoff normal forms methods, and the proof of convenient multi-linear estimates. In section 4 of these notes, we concentrate ourselves on the latter, giving instead of the proof of the general theorem, the demonstration of a weaker result for a toy model. We shall nevertheless encounter most of the difficulties that have to be solved in general. In particular, we shall see that some estimates necessary to implement the normal forms method follow from very special properties of the spectrum of the laplacian on the sphere. This explains why our result is limited to such compact manifolds (Actually, they hold true more generally for Zoll manifolds, which enjoy similar spectral properties). In the fifth section, we become more technical, outlining the modifications of the arguments needed to implement the Hamiltonian approach, and indicating the proof of the general theorem.

1. – The nonlinear Klein-Gordon equation.

Let us introduce the equation we want to study. Denote by (M,g) a Riemannian manifold, by $V : M \to \mathbb{R}_+$ a smooth nonnegative potential, by m an element of $]0,+\infty[$. A solution u to the nonlinear Klein-Gordon equation with nonlinearity u^p is a function u, defined for some $T > 0$ on $]-T,T[\times M$, with values in $\mathbb{R}$, satisfying

$$(\partial_t^2 - \varDelta_g + V + m^2)u = u^p. \tag{1}$$

We complement this equation by Cauchy data for u and $\partial_t u$ at time $t = 0$

$$u|_{t=0} = u_0, \partial_t u|_{t=0} = u_1 \tag{2}$$

where u_0, u_1 are given smooth functions from M to $\mathbb{R}$. It is well known that if u_0, u_1 are smooth enough, and have enough decay at infinity, there is some $T > 0$ such that problem (1)-(2) has a unique smooth solution defined on $]-T,T[\times M$. Our aim in these notes is to describe some results obtained in recent years concerning lower bounds for T in function of the size of the Cauchy data, when these data are small. We shall not try to give the most general statements, but instead we will describe only the main ideas.

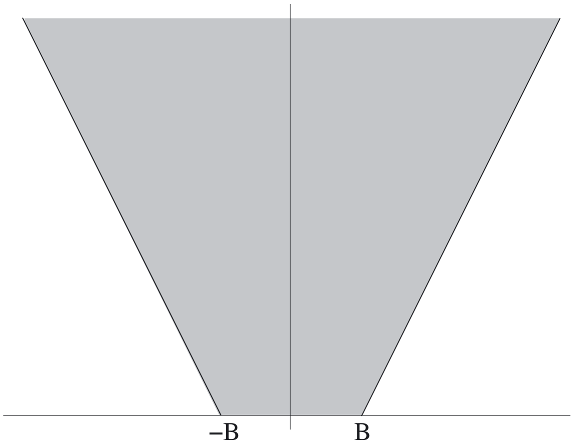
Let us consider the linear Klein-Gordon equation on $\mathbb{R}^d$ with zero potential $(V \equiv 0)$ and with smooth compactly supported Cauchy data

$$\begin{aligned}(\partial_t^2 - \varDelta + m^2)u &= 0\\ u|_{t=0} &= u_0\\ \partial_t u|_{t=0} &= u_1\end{aligned} \tag{3}$$

with $m > 0, u_0, u_1 \in C_0^\infty(\mathbb{R}^d)$. One of the main properties of this equation, holding true including for $m = 0$, is finite propagation speed: if u_0, u_1 are supported inside the ball centered at 0, of radius $B > 0$, then the solution is supported inside the domain

$$\{(t, x); |x| \leq |t| + B\}$$

whose upper part is represented below.



A fundamental quantity associated to (3) is the energy of the solution, defined by

$$E(u)(t) = \frac{1}{2} \int_{\mathbb{R}^d} \left(|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 + m^2 |u(t,x)|^2 \right) dx. \tag{4}$$

An elementary computation shows that $\frac{d}{dt} E(u)(t) = 0$ i.e. the energy of the solution is conserved. More generally, if one assumes that u solves

$$(\partial_t^2 - \Delta + m^2) u = g(t,x) \tag{5}$$

for a function g which is smooth enough and decaying enough when $|x| \to +\infty$, one gets

$$\frac{d}{dt} E(u)(t) \leq \int_{\mathbb{R}^d} |\partial_t u(t,x)| |g(t,x)| \, dx.$$

Moreover, if one makes act $\alpha \in \mathbb{N}^{d+1}$ derivatives on the equation, one gets for $t > 0$

$$E(\partial^\alpha u)(t) \leq E(\partial^\alpha u)(0) + \int_0^t \int_{\mathbb{R}} |\partial_t (\partial^\alpha u)(\tau, x)| |\partial^\alpha g(\tau, x)| \, dx d\tau \tag{6}$$

where $\partial^a = \partial_t^{a_0}\partial_{x_1}^{a_1}\cdots\partial_{x_d}^{a_d}$ if $a \in \mathbb{N}^{d+1}$. Finally, another elementary property of solutions of (3) is the dispersive estimate

$$\|u(t,\cdot)\|_{L^\infty} \le C(1+|t|)^{-d/2} \tag{7}$$

which can be deduced from the explicit representation of the solution of (3) obtained using Fourier transform.

Denote by $H^s(\mathbb{R}^d)$ the Sobolev space, defined in the special case of nonnegative integer s, as the space of functions $u \in L^2(\mathbb{R}^d)$ such that

$$\|u\|_{H^s}^2 \stackrel{\text{def}}{=} \Big(\sum_{a\in\mathbb{N}^d,\ |a|\le s} \|\partial^a u\|_{L^2}^2\Big)^{1/2} < +\infty. \tag{8}$$

Let ε be a small positive number. A very classical result, obtained combining the energy inequality (6) and the fixed point theorem, asserts that if $u_0 \in H^s(\mathbb{R}^d)$, $u_1 \in H^{s-1}(\mathbb{R}^d)$, with $s > \frac{d}{2}$, and if $p \in \mathbb{N}, p \ge 2$, the problem

$$\begin{aligned} (\partial_t^2 - \Delta + m^2)u &= u^p \\ u|_{t=0} &= \varepsilon u_0 \\ \partial_t u|_{t=0} &= \varepsilon u_1 \end{aligned} \tag{9}$$

has for some $T > 0$ a unique solution $u \in C^0(]-T,T[, H^s(\mathbb{R}^d)) \cap C^1(]-T,T[, H^{s-1}(\mathbb{R}^d))$. Moreover, the proof gives a lower bound for T in function of ε: $T \ge c\varepsilon^{-p+1}$. The natural question is then to know if a better lower bound can be obtained when say $u_0, u_1 \in C_0^\infty(\mathbb{R}^d)$.

2. – Global existence for NLKG on $\mathbb{R}^d$.

In this section, we shall recall results of global existence obtained independently by Klainerman [14] and Shatah [19] when the nonlinearity in (9) is a *short range* perturbation of the linear equation.

THEOREM 1. – *Assume in* (9) *that* $(p-1)\dfrac{d}{2} > 1$. *Then for any* $u_0, u_1 \in C_0^\infty(\mathbb{R}^d)$, *there is* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in]0, \varepsilon_0[$, *equation* (9) *has a unique smooth solution defined on* $]-\infty, +\infty[\times \mathbb{R}^d$.

This result, that asserts that (9) has global solutions if the Cauchy data are small enough, is proved by Klainerman [14] and Shatah [19] for much more general nonlinearities than those of (9): actually, one may allow any quasi-linear nonlinearity, vanishing at least at order p at 0. Let us recall the principles of the proof of that theorem. Consider inequality (6) with $g = u^p$. Write $\partial^a g = pu^{p-1}\partial^a u + \cdots$ and forget all terms in the right hand side but the first one (the forgotten terms can be treated by a modification of the estimates below). We can deduce from (6) that for $t \geq 0$

$$E(\partial^a u)(t) \leq E(\partial^a u)(0) + C\int_0^t \|u(\tau,\cdot)\|_{L^\infty}^{p-1} E(\partial^a u)(\tau)\,d\tau \tag{10}$$

whence by Gronwall inequality

$$E(\partial^a u)(t) \leq E(\partial^a u)(0)\exp\Big[C\int_0^t \|u(\tau,\cdot)\|_{L^\infty}^{p-1}\,d\tau\Big]. \tag{11}$$

To prove global existence, it is enough to show that there are $s > \frac{d}{2}, C > 0, \varepsilon_0 > 0$ so that for any $\varepsilon \in]0,\varepsilon_0[$, any $T > 0$ such that a solution exists on $]-T,T[\times\mathbb{R}^d$, one has

$$\sup_{]-T,T[}\big(\|u(t,\cdot)\|_{H^s} + \|\partial_t u(t,\cdot)\|_{H^{s-1}}\big) \leq C\varepsilon. \tag{12}$$

Actually, the fact that one may solve (9) locally in time with data in $H^s \times H^{s-1}$ implies that if (12) holds true, then the solution exists beyond time T. By the usual ODE argument, it follows that the solution may be extended to the whole real line. By definition of the energy, (12) will hold true if a uniform estimate can be obtained for the left hand side of (11), so if $C' > 0$ may be found so that

$$\int_0^t \|u(\tau,\cdot)\|_{L^\infty}^{p-1}\,d\tau \leq C' \tag{13}$$

for any $t \geq 0$ such that the solution exists up to time t. If we conjecture that (7) will hold true for the solution of the nonlinear equation – which is reasonable since for small solutions the right hand side of (9) is small – we bound (13) by $C\int_0^t (1+\tau)^{-(p-1)d/2}\,d\tau \leq C'$ under our assumption $(p-1)\frac{d}{2} > 1$. In other words, we have to control at the same time $\sum_{|a|\leq s} E(\partial^a u)(t)$ and $\|u(t,\cdot)\|_{L^\infty}$ over $]-T,T[$. More precisely, global existence would follow from the following two implications: there are large enough constants A_0, A_1 such that for ε small enough, and all T for which the solution exists over $]-T,T[$

$$\begin{aligned}
&\int_{-T}^{T} \|u(t,\cdot)\|_{L^\infty}^{p-1}\,dt < A_1\varepsilon^{p-1} \Rightarrow \sup_{|t|<T}\sum_{|a|\leq s} E(\partial^a u)(t) < \frac{A_0}{2}\varepsilon^2\\
&\sup_{|t|<T}\sum_{|a|\leq s} E(\partial^a u)(t) < A_0\varepsilon^2 \Rightarrow \int_{-T}^{T} \|u(t,\cdot)\|_{L^\infty}^{p-1}\,dt < \frac{A_1}{2}\varepsilon^{p-1}.
\end{aligned} \tag{14}$$

Actually, if the upper bound of those T for which the assumptions in the left hand side of (14) hold true is finite, one gets a contradiction: since the estimates in the right hand side are given in terms of constants $A_0/2, A_1/2$, the estimates of the left hand side have to hold by continuity beyond time T, contradicting the maximality of that number.

The first implication (14) follows from (11) if one assumes that A_0 has been chosen large enough relatively to the data u_0, u_1 and if ε is small enough, but the second implication in (14) is not an evidence. As a matter of fact, we can always estimate the L^∞ norm of u from the L^2 norms of its derivatives through the Sobolev inequality

$$\|u\|_{L^\infty} \leq C \sum_{|\alpha| \leq [d/2]+1} \|\partial^\alpha u\|_{L^2}, \tag{15}$$

but this provides no time decay to make converge the last integral in (14). To circumvent this problem, Klainerman [14] introduced the "Klainerman vector fields" given by

$$\begin{aligned} &t\partial_{x_j} + x_j \partial_t, \quad j = 1, \dots, d \\ &x_i \partial_{x_j} - x_j \partial_{x_i}, \quad 1 \leq i \neq j \leq d \end{aligned} \tag{16}$$

and proved a Klainerman-Sobolev estimate, saying roughly

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{C}{(1+|t|)^{d/2}} \sum_{|\alpha| \leq s} \|Z^\alpha u(t,\cdot)\|_{L^2}, \tag{17}$$

where in the right hand side s is a large enough integer and Z runs among all vector fields (16) and all usual derivatives $\partial_t, \partial_{x_j}$, $j = 1, \dots, d$. (Actually the right estimate is not as simple as (17): we refer to [14] or Hörmander [13] chapter 7, for the exact inequality). The idea of the proof of theorem 1 is to replace (14) by similar implications, but with $E(\partial^\alpha u)$ replaced by $E(Z^\alpha u)$. Since the vector fields (16) commute to $\partial_t^2 - \Delta + m^2$, (11) remains true if $\partial^\alpha u$ is replaced by $Z^\alpha u$, which implies the corresponding version of the first implication (14). The version of the second implication, in which ∂^α is replaced by Z^α, follows from (17), writing

$$\int_{-T}^{T} \|u(t,\cdot)\|_{L^\infty}^{p-1}\, dt \leq C \sup_{|t|<T} \Big(\sum_{|\alpha| \leq s} E(Z^\alpha u)(t) \Big)^{(p-1)/2} \int_{-T}^{T} \frac{dt}{(1+|t|)^{d(p-1)/2}}, \tag{18}$$

using that $\frac{d}{2}(p-1) > 1$ and taking $A_1 \gg A_0$.

Summing up, we see that the main ingredients of the proof of theorem 1 are an estimate of type (17) and a *short range* condition $(p-1)\frac{d}{2} > 1$. In the following section we shall consider one of the simplest *long range* cases $(p-1)\frac{d}{2} = 1$. This implies either $p = 2, d = 2$ or $p = 3, d = 1$. As mentioned in the introduction, the case of a quadratic nonlinearity in two space dimensions has been studied by Ozawa, Tsutaya and Tsutsumi [18] (see also [7]). They proved that global existence for small rapidly decaying Cauchy data holds true, using a normal forms method to reduce the problem to a short range one. We shall discuss the case of cubic nonlinearities in one space dimension below.

3. – Cubic Klein-Gordon equation on $\mathbb{R}$.

Let us state the following theorem proved in [6].

THEOREM 2. – *Let $u_0, u_1 \in C_0^\infty(\mathbb{R})$. There is $\varepsilon_0 > 0$ such that for $\varepsilon \in]0, \varepsilon_0[$, the problem*

$$(19)\qquad \begin{aligned} (\partial_t^2 - \Delta + 1)u &= u^3 \\ u|_{t=0} &= \varepsilon u_0 \\ \partial_t u|_{t=0} &= \varepsilon u_1 \end{aligned}$$

has a unique global solution u. Moreover

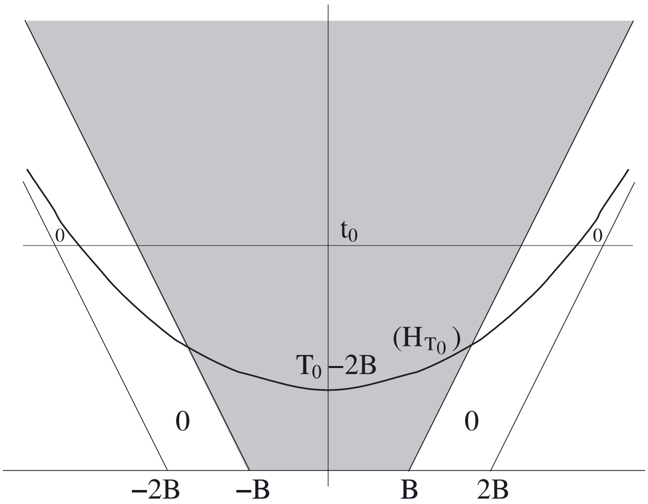
$$(20)\qquad u(t,x) \sim \mathrm{Re}\, \frac{\varepsilon}{\sqrt{t}} a_\varepsilon\Big(\frac{x}{t}\Big) \exp\left[i\sqrt{t^2-x^2} - i\frac{3}{2}\varepsilon^2 \log t \sqrt{1-\frac{x^2}{t^2}} \Big| a_\varepsilon\Big(\frac{x}{t}\Big)\Big|^2 \right],$$

when $t \to +\infty$, where $a_\varepsilon \in C^\infty(\mathbb{R})$, $\operatorname{Supp} a_\varepsilon \subset [-1, 1]$.

Let us make some comments. First, the result of [6] is much more general: it applies to quadratic or cubic quasi-linear nonlinearities satisfying a "null condition" i.e. a condition of compatibility to the linear part of the equation. For arbitrary cubic nonlinearities, the solution exists in general on an interval of length $\exp(c/\varepsilon^2)$ (see Moriyama, Tonegawa and Tsutsumi [17]) and there are examples for which blowing up in finite time happens (see references in the introduction of [6]).

In the asymptotics (20), the logarithmic perturbation in the exponential is due to the long range character of the nonlinearity. In higher space dimensions, or for linear solutions, this logarithmic modulation does not appear.

The proof of theorem 2 given in [6] is pretty long and technical, because general nonlinearities satisfying the null condition are considered. In the case of equation (19), the main ideas of [6] can be developpped in a fairly simple way. This is what we shall do below. For more details on this approach to global existence for nonlinearity u^3, we refer to Lindblad and Soffer [15, 16]. To prove theorem 2, let $B > 0$ be such that u_0, u_1 are supported inside $[-B, B]$. Local existence theory implies that for any fixed $t_0 > 0$ and ε small enough, the solution of (19) exists for $0 \leq t \leq t_0$. Let for $t_0 \gg T_0 > 2B$ (H_{T_0}) be the hyperbola $(t+2B)^2 - x^2 = T_0^2$, drawn on the following picture:



By finite propagation speed, any solution of (19) is supported, for $t \geq 0$, in $|x| \leq t + B$, and extending it by 0 outside this domain, one gets a solution defined under (H_{T_0}). In that way, one is reduced to solving (19) above (H_{T_0}), with Cauchy data on this hyperbola. On the region above (H_{T_0}) one can take hyperbolic coordinates (T, X) related to the old coordinates (t, x) by

$$t + 2B = T \cosh X, \;\; x = T \sinh X. \tag{21}$$

The equation of (H_{T_0}) in these new coordinates becomes $T = T_0$. Since the solution u is expected by (20) to decay like $t^{-1/2}$, we look for it in terms of a new unknown w as

$$u(t,x) = \frac{1}{\sqrt{T}} w(T, X). \tag{22}$$

The first equation in (19) becomes then

$$\Big(\partial_T^2 - \frac{\partial_X^2}{T^2} + 1\Big) w = \frac{1}{T} w^3 - \frac{1}{4T^2} w. \tag{23}$$

The last contribution, which has a time integrable coefficient when $T \to +\infty$, plays little role in the reasoning, and will be forgotten from now on. Equation (23) has a natural energy

$$E(w)(T) = \frac{1}{2} \int_{\mathbb{R}} \Big(|\partial_T w|^2 + \Big|\frac{\partial_X w}{T}\Big|^2 + |w|^2 \Big)\, dX \tag{24}$$

and one has $\left(\text{forgetting the } \frac{1}{T^2} \text{ terms in the right hand side of (23)}\right)$

$$\frac{d}{dT}E(w)(T) \leq \int_{\mathbb{R}} |\partial_T w||w|^3 \frac{dX}{T} \leq \frac{C}{T}\|w(T,\cdot)\|_{L^\infty}^2 E(w)(T). \tag{25}$$

Commuting ∂_X^a derivatives to (23), one gets in the same way as (10), (11), for $T \geq T_0$

$$\sum_{|a|\leq s} E(\partial_X^a w)(T) \leq \sum_{|a|\leq s} E(\partial_X^a w)(T_0) + C\int_{T_0}^{T} \|w(\tau,\cdot)\|_{L^\infty}^2 \Big(\sum_{|a|\leq s} E(\partial_X^a w)(\tau)\Big)\frac{d\tau}{\tau} \tag{26}$$

whence

$$\sum_{|a|\leq s} E(\partial_X^a w)(T) \leq \Big(\sum_{|a|\leq s} E(\partial_X^a w)(T_0)\Big)\exp\Big[C\int_{T_0}^{T} \|w(\tau,\cdot)\|_{L^\infty}^2 \frac{d\tau}{\tau}\Big]. \tag{27}$$

Let us make a small remark: in the new coordinates (T,X), ∂_X corresponds essentially to $Z = t\partial_x + x\partial_t$ in the old coordinates. In other words, the above estimates correspond to the estimates of the preceding section for the Klainerman vector fields. Remark also that, even if we can prove the best estimate we can hope for, namely $\|w(\tau,\cdot)\|_{L^\infty} \leq C\varepsilon$, then (27) will *not* give a uniform estimate for $E(\partial_X^a w)(T)$ when $T \to +\infty$, but only a $O(T^{C\varepsilon^2})$ bound. Actually the asymptotics (20), expressed in the new coordinates, imply that no ∂_X-derivative of w can be bounded uniformly in time. Consequently, we shall deduce global existence from a weakened version of (14): there are $s \in \mathbb{N}$, large enough constants $A_0, A_1, C > 0$, such that for $\varepsilon > 0$ small enough, one has the following two implications:

$$\begin{gathered}\forall T \geq T_0, \|w(T,\cdot)\|_{L^\infty} \leq A_1\varepsilon \Rightarrow \forall T \geq T_0, \sum_{|a|\leq s} E(\partial^a w)(T) < \frac{A_0}{2}\varepsilon^2 T^{CA_1^2\varepsilon^2} \\ \forall T \geq T_0, \sum_{|a|\leq s} E(\partial^a w)(T) < A_0\varepsilon^2 T^{CA_1^2\varepsilon^2} \Rightarrow \forall T \geq T_0, \|w(T,\cdot)\|_{L^\infty} \leq \frac{A_1}{2}\varepsilon.\end{gathered} \tag{28}$$

The first implication follows immediately from (27) if constant A_0 is large enough relatively to $\sum_{|a|\leq s} E(\partial^a w)(T_0)$. The second implication *cannot* follow from a Klainerman-Sobolev embedding i.e. a control of $\|w(T,\cdot)\|_{L^\infty}$ by $\|w\|_{L^2} + \|\partial_X w\|_{L^2}$ since this would give only a $O(T^\delta)$ bound for some $\delta > 0$. The idea to prove (28) is to plug the information coming from the assumption of the second implication (28) inside equation (23). Actually, Sobolev embedding together with that assumption show that $|\partial_X^2 w| \leq CT^\delta$ for some small $\delta > 0$ (if s in (28) has been taken large enough). Consequently we deduce from (23) that w satisfies an ODE, for any fixed X,

$$(\partial_T^2 + 1)w = \frac{1}{T}w^3 + \frac{1}{T^{2-\delta}}\, g \tag{29}$$

where g depends on w and is uniformly bounded. One has to show then that for Cauchy data at time $T = T_0$ small enough, this ODE has global *bounded* solutions. If we had in front of w^3 an integrable power of T, such a result would be essentially a consequence of Gronwall inequality, and the solution would have the same asymptotics as solutions of the linear equation $(\partial_T^2 + 1)w = 0$. In our case, one uses instead a Poincaré normal form, which, through a change of unknown $w = v + \frac{1}{T}B(v, \partial_T v)$, where B is a convenient cubic perturbation, allows one to eliminate most of the cubic terms, reducing the situation to a first order complex ODE of type $\partial_T a + ia = a\frac{i}{T}|a|^2 a$, with a real parameter a. It is then elementary to check that this equation has global bounded small solutions, from which one deduces the same property for solutions of (29). We shall not give here more hints about this normal forms method, for which we refer to [6, 15, 16], since we shall describe below similar reasoning in a more complicated context. Once the boundedness of the solution w of (29) has been obtained, the second implication (28) holds true, if one keeps track of the constants along the proof. Theorem 2 follows from that, and from the asymptotics of solutions of (29).

4. – Long time existence on some compact manifolds.

The results described above rely on the dispersive properties of the Klein-Gordon operator on $\mathbb{R}^d$, which makes the L^∞ norm of the solution go to zero as time goes to infinity. The aim of this section is to describe some results obtained during the last few years in collaboration with several authors, about long time existence for solutions of nonlinear Klein-Gordon equations with small Cauchy data on convenient compact manifolds. In this case, we do not have any time decay for linear solutions, and long time existence follows from normal form methods performed on the PDE. Let us state the main result we shall present, which was obtained with D. Bambusi, B. Grébert and J. Szeftel [2]:

THEOREM 3. – *Let M be the unit sphere of dimension d, $V : M \to \mathbb{R}_+$ be a C^∞ potential. Let $f : M \times \mathbb{R} \to \mathbb{R}$ be a smooth function $(x, u) \to f(x, u)$ vanishing at least at order 2 at $u = 0$. There is $\mathcal{N} \subset]0, +\infty[$, a subset of zero measure, and for any $m \in]0, +\infty[- \mathcal{N}$, any $N \in \mathbb{N}$, any $u_0, u_1 \in C^\infty(M, \mathbb{R})$, there are $\varepsilon_0 > 0, c > 0$, such that for any $\varepsilon \in]0, \varepsilon_0[$ the nonlinear Klein-Gordon* *equation on* $\mathbb{R}\times M$

$$
\begin{aligned}
(\partial_t^2 - \Delta_g + V + m^2)u &= f(x,u)\\
u|_{t=0} &= \varepsilon u_0 \\
\partial_t u|_{t=0} &= \varepsilon u_1
\end{aligned}
\tag{30}
$$

has a unique smooth solution u *defined on* $]-c\varepsilon^{-N}, c\varepsilon^{-N}[$. *Moreover, for* s *large enough,* $\|u(t,\cdot)\|_{H^s}$ *and* $\|\partial_t u(t,\cdot)\|_{H^{s-1}}$ *are bounded uniformly on that interval.*

REMARKS, • The theorem holds true actually on more general compact manifolds than spheres, namely on Zoll manifolds i.e. compact manifolds without boundary for which the geodesic flow is periodic.

• The result of theorem 3 was known yet in one space dimension i.e. when M is the circle, or an interval (with Dirichlet or Neumann boundary conditions). This was proved by Bourgain [4] (on a slightly weaker form than in the above statement) and by Bambusi [1] and Bambusi-Grébert [3]. Actually, in one space dimension, global existence holds true as a consequence of conservation of energy, and the statement of the theorem is then about uniform control of Sobolev norms on intervals of length ε^{-N} for any N.

• Let us point out a major difference with the results of sections 2 and 3: these results did hold true for any value of the mass $m > 0$. We fixed it to be $m = 1$ in several statements since the homogeneity of the equation on $\mathbb{R}\times\mathbb{R}^d$ allows one to reduce to that case through rescaling. On the other hand, theorem 3 will be proved only for m outside an exceptional subset of zero measure.

The proof of theorem 3 combines methods developed in previous work with J. Szeftel [8, 9], with a Hamiltonian approach similar to the one used by Bourgain [4], Bambusi [1], Bambusi-Grébert [3] (we refer also to the lectures of Grébert [12]). Actually, theorem 3 is a corollary of a more "Hamiltonian" result proved in [2], relying on the use of Birkhoff normal forms and on the construction of approximate canonical coordinates. In this section, we shall discuss instead of the full Klein-Gordon equation a simplified model, for which we shall prove a weaker result that theorem 3. We shall give some hints about the proof of theorem 3 in the last section.

Consider

$$
\begin{aligned}
(D_t - \Lambda_m)u &= u^\ell \bar{u}^{p-\ell}\\
u|_{t=0} &= \varepsilon u_0
\end{aligned}
\tag{31}
$$

where $D_t = \dfrac{1}{i}\dfrac{\partial}{\partial t}$, $\Lambda_m = \sqrt{-\Delta_g + V + m^2}$ (defined through spectral theory), $p\in\mathbb{N}, p$ even, $p\geq 2, 0\leq \ell\leq p$ and $u_0\in C^\infty(M,\mathbb{C})$. Local existence theory provides a solution defined over an interval $]-c\varepsilon^{-p+1}, c\varepsilon^{-p+1}[$, and our aim is to prove that actually the solution extends to $]-c\varepsilon^{-2p+2}, c\varepsilon^{-2p+2}[$ (for another $c > 0$, and for $\varepsilon > 0$ small enough). Remark that the operator $D_t - \Lambda_m$ is "one-half" of the Klein-Gordon operator since

$$(D_t - \Lambda_m)(D_t + \Lambda_m) = -\partial_t^2 + \Delta_g - V - m^2.$$

Equation (31) has local in time solutions if $u_0 \in H^s(M)$ with $s > d/2$. Consequently, it is sufficient to find s large enough, $c > 0, C > 0, \varepsilon_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0[$ and any $T < c\varepsilon^{-2p+2}$ for which a solution exists on $]-T, T[$, one has

$$\sup_{t \in]-T,T[} \|u(t,\cdot)\|_{H^s} \leq C\varepsilon. \tag{32}$$

Instead of trying to control directly $\|u(t,\cdot)\|^2_{H^s}$ we shall introduce an equivalent quantity. Denote

$$\Theta_s(u) = \frac{1}{2}\langle u, u\rangle_{H^s} - \operatorname{Re}\langle B(\underbrace{u,\dots,u}_{\ell}, \underbrace{\bar{u},\dots,\bar{u}}_{p-\ell}), u\rangle_{H^s} \tag{33}$$

where for s large enough, $\langle\cdot,\cdot\rangle_{H^s}$ stands for the scalar product associated to the H^s norm, and $(u_1,\dots,u_p) \to B(u_1,\dots,u_p)$ will be a convenient bounded p-linear operator from $H^s \times \cdots \times H^s$ to H^s.

Remark that the boundedness of B, and the fact that $p \geq 2$, imply that for small u, $\Theta_s(u(t,\cdot)) \sim \|u(t,\cdot)\|^2_{H^s}$, and so that one has just to control this new quantity. Actually, we shall construct B so that

$$\frac{d}{dt}\Theta_s(u(t,\cdot)) \leq C\|u(t,\cdot)\|^{2p}_{H^s}, \tag{34}$$

whence for $t \geq 0$ an estimate

$$\Theta_s(u(t,\cdot)) \leq \Theta_s(u(0,\cdot)) + C\int_0^t \|u(\tau,\cdot)\|^{2p}_{H^s}\,d\tau. \tag{35}$$

As long as $u(t,\cdot)$ remains small in H^s, this inequality implies

$$\|u(t,\cdot)\|^2_{H^s} \leq C\|u(0,\cdot)\|^2_{H^s} + C\int_0^t \|u(\tau,\cdot)\|^{2p}_{H^s}\,d\tau,$$

and since the first term in the right hand side is of size of order ε^2, the same will be true for the left hand side when $0 \leq t \leq c\varepsilon^{-2p+2}$. This provides the wanted estimate (32).

When one computes the left hand side of (34), plugging the expression for $\partial_t u$

given by (31) in the time derivative of (33), one gets

$$\begin{aligned}\frac{d}{dt}\Theta_s(u(t,\cdot)) &= \operatorname{Re} i\langle u^\ell \bar{u}^{p-\ell}, u\rangle_{H^s}\\ &\quad - \operatorname{Re} i\sum_{j=1}^{\ell}\langle B(u,\dots,\Lambda_m u,\dots,u,\bar{u},\dots,\bar{u}),u\rangle_{H^s}\\ &\quad + \operatorname{Re} i\sum_{j=1}^{\ell}\langle B(u,\dots,u,\bar{u},\dots,\Lambda_m\bar{u},\dots,\bar{u}),u\rangle_{H^s}\\ &\quad + \operatorname{Re} i\langle \Lambda_m B(u,\dots,\bar{u}),u\rangle_{H^s} + O(\|u(t,\cdot)\|_{H^s}^{2p}).\end{aligned} \tag{36}$$

Actually, we used that the contribution $\operatorname{Re} i\langle \Lambda_m u, u\rangle_{H^s}$ coming from (31) vanishes by self-adjointness of Λ_m, and that the contributions coming from the nonlinearity in (31), of type $\langle B(u,\dots,u^\ell\bar{u}^{p-\ell},\dots,u,\bar{u},\dots,\bar{u}),u\rangle_{H^s}$ can be absorbed in the last term of (36). To get (34), it is then enough to construct a B, bounded on H^s, and such that for any $u_1,\dots,u_p$ in $L^2(M)$

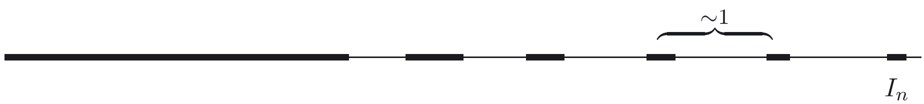
$$\begin{aligned}\sum_{j=1}^{\ell} B(u_1,\dots,\Lambda_m u_j,\dots,u_p) - \sum_{j=\ell+1}^{p} B(u_1,\dots,\Lambda_m u_j,\dots,u_p) - \Lambda_m B(u_1,\dots,u_p)\\ = u_1\cdots u_p.\end{aligned} \tag{3.7}$$

Construction of operator B.

The construction of B relies on the assumptions made on M. Actually, if $M=\mathbb{S}^d$ and $P=\sqrt{-\Delta_g+V}=\Lambda_0$, the spectrum of P is made of a family of clusters i.e. $\sigma(P)\subset\bigcup_{n=1}^{+\infty} I_n$ where I_1 is a bounded interval containing the small eigenvalues and

$$I_n=\Big[n+a-\frac{c_0}{n^\delta},\, n+a+\frac{c_0}{n^\delta}\Big] \tag{38}$$

where $a\in\mathbb{R}, c_0>0, \delta>0$. This structure of the spectrum is illustrated by the following picture:



This result is due to Weinstein [20] and one can even take $\delta=1$ (even if $\delta>0$ is the only information we shall need). Actually, the above property of the spectrum is true more generally if M is a Zoll manifold i.e. a compact manifold on which the geodesic flow is 2π-periodic, as was proved by Colin de Verdière [5] and Guillemin [12]. Moreover, the number of eigenvalues contained in each interval I_n grows at most polynomially when n goes to infinity.

Denote by Π_n the spectral projector associated to I_n, $\Pi_n = 1_{I_n}(P)$. Since $\Lambda_m = \sqrt{P^2+m^2}$, and since the eigenvalues of P restricted to the range of Π_n differ from each other by at most $O(n^{-\delta})$, $(\Lambda_m - \lambda_n \mathrm{Id})\Pi_n$ has operator norm $O(n^{-\delta})$ if λ_n is some fixed element of $\sigma(P)\cap I_n$. This allows one, modulo some small remainders, to replace $\Lambda_m\Pi_n$ by a scalar multiple $\sqrt{m^2+\lambda_n^2}$ of Π_n. To avoid technicalities, for which we refer to [9], we shall assume in what follows that

$$\Lambda_m\Pi_n = \sqrt{m^2+\lambda_n^2}\,\Pi_n \tag{39}$$

for some $\lambda_n \in \mathbb{R}_+^*$ with $\lambda_n \sim n+a$ (which reflects the fact that $\lambda_n \in I_n$). Actually, (39) holds true when $V\equiv 0$ and so $P=\sqrt{-\Delta_g}$, since each I_n is then reduced to one multiple eigenvalue, and (39) is almost true in general, by what we saw above. Moreover, the structure of the spectrum implies that

$$\mathrm{Id} = \sum_{n=1}^{+\infty}\Pi_n. \tag{40}$$

Let us in (37) decompose $u_j = \sum_{n_j=1}^{+\infty}\Pi_{n_j}u_j$. We get

$$\begin{aligned}\sum_{n_1}\cdots\sum_{n_p}\Big[\sum_{j=1}^{\ell} B(\Pi_{n_1}u_1,\dots,\Lambda_m\Pi_{n_j}u_j,\dots,\Pi_{n_p}u_p)&\\ -\sum_{j=\ell+1}^{p} B(\Pi_{n_1}u_1,\dots,\Lambda_m\Pi_{n_j}u_j,\dots,\Pi_{n_p}u_p)&\\ -\Lambda_m B(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p) - (\Pi_{n_1}u_1)\cdots(\Pi_{n_p}u_p)\Big] &= 0.\end{aligned}$$

Applying then $\Pi_{n_{p+1}}$ to this equality, and using (39), we see that we are reduced to constructing B such that for any $n_1,\dots,n_{p+1}$

$$F_m(\lambda_{n_1},\dots,\lambda_{n_{p+1}})\Pi_{n_{p+1}}[B(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p)] = \Pi_{n_{p+1}}[(\Pi_{n_1}u_1)\cdots(\Pi_{n_p}u_p)] \tag{41}$$

where

$$F_m(\xi_1,\dots,\xi_{p+1}) = \sum_{j=1}^{\ell}\sqrt{m^2+\xi_j^2} - \sum_{j=\ell+1}^{p+1}\sqrt{m^2+\xi_j^2}. \tag{42}$$

We deduce from (41) that we shall be able to define B if we are allowed to divide by function $F_m(\lambda_{n_1},\dots,\lambda_{n_{p+1}})$, and if the resulting expression has good enough estimates to ensure that B will be bounded from $H^s\times\cdots\times H^s \to H^s (s\gg 1)$.

Since p is even, for every fixed $\xi_1,\dots,\xi_{p+1}$, $m \to F_m(\xi_1,\dots,\xi_{p+1})$ is a non zero analytic function, and so has only a discrete set of zeros. The union of these zeros for ξ_j describing the (countable) spectrum of P is a countable subset of $]0,+\infty[$. Consequently, for m outside this set, $F_m(\lambda_{n_1},\dots,\lambda_{n_{p+1}}) \neq 0$ for any $n_1,\dots,n_{p+1}$, which allows one to define B from (41). Nevertheless, to get nice estimates for B, we need lower bounds for $F_m(\lambda_{n_1},\dots,\lambda_{n_{p+1}})$. Let us state the following lemma:

LEMMA 4. – *There is a zero measure subset $\mathcal{N}$ of $]0,+\infty[$, and for any $m \in]0,+\infty[- \mathcal{N}$, there are $c>0, N_0 \in \mathbb{N}$ such that for any $n_1 \leq \cdots \leq n_{p+1}$*

$$|F_m(\lambda_{n_1},\dots,\lambda_{n_{p+1}})| \geq c(1+n_{p-1})^{-N_0}. \tag{43}$$

This lemma, together with the following one, is the main technical result to be proved to get theorem 3. Its meaning is that, for m outside $\mathcal{N}$, the left hand side of (43) may be bounded from below by a convenient negative power of the *third largest* among $\lambda_{n_1},\dots,\lambda_{n_{p+1}}$ (or among $n_1,\dots,n_{p+1}$ since $\lambda_n \sim n+a$). The proof, which may be found in [8, 9], uses geometric tools and the special structure of the spectrum of P discussed above. For an alternate proof in dimension 1, that could be extended to our framework, see Bambusi [1], Bourgain [4], Bambusi-Grébert [3]. To exploit (43) to prove that operator B is bounded on H^s, we need also:

LEMMA 5. – *There is $\upsilon \in \mathbb{R}_+$, and for every $N \in \mathbb{N}$ there is $C_N > 0$, such that for any $n_1 \leq \cdots \leq n_{p+1}$, any $u_1,\dots,u_p \in L^2(M)$*

$$\|\varPi_{n_{p+1}}(\varPi_{n_1}u_1 \cdots \varPi_{n_p}u_p)\|_{L^2} \leq C_N \frac{(1+n_{p-1})^{\upsilon+N}}{((n_{p+1}-n_p)+n_{p-1}+1)^N} \prod_{j=1}^{p} \|\varPi_{n_j}u_j\|_{L^2}. \tag{44}$$

Let us indicate the idea of the proof of (44) when u_j is an eigenfunction of P associated to an eigenvalue $\lambda_{n_j} \in I_{n_j}$. By duality, it is enough to estimate by the right hand side of (44)

$$\int (\varPi_{n_1}u_1)\cdots(\varPi_{n_p}u_p)(\varPi_{n_{p+1}}u_{p+1})\,dx \tag{45}$$

for any function $u_{p+1} \in L^2(M)$, $\|u_{p+1}\|_{L^2} = 1$. Denote by a the product $\prod_{j=1}^{p-1} \varPi_{n_j}u_j$ so that

$$\|\partial^\beta a\|_{L^\infty} \leq C(1+n_{p-1})^{\upsilon+|\beta|} \prod_{j=1}^{p-1} \|\varPi_{n_j}u_j\|_{L^2} \tag{46}$$

from Sobolev embedding, for a υ depending only on the dimension. To further simplify, assume that u_{p+1} is also an eigenfunction associated to an eigenvalue

$\lambda_{n_{p+1}}$. We have then

$$
\begin{aligned}
(\lambda_{n_p} - \lambda_{n_{p+1}}) \int a(x)(\Pi_{n_p} u_p)(\Pi_{n_{p+1}} u_{p+1})\, dx \\
= \int a(x)[(P\Pi_{n_p} u_p)(\Pi_{n_{p+1}} u_{p+1}) - (\Pi_{n_p} u_p)(P\Pi_{n_{p+1}} u_{p+1})]\, dx \\
= \int ([a,P]\Pi_{n_p} u_p)(\Pi_{n_{p+1}} u_{p+1})\, dx
\end{aligned}
\tag{47}
$$

and since $[a,P]$ is an operator bounded on L^2, with operator norm smaller than $\sum_{|\beta|\le k} \|\partial^\beta(\partial a)\|_{L^\infty}$ for a large enough k, one estimates (45) from (47) and (46) by

$$
C(\lambda_{n_{p+1}} - \lambda_{n_p})^{-1}(1+n_{p-1})^{\upsilon+k+1} \prod_{j=1}^{p} \|\Pi_{n_j} u_j\|_{L^2}. \tag{48}
$$

Since $\lambda_n \sim n + a$, this gives (44) when $N = 1$. The general case follows by induction. We refer to [9] for the proof of (44) when the (unjustified) simplifying assumptions that we made here are not satisfied.

Let us explain the origin of (44). In the special case $P = \sqrt{-\Delta_g}$ on $M = \mathbb{S}^d$, the image of each Π_n is a space of spherical harmonics. It follows from elementary properties of these functions that the left hand side of (44) is identically zero when $n_{p+1} - n_p \gg n_{p-1}$ i.e. when the right hand side of (44) is small. Inequality (44) is a generalization of a weaker form of that property to $P = \sqrt{-\Delta_g + V}$ on a Zoll manifold M. Actually, it is proved in [9] that (44) can even be generalized to any compact manifold without boundary.

Finally, let us indicate how lemma 5 can be used to prove that operator B defined by (41) is bounded on H^s ($s \gg 1$). Actually, (43) and (41) imply that $\|\Pi_{n_{p+1}}[B(\Pi_{n_1} u_1, \dots, \Pi_{n_p} u_p)]\|_{L^2}$ is bounded from above by the right hand side of (44) (with υ replaced by $\upsilon + N_0$). If $u_j \in H^s$, one has that $\|\Pi_{n_j} u_j\|_{L^2} \le C c_{n_j}(1+n_j)^{-s}$ for a sequence $(c_{n_j})_{n_j}$ in ℓ^2. Then the quantity $\|\Pi_{n_{p+1}}(\Pi_{n_1} u_1 \cdots \Pi_{n_p} u_p)\|_{L^2}$ is bounded from above by

$$
C_N c_{n_{p-1}} (1+n_{p-1})^{\upsilon+N_0-s} \frac{(1+n_{p-1})^N}{((n_{p+1}-n_p)+n_{p-1}+1)^N} \|\Pi_{n_p} u_p\|_{L^2} \prod_{j=1}^{p-1} \|u_j\|_{H^s} \tag{49}
$$

where the negative power $-s$ comes from the smoothness of u_{p-1}, and may be used to compensate the power $\upsilon + N_0$ of the small frequency $1 + n_{p-1}$. One has then to prove that the sum in $n_1 \le \cdots \le n_p$ of (49) may be bounded by $C(1+n_{p+1})^{-s} c_{n_{p+1}}$ for a sequence $(c_{n_{p+1}})_{n_{p+1}}$ of ℓ^2. This may be done for large enough s, N using that (49) gives essentially a convolution operator relatively to indices n_p, n_{p+1}. We refer to [8, 9] for technical details. This concludes the proof of the boundedness of operator B on H^s, and thus the proof of long time existence for equation (31).

5. – Hamiltonian approach.

We shall describe the arguments that have to be added to the preceding ones to get not only long time existence for equation (31), but the almost global result stated in theorem 3.

Let us make first some remarks. We assumed in (31) that p was even, and we used this, before the statement of lemma 4, to ensure that for any fixed $(\xi_1,\dots,\xi_{p+1})$, $m\to F_m(\xi_1,\dots,\xi_{p+1})$ is a *non zero* analytic function. It is also easy to check that when p is odd and $\ell\neq\frac{p+1}{2}$, function (42) is again, for any fixed $(\xi_1,\dots,\xi_{p+1})$ a non zero analytic function of m. Because of that, the lower bound of lemma 4 still holds true in that case, which implies the extension of the solution of (31) to $]-c\varepsilon^{-2p+2},c\varepsilon^{-2p+2}[$ for these values of (p,ℓ) as well. On the other hand, if we consider now the case of odd p and $\ell=\frac{p+1}{2}$, we see that $F_m(\xi_1,\dots,\xi_{p+1})\equiv 0$ as a function of m if $\{\xi_1,\dots,\xi_\ell\}=\{\xi_{\ell+1},\dots,\xi_{p+1}\}$. Consequently, equation (41) *cannot* be solved if $\{n_1,\dots,n_\ell\}=\{n_{\ell+1},\dots,n_{p+1}\}$. Define anyway an operator B by

$$B(u_1,\dots,u_p)=\textstyle\sum'_{n_1,\dots,n_{p+1}}F_m(\lambda_{n_1},\dots,\lambda_{n_{p+1}})^{-1}\Pi_{n_{p+1}}\big((\Pi_{n_1}u_1)\cdots(\Pi_{n_p}u_p)\big) \tag{50}$$

where $\sum'$ means that we restrict summation to $\{n_1,\dots,n_\ell\}\neq\{n_{\ell+1},\dots,n_{p+1}\}$, that is to those indices for which the lower bound (43) holds true. This defines a bounded operator on $H^s\times\cdots\times H^s\to H^s$, but instead of (37), we shall get

$$\begin{aligned}\sum_{j=1}^{\ell}B(u_1,\dots,\Lambda_m u_j,\dots,u_p)-\sum_{j=\ell+1}^{p}B(u_1,\dots,\Lambda_m u_j,\dots,u_p)-\Lambda_m B(u_1,\dots,u_p)\\ =u_1\cdots u_p-R(u_1,\dots,u_p)\end{aligned} \tag{51}$$

with

$$R(u_1,\dots,u_p)=\textstyle\sum''_{n_1,\dots,n_{p+1}}\Pi_{n_{p+1}}\big((\Pi_{n_1}u_1)\cdots(\Pi_{n_p}u_p)\big), \tag{52}$$

the sum $\sum''$ being by definition restricted to indices satisfying $\{n_1,\dots,n_\ell\}=\{n_{\ell+1},\dots,n_{p+1}\}$. If we go back to (36), we thus get

$$\frac{d}{dt}\Theta_s(u(t,\cdot))=\operatorname{Re} i\langle R(u,\dots,u,\bar u,\dots,\bar u),u\rangle_{H^s}+O(\|u(t,\cdot)\|^{2p}_{H^s}). \tag{53}$$

Instead of the standard H^s scalar product, we could as well have defined

$$\langle u,v\rangle_{H^s}=\int_M \Lambda_m^s u\cdot\overline{\Lambda_m^s v}\,dx,$$

in which case the first term in the right hand side of (53) would be minus the imaginary part of

$$(54)\quad {\sum}''_{n_1,\dots,n_{p+1}} \int_M (\Pi_{n_1}u_1)\cdots(\Pi_{n_\ell}u_\ell)(\overline{\Pi_{n_{\ell+1}}u_{\ell+1}})\cdots(\overline{\Pi_{n_p}u_p})(\Lambda_m^{2s}\overline{\Pi_{n_{p+1}}u_{p+1}})dx.$$

By definition of $\sum''$ and by (39), each term in this sum is *real*, since we have a pairing of each $\Pi_{n_j}u, j=1,\dots,\ell$ with a $\overline{\Pi_{n_k}u}, k\in\{\ell+1,\dots,p+1\}$ with $n_j=n_k$. Consequently, (53) reduces to (34), which shows as before that, including in the case p odd and $\ell=\frac{p+1}{2}$, we get an extension of our solution to an interval of type $]-c\varepsilon^{-2p+2}, c\varepsilon^{-2p+2}[$.

An idea, to try to prove an almost global existence result for the solution of equation (31), would be to write explicitly the $2p$-homogeneous remainder in the right hand side of (36), and to try to get rid of it, modulo remainders of higher order, modifying the definition (33) of $\Theta_s(u)$ through a new contribution of order $2p$. One faces then the following difficulty: the new expression one would have to eliminate is now given by a much more complicated nonlinear operator than the one in the right hand side of (37). Because of that, we are unable to perform a simple symmetry argument, like the one indicated after formula (54), to show that contributions that could not be eliminated by a normal forms method always give a zero contribution to $\frac{d}{dt}\Theta_s(u(t,\cdot))$.

The idea used in [2] to get around such difficulties, and prove almost global existence for the solution of (30), is to make use of the Hamiltonian character of that equation. Consider the following Hamiltonian framework. On the phase space $H^s(M,\mathbb{R})\times H^s(M,\mathbb{R}), (s\gg 1)$, define a symplectic structure setting

$$(55)\quad \begin{gathered} J=\begin{bmatrix} 0 & -\mathrm{Id}\\ \mathrm{Id} & 0\end{bmatrix}\\ \Omega((p,q),(p',q'))=\langle J^{-1}(p,q),(p',q')\rangle=\langle q,p'\rangle-\langle q',p\rangle,\end{gathered}$$

where $\langle\cdot,\cdot\rangle$ stands for the L^2 scalar product and (p,q), (p',q') belong to $H^s(M,\mathbb{R})\times H^s(M,\mathbb{R})$. For a smooth function G defined on an open subset of the phase space, introduce its Hamiltonian vector field

$$(56)\quad X_G(p,q)=J\nabla G(p,q)=(-\nabla_q G(p,q),\nabla_p G(p,q)),$$

which is an element of $H^{-s}(M,\mathbb{R})\times H^{-s}(M,\mathbb{R})$ (and for nice G's, lies actually in $H^s(M,\mathbb{R})\times H^s(M,\mathbb{R})$). Set also for two smooth functions G_1,G_2

$$\{G_1,G_2\}=\partial G_2\cdot X_{G_1}=\Omega(X_{G_2},X_{G_1}).$$

Define for a smooth enough real valued function u

$$p=\Lambda_m^{-1/2}\partial_t u,\ q=\Lambda_m^{1/2}u.$$

If $\tilde{f}(x,u)$ is a primitive in u of $-f(x,u)$ (where f is defined in (30)), vanishing at 0, set

$$
\begin{aligned}
G_2(p,q) &= \frac{1}{2}\int_M \left(\left|\Lambda_m^{1/2}p\right|^2 + \left|\Lambda_m^{1/2}q\right|^2\right) \mathrm{d}x,\\
\tilde{G}(p,q) &= \int_M \tilde{f}(x,\Lambda_m^{-1/2}q)\,\mathrm{d}x, \qquad (57)\\
G(p,q) &= G_2(p,q) + \tilde{G}(p,q).
\end{aligned}
$$

Then equation (30) may be rewritten

$$(\dot{p},\dot{q}) = X_G(p,q). \qquad (58)$$

If we use on the phase space complex coordinates given by

$$
\begin{aligned}
H^s(M,\mathbb{R}) \times H^s(M,\mathbb{R}) &\to H^s(M,\mathbb{C})\\
(p,q) &\to \frac{1}{\sqrt{2}}(p+iq) = v,
\end{aligned} \qquad (59)
$$

we may write also (58) as

$$\dot{v} = i\nabla_{\bar{v}} G(v,\bar{v}). \qquad (60)$$

To prove almost global existence, one has to construct for any $N \in \mathbb{N}$ a function $\Theta_s(v,\bar{v})$, equivalent to the square of the Sobolev norm, such that

$$\frac{d}{dt}\Theta_s(v(t,\cdot),\bar{v}(t,\cdot)) \le C\|v(t,\cdot)\|_{H^s}^{N+2}. \qquad (61)$$

Denote using (40)

$$\Theta_s^0(v,\bar{v}) = \sum_{n=1}^{+\infty} n^{2s}\|\Pi_n v\|_{L^2}^2 \sim \|v\|_{H^s}^2.$$

LEMMA 6. – *There is $\mathcal{N} \subset]0,+\infty[$ of zero measure and for any $m \in]0,+\infty[- \mathcal{N}$, for any $N \in \mathbb{N}$, there are $s \gg 1$ and a canonical transformation χ, defined on a neighborhood of 0 in $H^s(M,\mathbb{C})$, with $\chi(0)=0, \chi'(0)=Id$, such that $\Theta_s(v,\bar{v}) \stackrel{\text{def}}{=} \Theta_s^0 \circ \chi(v,\bar{v})$ satisfies* (61).

One deduces from that lemma that the solution of (60) may be extended over an interval of length $c\varepsilon^{-N}$, reasoning like after (34).

The proof of lemma 6 relies on Birkhoff normal forms, and is a variation on similar results proved by Bourgain [4], Bambusi [1] and Bambusi-Grébert [3] in one space dimension. The idea is as follows. Consider real valued homogeneous functions of degree $k+1$ $h_k(v,\bar{v})$, set $h(v,\bar{v}) = \sum_{k\ge 2} h_k(v,\bar{v})$ (where the sum if finite, but involves enough terms) and denote by $\Phi^t(v,\bar{v})$ the flow of X_h at time t. One looks for χ as $\chi(v,\bar v) = \Phi^1(v,\bar v)$. Since χ will automatically be canonical, we get using (58) or (60)

$$\begin{aligned}\frac{d}{dt}(\Theta_s^0 \circ \chi^{-1})(v(t),\bar v(t)) &= \{G, \Theta_s^0 \circ \chi^{-1}\}(v(t),\bar v(t)) \\ &= \{G\circ\chi, \Theta_s^0\} \circ \chi^{-1}(v(t),\bar v(t)).\end{aligned} \tag{62}$$

Since χ vanishes at order 1 at 0, conclusion (61) will follow if we prove that

$$\{\Theta_s^0, G\circ\chi\}(v,\bar v) = O(\|v\|_{H^s}^{N+2}),\ v \to 0. \tag{63}$$

The definition of $\chi = \Phi^1$ allows one to verify that

$$G\circ\chi \sim \sum_{q\geq 0} \frac{1}{q!}(\mathrm{Ad}\, h)^q \cdot G \tag{64}$$

where $(\mathrm{Ad}\, h)\cdot G = \{h, G\}$. We thus need to construct h so that

$$\sum_{q\geq 0} \frac{1}{q!}\{\Theta_s^0, (\mathrm{Ad}\, h)^q \cdot G\} = O(\|v\|_{H^s}^{N+2}),\ v \to 0. \tag{65}$$

Remind that $G_2(v,\bar v) = \int_M |\Lambda_m^{1/2} v|^2\, \mathrm{d}x$ and expanding in Taylor series $\tilde G$ write

$$\tilde G(v,\bar v) = \sum_{k\geq 2} \tilde G_k(v,\bar v),$$

where $\tilde G_k$ is homogeneous of degree $k+1$. Sorting by homogeneity the contributions in the left hand side of (65), one gets a set of equations

$$\{\Theta_s^0, \{h_k, G_2\} + \Gamma_k\} = 0 \tag{66}$$

where $\Gamma_k(v,\bar v) = \tilde G_k(v,\bar v) + R_k(v,\bar v)$ is homogeneous of degree $k+1$, R_k depending on $h_{k'}, k' < k$. Moreover, using a modification of lemma 5, one may prove that each Γ_k may be written

$$\Gamma_k(v,\bar v) = \sum_{\ell=0}^{k+1} \Gamma_k^\ell(\underbrace{v,\dots,v}_{\ell},\underbrace{\bar v,\dots,\bar v}_{k+1-\ell}) \tag{67}$$

where Γ_k^ℓ is a $(k+1)$-linear form such that for some $\upsilon > 0$, any $N \in \mathbb{N}$ and any $n_1 \leq \cdots \leq n_{k+1}$,

$$|\Gamma_k^\ell(\Pi_{n_1}u_1,\dots,\Pi_{n_{k+1}}u_{k+1})| \leq C_N \frac{(1+n_{k-1})^{\upsilon+N}}{((n_{k+1}-n_k)+n_{k-1}+1)^N} \prod_{j=1}^{k+1} \|\Pi_{n_j}u_j\|_{L^2} \tag{68}$$

(compare with the statement of lemma 5).

To solve (66), one would like to find h_k such that $\{h_k, G_2\} + \Gamma_k = 0$. This is not possible if k is odd, because of the contribution $\Gamma_k^{(k+1)/2}$ for the same reason as the one we described at the beginning of this section. Anyway, if when $\ell = \dfrac{k+1}{2}$ one sets

$$\Gamma_k''^{\ell}(u_1, \dots, u_{k+1}) = \textstyle\sum''_{n_1,\dots,n_{k+1}} \Gamma_k^{\ell}(\Pi_{n_1} u_1, \dots, \Pi_{n_{k+1}} u_{k+1}),$$

where the sum $\sum''$ is extended to those indices for which $\{n_1, \dots, n_\ell\} = \{n_{\ell+1}, \dots, n_{k+1}\}$, one checks easily that

$$\{\Theta_s^0, \Gamma_k''^{(k+1)/2}(v, \dots, \bar{v})\} \equiv 0.$$

Consequently, equation (66) may be rewritten when k is odd

$$\left\{\Theta_s^0, \{h_k, G_2\} + \Gamma_k'\right\} = 0 \tag{68}$$

where $\Gamma_k'(v, \bar{v}) = \Gamma_k(v, \bar{v}) - \Gamma_k''^{(k+1)/2}(v, \dots, \bar{v})$. One has thus to solve $\{h_k, G_2\} + \Gamma_k' = 0$ which, because of the elimination of the $\Gamma_k''^{(k+1)/2}$ term just performed, may be treated by similar methods as the ones used to solve equation (51).

To conclude, let us say that the result of [2] is actually more precise than what we described above. The construction of canonical transformation χ of lemma 6 allows one to bring our Hamiltonian equation (58) to canonical form, up to any given order. We refer to [2] for precise statements.

REFERENCES

[1] D. Bambusi, *Birkhoff normal form for some nonlinear PDEs*, Comm. Math. Phys., **234**, no. 2 (2003), 253-285.

[2] D. Bambusi - J.-M. Delort - B. Grébert - J. Szeftel, *Almost global existence for Hamiltonian semi-linear Klein-Gordon equations with small Cauchy data on Zoll manifolds*, to appear, Comm. Pure Appl. Math.

[3] D. Bambusi - B. Grébert, *Birkhoff normal form for pdes with tame modulus*, Duke Math. J., **135**, no. 3 (2006), 507-567.

[4] J. Bourgain, *Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations*, Geom. Funct. Anal., **6**, no. 2 (1996), 201-230.

[5] Y. Colin de Verdière, *Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques*, Comment. Math. Helv., **54**, no. 3 (1979), 508-522.

[6] J.-M. Delort, *Existence globale et comportement asymptotique pour l'équation de Klein-Gordon quasi linéaire à données petites en dimension 1*, Ann. Sci. École Norm. Sup. (4) **34**, no. 1 (2001), 1-61. Erratum, Ann. Sci. École Norm. Sup. (4) **39**, no. 2 (2006), 335-345.

[7] J.-M. Delort - D. Fang - R. Xue, *Global existence of small solutions for quadratic quasilinear Klein-Gordon systems in two space dimensions*, J. Funct. Anal., **211**, no. 2 (2004), 288-323.

[8] J.-M. Delort - J. Szeftel, *Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres*, Int. Math. Res. Not., no. 37 (2004), 1897-1966.

[9] J.-M. Delort - J. Szeftel, *Long-time existence for semi-linear Klein-Gordon equations with small Cauchy data on Zoll manifolds*, Amer. J. Math., **128**, no. 5 (2006), 1187-1218.

[10] J. DUISTERMAAT - V. GUILLEMIN, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math., **29**, no. 1 (1975), 39-79.
[11] B. GRÉBERT, *Birkhoff normal form and hamiltonian PDEs*, preprint, (2006).
[12] V. GUILLEMIN, *Lectures on spectral theory of elliptic operators*, Duke Math. J., **44**, no. 3 (1977), 485-517.
[13] L. HÖRMANDER, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications, 26, Springer-Verlag, Berlin, 1997. viii+289 pp.
[14] S. KLAINERMAN, *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions*, Comm. Pure Appl. Math., **38**, no. 5 (1985), 631-641.
[15] H. LINDBLAD - A. SOFFER, *A remark on Long Range Scattering for the critical nonlinear Klein-Gordon equation*, J. Hyperbolic Differ. Equ. 2, no. 1 (2005), 77-89.
[16] H. LINDBLAD - A. SOFFER, *A remark on asymptotic completeness for the critical nonlinear Klein-Gordon equation*, Lett. Math. Phys., **73**, no. 3 (2005), 249-258.
[17] K. MORIYAMA - S. TONEGAWA - Y. TSUTSUMI, *Almost global existence of solutions for the quadratic semilinear Klein-Gordon equation in one space dimension*, Funkcial. Ekvac., **40**, no. 2 (1997), 313-333.
[18] T. OZAWA - K. TSUTAYA - Y. TSUTSUMI, *Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions*, Math. Z., **222**, no. 3 (1996), 341-362.
[19] J. SHATAH, *Normal forms and quadratic nonlinear Klein-Gordon equations*, Comm. Pure Appl. Math., **38** (1985), 685-696.
[20] A. WEINSTEIN, *Asymptotics of eigenvalue clusters for the Laplacian plus a potential*, Duke Math. J., **44**, no. 4 (1977), 883-892.

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