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A Note on Strong Lie Derived Length of Group Algebras.

FRANCESCO CATINO - ERNESTO SPINELLI

Sunto. – *Per un'algebra gruppale KG di un gruppo non-abeliano G su di un campo K di caratteristica positiva p si studia la lunghezza derivata forte di Lie dell'algebra di Lie associata.*

Summary. – *For a group algebra KG of a non-abelian group G over a field K of positive characteristic p we study the strong Lie derived length of the associated Lie algebra.*

1. – Introduction.

Let G be a group and let KG be the group algebra of G over a field K. We consider the Lie algebra associated with KG by setting $[x, y] := xy - yx$ for every $x, y \in KG$. We define by induction $\delta^{(0)}(KG) := KG$ and $\delta^{(n+1)}(KG)$ the associative ideal generated by $[\delta^{(n)}(KG), \delta^{(n)}(KG)]$, where this symbol denotes the additive subgroup generated by all the Lie commutators $[a, b]$ with $a, b \in \delta^{(n)}(KG)$. KG is *strongly Lie solvable* if there exists an integer m such that $\delta^{(m)}(KG) = 0$ and the minimal m with this property is called the *strong Lie derived length* of KG. Such an m is usually denoted by $dl^L(KG)$.

If K has characteristic $p > 0$ and G is non-abelian, it is well-known that the group algebra KG is strongly Lie solvable if, and only if, the commutator subgroup G' of G is a finite p-group (see Theorem V.5.1 of [5]).

If $t(G')$ denotes the nilpotency index of the augmentation ideal $\Delta(G')$ of KG', as an immediate consequence of Lemma 2.1 of [6] and of Lemma 2.2 of [4] we have the following elementary bounds

$$\lceil \log_2(t(G') + 1) \rceil \leq dl^L(KG) \leq \lceil \log_2(2t(G')) \rceil,$$

where $\lceil r \rceil$ denotes the upper integral part of a real number r.

Very little it is known about the strong Lie derived length of group algebras. The most remarkable works in this area are the papers by C. Baginski [1], M. Sahai [4] and A. Shalev [6], [7].

With the extra assumption that G is nilpotent, the evaluation of $dl^L(KG)$ is more accurate. Denote by $KG^{(1)} := KG$ and $KG^{(m+1)}$ the associative ideal generated by $[KG^{(m)}, KG]$; we denote by $cl^L(KG)$ the minimal integer n such that $KG^{(n+1)} = 0$, the strong Lie nilpotency class of KG. An easy induction allows to verify that $\delta^{(m)}(KG) \subseteq KG^{(2^m)}$ for every non-negative integer m. So we have

$$\lceil \log_2(t(G') + 1) \rceil \leq dl^L(KG) \leq \lceil \log_2(cl^L(KG) + 1) \rceil. \tag{1}$$

An easy induction shows that if G is any group with G/G'^p nilpotent for some prime p, then also G/G'^{p^n} is nilpotent for all $n \geq 1$. In particular, if the condition $\gamma_3(G) \leq G'^p$ is satisfied, G is nilpotent, provided that G' is a finite p-group. Under the same assumptions, by Theorem 3.1 of [2], $cl^L(KG) = t(G')$ and by (1) we obtain

$$dl^L(KG) = \lceil \log_2(t(G') + 1) \rceil. \tag{2}$$

This result extends Corollary 4 of [1], which deals with finite p-groups whose commutator subgroup is cyclic.

We define by induction $\delta^{[0]}(KG) := KG$ and $\delta^{[n+1]}(KG) := [\delta^{[n]}(KG), \delta^{[n]}(KG)]$. Recall that KG is *Lie solvable* if there exists an integer n such that $\delta^{[n]}(KG) = 0$ and the minimal n with this property is called the *Lie derived length* of KG. Such an n is usually denoted by $dl_L(KG)$. Clearly $\delta^{[n]}(KG) \subseteq \delta^{(n)}(KG)$ for all non-negative integer n. Thus a strongly Lie solvable group algebra KG is Lie solvable and $dl_L(KG) \leq dl^L(KG)$. But equality does not always hold. In fact, let G be a 2-group of maximal class of order 2^n with $n \geq 5$ and let K be a field of characteristic 2. Then G contains an abelian subgroup of index 2 and, by Theorem 1 of [3], $dl_L(KG) \leq 3$, whereas $dl^L(KG) = n - 1 > 3$ since G' is cyclic of order 2^{n-2}.

We remark that, obviously, we obtain with precision the strong Lie derived length of a group algebra KG also when the Lie derived length reaches its upper bound $\lceil \log_2(2t(G')) \rceil$. For example, when the group G is a semidirect product of an elementary abelian p-group by an automorphism of prime order q, where $p \neq q$ and $q > 2$ (Theorem C(1) of [6]).

If the group G is as above, but with $q = 2$, A. Shalev in [6] proved that the Lie derived length of KG is $\lceil \log_2(3t(G')/2) \rceil$.

The aim of the sequel is to prove that this is also the strong Lie derived length of KG.

THEOREM. – *Let K be a field of characteristic $p > 2$ and let $G = E \rtimes \langle a \rangle$ be a split extension of an elementary abelian p-group E by an automorphism a of order 2. Then*

$$dl^L(KG) = \lceil \log_2(3t(G')/2) \rceil.$$

2. – Proof of the Theorem.

LEMMA. – *Let K be a field of characteristic $p > 2$ and let $G = E \rtimes \langle a \rangle$ be a split extension of an elementary abelian p-group E by an automorphism a that acts on E by inversion. The following holds*:

(1) $[\Delta(G')^m KG, \Delta(G')^m KG] \subseteq \Delta(G')^{2m} KG$ *provided m is odd;*
(2) $[\Delta(G')^m KG, \Delta(G')^m KG] \subseteq \Delta(G')^{2m+1} KG$ *provided m is even.*

PROOF. – The first statement is trivial. In order to prove the other one, we preliminarily observe that

$$[\Delta(G')^2, KG] \subseteq \Delta(G')^3 KG. \tag{3}$$

Now let m be an even integer. If $m = 2$, using (3) we have

$$\begin{aligned}
&[\Delta(G')^2 KG, \Delta(G')^2 KG] \\
&\qquad \subseteq \Delta(G')^2 [KG, \Delta(G')^2 KG] KG \\
&\qquad + [\Delta(G')^2, \Delta(G')^2 KG] KG \\
&\qquad \subseteq \Delta(G')^4 [KG, KG] KG + \Delta(G')^2 [KG, \Delta(G')^2] KG \\
&\qquad \subseteq \Delta(G')^5 KG.
\end{aligned}$$

Finally, let $m > 2$. Then

$$\begin{aligned}
&[\Delta(G')^m KG, \Delta(G')^m KG] \\
&\qquad \subseteq \Delta(G')^2 [\Delta(G')^{m-2} KG, \Delta(G')^m KG] KG \\
&\qquad + [\Delta(G')^2, \Delta(G')^m KG] \Delta(G')^{m-2} KG \\
&\qquad \subseteq \Delta(G')^4 [\Delta(G')^{m-2} KG, \Delta(G')^{m-2} KG] KG \\
&\qquad + \Delta(G')^2 [\Delta(G')^{m-2} KG, \Delta(G')^2] \Delta(G')^{m-2} KG \\
&\qquad \subseteq \Delta(G')^4 \Delta(G')^{2m-3} KG \\
&\qquad + \Delta(G')^m [KG, \Delta(G')^2] \Delta(G')^{m-2} KG \\
&\qquad \subseteq \Delta(G')^{2m+1} KG,
\end{aligned}$$

by combining (3) and by the induction hypothesis. □

PROOF OF THE THEOREM. – As just remarked in [6] by A. Shalev, we may assume that $C_E(a) = 1$ and so $G' = E$.

Let n be a non-negative integer. In the sequel we put $s_0 := 1$ and

$$s_n := \begin{cases} (2^{n+2} - 1)/3 & \text{provided } n \text{ is even,} \\ (2^{n+2} - 2)/3 & \text{provided } n \text{ is odd.} \end{cases}$$

First we proceed by induction to prove that

$$\forall n \in \mathbb{N}_0 \qquad \delta^{(n+1)}(KG) \subseteq \Delta(G')^{s_n} KG. \tag{4}$$

Since $G' = \gamma_3(G)$, it follows that

$$\delta^{(2)}(KG) = \Delta(G')^2 KG.$$

Let $n > 2$. By induction and by the Lemma, we obtain

$$\begin{aligned}\delta^{(n+1)}(KG) &= [\delta^{(n)}(KG), \delta^{(n)}(KG)]KG \\ &\subseteq [\Delta(G')^{s_{n-1}} KG, \Delta(G')^{s_{n-1}} KG]KG \\ &\subseteq \Delta(G')^{s_n} KG.\end{aligned}$$

Now, let $d := dl^L(KG)$. Then, by (4), $\Delta(G')^{s_{d-2}} \neq 0$ and so we have $s_{d-2} < t(G')$.

If d is even, then $d < \log_2(3t(G')/2 + 1/2) + 1$ and so $d \leq \lceil \log_2(3t(G')/2) \rceil$, since $t(G')$ is odd.

If d is odd, since $2^{d-1} < \lceil 3t(G')/2 \rceil$, it follows that $2^{d-1} < 3t(G')/2 + 1/2$ and, as above, $d \leq \lceil \log_2(3t(G')/2) \rceil$.

Finally, by Theorem C(2) of [6], the result follows. □

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