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On the Variance Associated to a Family of Ovaloids in the Euclidean Space $\mathbf{E}_3$.

GIUSEPPE CARISTI - GIOVANNI MOLICA BISCI

Sunto. – *In questo lavoro si cosidera una variabile aleatoria η associata a un sistema di ovaloidi indipendenti e uniformemente distribuiti nello spazio Euclideo tridimensionale ed a un fissato corpo convesso $\mathbf{K}_0$.*

Summary. – *In this paper we consider a random variable η arising from an intersection problem between a fixed convex body $\mathbf{K}_0$ and a system of random independent and uniformly distributed ovaloids in $\mathbf{E}_3$.*

1. – Introduction.

Let us consider $\mathbf{E}_3$ be the Euclidean three dimensional space of coordinates x_1, x_2, x_3. The elementary Kinematic measure in $\mathbf{E}_3$ is given by the following formula

$$d\mathbf{K} = dP \wedge d\Omega \wedge d\psi, \tag{1}$$

where $dP = dx_1 \wedge dx_2 \wedge dx_3$, $d\Omega = |\sin\theta| d\varphi \wedge d\theta$ is the elementary measure on the unit sphere and ψ is an angle of rotation. We consider a fixed ovaloid $\mathbf{K}_0$ of area S_0, volume V_0 and such that the integral of the mean curvature is $\overline{\mathbf{H}}_0$. Let now $\mathbf{K}_1, \ldots, \mathbf{K}_m$ be a family of aleatory (respect to position) independent and uniformly distributed ovaloids in $\mathbf{E}_3$. We assume that $\mathbf{K}_i = \mathbf{K}$, $\forall i = 1, \ldots, m$, where $\mathbf{K}$ is an ovaloid of volume V, area S and such that the integral of the mean curvature is $\overline{\mathbf{H}}_0$. We denote by $\eta(r, m)$ the random volume obtained intersecting r ovaloids ($r \leq m$) in the family $\{\mathbf{K}_1, \ldots, \mathbf{K}_m\}$ and $\mathbf{K}_0$.

Santaló show that the mean value of this random variable is given by

$$E(\eta(r,m)) = \binom{m}{r} \frac{(8\pi^2 V)^r (8\pi^2 V_0 + 2\pi(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0))^{m-r} V_0}{[8\pi^2(V_0 + V) + 2\pi(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0)]^m}. \tag{2}$$

When $r = m$, of course

$$E(\eta(m)) = \frac{(8\pi^2 V)^m V_0}{[8\pi^2(V_0 + V) + 2\pi(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0)]^m}. \tag{3}$$

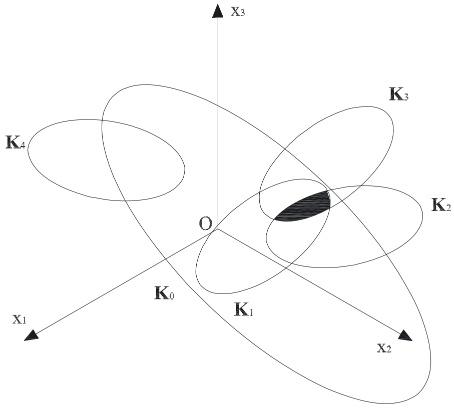

Fig. 1. – A case when $(m, r) = (4,3)$.

In this note we give the expression of the variance (that we denote σ^2) of $\eta(r, m)$. When $m = r$ we have the result of Stoka in [6]. We also compute the mean value $E(\eta(\beta_1, m)\eta(\beta_2, m))$ when $\beta_1 \leq \beta_2$. An application is obtained for a family of cubes of side a and considering as fixed convex body $\mathbf{K}_0$ a sphere Σ_0 of constant radius δ_0.

2. – Main Results.

In all this paper μ denotes the Lebesgue measure. With the same notations in Introduction we can give the Santaló results

$$\mu(\mathbf{K} \cap \mathbf{K}_0 \neq \emptyset) = 8\pi^2(V_0 + V) + 2\pi\big(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\big), \tag{4}$$

$$\mu(\mathbf{K} \cap \mathbf{K}_0 \neq \emptyset, P \in \mathbf{K}) = 8\pi^2 V, \tag{5}$$

$$\mu(\mathbf{K} \cap \mathbf{K}_0 \neq \emptyset, P \notin \mathbf{K}) = 8\pi^2 V_0 + 2\pi\big(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\big), \tag{6}$$

where $P \in Int(\mathbf{K}_0)$.

If we denote by $\mu(\mathbf{K}, l)$ the measure of the set of segments of length l entirely contained in $\mathbf{K}$, using the expression of the elementary Kinematic measure (1) we have

$$\mu\big(\mathbf{K} \cap \mathbf{K}_0 \neq \emptyset, \overline{P_1P_2} \subset \mathbf{K}\big) = \mu\big(\mathbf{K}, \overline{P_1P_2}\big), \tag{7}$$

with P_1,$P_2 \in Int(K_0)$.

We want to show the following

THEOREM 1. – *Let $\mathbf{K}_0$ be a fixed ovaloid in $\mathbf{E}_3$ of volume V_0, area S_0 and with integral of mean curvature $\overline{\mathbf{H}}_0$. We denote with $\mathcal{F} = \{\mathbf{K}_1, \ldots, \mathbf{K}_m\}$ a family of m*

independent, uniformly distributed ovaloids of volume V, area S, such that $\mathbf{K}_i \cap \mathbf{K}_0 \neq \emptyset$ and $\mathbf{K}_i = \mathbf{K}$ for $i = 1, \dots, m$. We assume that $\mathbf{K}$ has integral of mean curvature $\overline{\mathbf{H}}$. If $\eta(r,m)$ is the random volume $\mathbf{V}ol(\mathbf{K}_0 \cap \dots \cap \mathbf{K}_{i_r})$ obtained intersecting $r \leq m$ ovaloids in $\mathcal{F}$, the expression of the variance of this random variable is

$$\sigma^2(\eta(r,m)) = \binom{m}{r} \frac{\Theta(\mathbf{K}_0, \mathbf{K}; r, m)}{\left[8\pi^2(V_0+V) + 2\pi\left(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\right)\right]^m}, \tag{8}$$

with

$$\Theta(\mathbf{K}_0, \mathbf{K}_j; r, m) := 2\sum_{j=0} \binom{r}{j}\binom{m-r}{r-j} \int\limits_{\{\mathbf{G}\cap\mathbf{K}_0\neq\emptyset\}} \phi_j(\mathbf{K}, m, r; \lambda) d\mathbf{G}$$
$$-\binom{m}{r} \frac{(8\pi^2 V)^{2r}(8\pi^2 V_0 + 2\pi(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0))^{2m-2r} V_0^2}{\left[8\pi^2(V_0 - V) + 2\pi\left(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\right)\right]^m},$$

where $d\mathbf{G}$ is the elementary measure of the lines in the Euclidean Space $\mathbf{E}_3$ and

$$\phi_j(\mathbf{K}, m, r; \lambda) = \int_0^{\lambda}\int_0^{v} [\mu(\mathbf{K}, u)]^j \left[8\pi^2 V - \mu(\mathbf{K}, u)\right]^{2r-2j}$$
$$\times\left[8\pi^2(V_0+V) + 2\pi\left(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\right) + \mu(\mathbf{K}, u)\right]^{m-2r+j} u^2 du dv.$$

PROOF. – We have the following relation

$$\begin{aligned} \{\mathbf{K}/\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset\} = \{&\mathbf{K}/\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, P_1 \in \mathbf{K}, P_2 \in \mathbf{K}\} \\ &\cup\{\mathbf{K}/\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, P_1 \in \mathbf{K}, P_2 \in \mathbf{K}\} \\ &\cup\{\mathbf{K}/\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, P_1 \notin \mathbf{K}, P_2 \in \mathbf{K}\} \cup \{\mathbf{K}/\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, P_1 \notin \mathbf{K}, P_2 \notin \mathbf{K}\} \end{aligned} \tag{9}$$

Taking the measures we write

$$\begin{aligned} \mu(\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset) = \ &\mu(\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, P_1 \in \mathbf{K}, P_2 \in \mathbf{K}) \\ &+\mu(\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, P_1 \in \mathbf{K}, P_2 \in \mathbf{K}) \\ &+\mu(\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, P_1 \notin \mathbf{K}, P_2 \in \mathbf{K}) + \mu(\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, P_1 \notin \mathbf{K}, P_2 \notin \mathbf{K}) \end{aligned}$$

Now $\mathbf{K}$ is a convex body, hence

$$\mu(\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, P_1 \in \mathbf{K}, P_2 \in \mathbf{K}) = \mu\left(\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, \overline{P_1P_2} \subset \mathbf{K}\right).$$

We have

$$\begin{aligned} \{\mathbf{K}/\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, P_1 \in \mathbf{K}\} = \ &\{\mathbf{K}/\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, P_1 \in \mathbf{K}, P_2 \in \mathbf{K}\} \\ &\cup\{\mathbf{K}/\mathbf{K}\cap\mathbf{K}_0 \neq \emptyset, P_1 \in \mathbf{K}, P_2 \notin \mathbf{K}\}. \end{aligned}$$

Then

$$\mu(\mathbf{K}\cap\mathbf{K}_0\neq\emptyset, P_1\in\mathbf{K}, P_2\notin\mathbf{K}) = \mu(\mathbf{K}\cap\mathbf{K}_0\neq\emptyset, P_1\in\mathbf{K}) - \mu\big(\mathbf{K}\cap\mathbf{K}_0\neq\emptyset, \overline{P_1P_2}\subset\mathbf{K}\big); \tag{10}$$

We can write

$$\mu(\mathbf{K}\cap\mathbf{K}_0\neq\emptyset, P_1\notin\mathbf{K}, P_2\notin\mathbf{K}) = \mu(\mathbf{K}\cap\mathbf{K}_0\neq\emptyset) - \mu(\mathbf{K}\cap\mathbf{K}_0\neq\emptyset, P_1\in\mathbf{K})$$

$$-\mu(\mathbf{K}\cap\mathbf{K}_0\neq\emptyset, P_2\in\mathbf{K}) + \mu\big(\mathbf{K}\cap\mathbf{K}_0\neq\emptyset, \overline{P_1P_2}\subset\mathbf{K}\big).$$

In order to compute the variance we consider the following integral

$$I = \int\limits_{\{\mathbf{K}_i\cap\mathbf{K}_0\neq\emptyset, P_1\in\mathbf{K}(r,m), P_2\in\mathbf{K}(r,m)\}} dP_1 dP_2 d\mathbf{K}_1 ... d\mathbf{K}_m \tag{11}$$

where $\mathbf{K}(r,m) := \mathbf{K}_0\cap\mathbf{K}_{i_1}...\cap\mathbf{K}_{i_r}$.

We have

$$\int\limits_{\{\mathbf{K}_i\cap\mathbf{K}_0\neq\emptyset\}} d\mathbf{K}_1...d\mathbf{K}_m \int\limits_{\{P_1\in\mathbf{K}(r,m)\}} dP_1 \int\limits_{\{P_2\in\mathbf{K}(r,m)\}} dP_2 = \int\limits_{\{\mathbf{K}_i\cap\mathbf{K}_0\neq\emptyset\}} \eta^2(r,m) d\mathbf{K}_1...d\mathbf{K}_m. \tag{12}$$

That means

$$I = \int\limits_{\{\overline{P_1P_2}\subset\mathbf{K}_0\}} dP_1 dP_2 \int\limits_D d\mathbf{K}_1...d\mathbf{K}_m, \tag{13}$$

where

$$D = \{\{\mathbf{K}_1,\dots,\mathbf{K}_m\}/\mathbf{K}_i\cap\mathbf{K}\neq\emptyset, P_1\in\mathbf{K}_{j_1},\dots,\mathbf{K}_{j_r}, P_1\notin\mathbf{K}_{j_{r+1}},\dots,\mathbf{K}_{j_m},$$

$$P_2\in\mathbf{K}_{h_1},...\mathbf{K}_{h_r}, P_2\notin\mathbf{K}_{h_{r+1}},\dots,\mathbf{K}_{h_m}\},$$

with $\{j_1,\dots,j_m\}$ and $\{h_1,\dots,h_m\}$ permutations of $\{1,\dots,m\}$.

By hypothesis $\mathbf{K}_1,\dots,\mathbf{K}_m$ are independent and equal to an ovaloid $\mathbf{K}$, hence

$$\int\limits_D d\mathbf{K}_1...d\mathbf{K}_m = \binom{m}{r}\sum_{j=0}^{r}\binom{r}{j}\binom{m-r}{r-j}$$

$$\times\Bigg(\int\limits_{\{\overline{P_1P_2}\subset K\}} d\mathbf{K}\Bigg)^j \Bigg(\int\limits_{\{P_1\in\mathbf{K}, P_2\notin\mathbf{K}\}} d\mathbf{K}\Bigg)^{r-j} \Bigg(\int\limits_{\{P_1\notin\mathbf{K}, P_2\in\mathbf{K}\}} d\mathbf{K}\Bigg)^{r-j} \Bigg(\int\limits_{\{P_1\notin\mathbf{K}, P_2\notin\mathbf{K}\}} d\mathbf{K}\Bigg)^{m-2r+j},$$

where if $2r - j > m$ we put

$$\binom{m-r}{r-j} = 0.$$

Taking into account the previous calculations we write

$$\int\limits_D d\mathbf{K}_1 ... d\mathbf{K}_m = \binom{m}{r} \sum_{j=0}^{r} \binom{r}{j} \binom{m-r}{r-j} \times \left[\mu(\mathbf{K}, \overline{P_1P_2})\right]^j$$

$$\left[8\pi^2 V - \mu(\mathbf{K}, \overline{P_1P_2})\right]^{2r-2j} \times \left[8\pi^2\left(V_0 - V\right) + 2\pi(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0) + \mu(\mathbf{K}, \overline{P_1P_2})\right]^{m-2r+j}.$$

This means that

$$I = \binom{m}{r} \sum_{j=0}^{r} \binom{r}{j} \binom{m-r}{r-j} \int\limits_{\left\{\overline{P_1P_2} \subset \mathbf{K}_0\right\}} \left[\mu(\mathbf{K}, \overline{P_1P_2})\right]^j \left[8\pi^2 V - \mu(\mathbf{K}, \overline{P_1P_2})\right]^{2r-2j}$$

$$\times \left[8\pi^2(V_0 - V) + 2\pi(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0) + \mu(\mathbf{K}, \overline{P_1P_2})\right]^{m-2r+j} dP_1 dP_2.$$

But [1]

$$[dP_1 dP_2] = |t_2 - t_1|^2 [d\mathbf{G} dt_1 dt_2], \tag{14}$$

where $\mathbf{G}$ is the line for P_1 and P_2, $d\mathbf{G}$ is the elementary measure of the lines in the Euclidean Space, t_1 (respectively t_2) the distance between P_1 (respectively P_2) and the projection O^* of the origin of coordinates O on the line $\mathbf{G}$. We denote with α and β the distances between the intersection of $\mathbf{G}$ with $\mathbf{K}_0$ and O^* (as in figure)

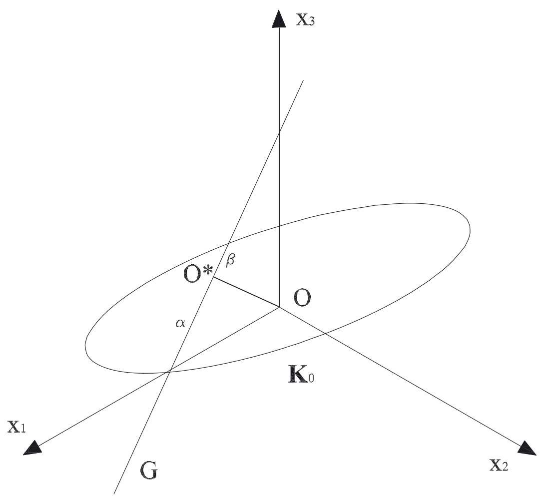

Fig. 2.

$$I = \binom{m}{r} \sum_{j=0}^{r} \binom{r}{j} \binom{m-r}{r-j} \int_{\{\mathbf{G}\cap\mathbf{K}_0\neq\emptyset\}} d\mathbf{G} \int_a^\beta \int_a^\beta [\mu(\mathbf{K}, |t_2 - t_1|)]^j$$

$$\times \big[8\pi^2 V - \mu(\mathbf{K}, |t_2 - t_1|)\big]^{2r-2j}$$

$$\big[8\pi^2(V_0 - V) + 2\pi\big(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\big) + \mu(\mathbf{K}, |t_2 - t_1|)\big]^{m-2r+j} |t_2 - t_1|^2 dt_1 dt_2.$$

We compute

$$J = \int_{t_2}^{\beta} [\mu(\mathbf{K}, |t_2 - t_1|)]^j \big[8\pi^2 V - \mu(\mathbf{K}, |t_2 - t_1|)\big]^{2r-2j}$$

$$\times \big[8\pi^2(V_0 - V) + 2\pi\big(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\big) + \mu(\mathbf{K}, t_1 - t_2)\big]^{m-2r+j} (t_1 - t_2)^2 dt_1$$

$$+ \int_a^{t_2} [\mu(\mathbf{K}, t_2 - t_1)]^j \big[8\pi^2 V - \mu(\mathbf{K}, t_2 - t_1)\big]^{2r-2j}$$

$$\times \big[8\pi^2(V_0 + V) + 2\pi\big(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\big) + \mu(K, t_1 - t_2)\big]^{m-2r+j} (t_2 - t_1)^2 dt_1$$

$$= \int_0^{\beta - t_2} [\mu(\mathbf{K}, u)]^j \big[8\pi^2 V - (\mathbf{K}, u)\big]^{2r-2j}$$

$$\times \big[8\pi^2(V_0 - V) + 2\pi\big(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\big) + \mu(\mathbf{K}, u)\big]^{m-2r+j} u^2 du$$

$$+ \int_0^{t_2 - a} [\mu(\mathbf{K}, u)]^j \big[8\pi^2 V - \mu(\mathbf{K}, u)\big]^{2r-2j}$$

$$\times \big[8\pi^2(V_0 - V) + 2\pi\big(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\big) + \mu(\mathbf{K}, u)\big]^{m-2r+j} u^2 du.$$

Putting

$$\int_0^{\xi} [\mu(\mathbf{K}, u)]^j \big[8\pi^2 V - \mu(\mathbf{K}, u)\big]^{2r-2j}$$

$$\times \big[8\pi^2(V_0 - V) + 2\pi\big(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\big) + \mu(\mathbf{K}, u)\big]^{m-2r+j} u^2 du = f_j(\mathbf{K}, m; \xi).$$

we write

(15) $$J = f_i(\mathbf{K}, m, r; \beta - t_2) + f_j(\mathbf{K}, m, r; t_2 - a),$$

Hence

$$\int_a^\beta J dt_2 = \int_a^\beta f_j(\mathbf{K}, m, r; \beta - t_2) dt_2 + \int_a^\beta f_j(\mathbf{K}, m, r; t_2 - a) dt_2 = \tag{16}$$

$$= 2\int_0^\lambda f_j(\mathbf{K}, m, r; v) dv, \tag{17}$$

where $\lambda = \beta - a$ is the length of the chord obtained as intersection between $\mathbf{K}_0$ and $\mathbf{G}$. We denote

$$\phi_j(\mathbf{K}, m, r; \lambda) = \int_0^\lambda f_j(\mathbf{K}, m, r; v) dv. \tag{18}$$

Using this positions we have that

$$I = 2\binom{m}{r} \sum_{j=0}^{r} \binom{r}{j} \binom{m-r}{r-j} \int_{\{\mathbf{G}\cap\mathbf{K}_0\neq\emptyset\}} \phi_j(\mathbf{K}, m, r; \lambda) d\mathbf{G}. \tag{19}$$

Then

$$\int_{\{\mathbf{K}_i\cap\mathbf{K}_0\neq\emptyset\}} \eta^2(r, m) d\mathbf{K} ... d\mathbf{K}_m = 2\binom{m}{r} \sum_{j=0}^{r} \binom{r}{j} \binom{m-r}{r-j} \int_{\{\mathbf{G}\cap\mathbf{K}_0\neq\emptyset\}} \phi_j(\mathbf{K}, m, r; \lambda) d\mathbf{G} \tag{20}$$

where

$$\phi_j(\mathbf{K}, m, r; \lambda) = \int_0^\lambda \int_0^v [\mu(\mathbf{K}, u)]^j \left[8\pi^2 V - \mu(\mathbf{K}, u)\right]^{2r-2j}$$
$$\times \left[8\pi^2(V_0 - V) + 2\pi\left(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\right) + \mu(\mathbf{K}, u)\right]^{m-2r+j} u^2 du dv.$$

Taking into account that the convex sets $\mathbf{K}_1, \dots, \mathbf{K}_m$ are independents,

$$\int_{\{\mathbf{K}_i\cap\mathbf{K}_0\neq\emptyset\}} d\mathbf{K}_1 ... d\mathbf{K}_m = \left(\int_{\{\mathbf{K}\cap\mathbf{K}_0\neq\emptyset\}} d\mathbf{K} \right)^m = \left[8\pi^2(V_0 + V) + 2\pi\left(S_0\overline{\mathbf{H}} + S\overline{\mathbf{H}}_0\right)\right]^m. \tag{21}$$

Definitively the expression of the mean value of $\eta^2(r, m)$ is:

$$E\left[\eta^2(r, m)\right] = \frac{\displaystyle\int_{\{\mathbf{K}_i\cap\mathbf{K}_0\neq\emptyset\}} \eta^2(r, m) d\mathbf{K}_1 ... d\mathbf{K}_m}{\displaystyle\int_{\{\mathbf{K}_i\cap\mathbf{K}_0\neq\emptyset\}} d\mathbf{K}_1 ... d\mathbf{K}_m} = \tag{22}$$

$$
(23) \qquad = \frac{2\binom{m}{r}\sum_{j=0}^{r}\binom{r}{j}\binom{m-r}{r-j}\int\limits_{\{\mathbf{G}\cap\mathbf{K}_0\neq\emptyset\}}\phi_j(\mathbf{K},m,r;\lambda)d\mathbf{G}}{\left[8\pi^2(V_0+V)+2\pi\left(S_0\overline{\mathbf{H}}+S\overline{\mathbf{H}}_0\right)\right]^m}.
$$

And so we have the assertion. □

In the sequel, we consider the case of two random variables.

THEOREM 2. – *Let $\mathbf{K}_0$ be a fixed ovaloid in $\mathbf{E}_3$ of volume V_0, area S_0 and with integral of mean curvature $\overline{\mathbf{H}}_0$. We denote with $\mathcal{F}=\{\mathbf{K}_1,\dots,\mathbf{K}_m\}$ a family of m independent, uniformly distributed ovaloids of volume V, area S such that $\mathbf{K}_i\cap\mathbf{K}_0\neq\emptyset$ and $\mathbf{K}_i=\mathbf{K}$ for $i=1,\dots,m$. We assume that $\mathbf{K}$ has integral of mean curvature $\overline{\mathbf{H}}$, that $\eta(\beta_1,m)$ is the random volume $\mathbf{V}ol(\mathbf{K}_0\cap\ldots\cap\mathbf{K}_{i_{\beta_1}})$ obtained intersecting $\beta_1\leq m$ ovaloids in $\mathcal{F}$ and that $\eta(\beta_2,m)$ is the random volume $\mathbf{V}ol(\mathbf{K}_{i_1}\cap\ldots\cap\mathbf{K}_{i_{\beta_2}})$ obtained intersecting $\beta_2\leq m$ ovaloids in $\mathcal{F}$, with $\beta_1\leq\beta_2$. Then*

$$
(24) \qquad E(\eta(\beta_1,m)\eta(\beta_2,m)) = \frac{2\binom{m}{\beta_1}\sum_{l=0}^{\beta_1}\binom{\beta_1}{l}\binom{m-\beta_1}{\beta_2-l}\int\limits_{\{\mathbf{G}\cap\mathbf{K}_0\neq\emptyset\}}\phi_l(\mathbf{K};m,\lambda)d\mathbf{G}}{\left[8\pi^2V_0+V+2\pi\left(S_0\overline{\mathbf{H}}+S\overline{\mathbf{H}}_0\right)\right]^m},
$$

with

$$
(25) \qquad \phi_l(\mathbf{K};m,\lambda)=\int_0^\lambda\int_0^v(\mathbf{K},u)\left[8\pi^2V-\mu(\mathbf{K},u)\right]^{\beta_1+\beta_2-2l}
$$

$$
\times\left[8\pi^2(V_0-V)+2\pi\left(S_0\overline{\mathbf{H}}+S\overline{\mathbf{H}}_0\right)+\mu(\mathbf{K},u)\right]^{m-\beta_1-\beta_2+l}u^2dudv,
$$

and where $d\mathbf{G}$ is the elementary measure of the lines in the 3-dimensional Euclidean space.

PROOF. – After a simple calculation

$$
(26) \qquad \int_D d\mathbf{K}_1\ldots d\mathbf{K}_m=\binom{m}{\beta_1}\sum_{l=0}^{\beta_1}\binom{\beta_1}{m}\binom{m-\beta_1}{\beta_2-l}
$$

$$
\times\Bigg(\int\limits_{\{\mathbf{K}\cap\mathbf{K}_0\neq\emptyset,P_1,P_2\in\mathbf{K}\}}d\mathbf{K}\Bigg)^l\Bigg(\int\limits_{\{\mathbf{K}\cap\mathbf{K}_0\neq\emptyset,P_1,P_2\notin\mathbf{K}\}}d\mathbf{K}\Bigg)^{\beta_1-l}
$$

$$\times\left(\int_{\{\mathbf{K}\cap\mathbf{K}_0\neq\emptyset, P_1\notin\mathbf{K}, P_2\in\mathbf{K}\}} d\mathbf{K}\right)^{\beta_2-l}\left(\int_{\{\mathbf{K}\cap\mathbf{K}_0\neq\emptyset, P_1\notin\mathbf{K}, P_2\notin\mathbf{K}\}} d\mathbf{K}\right)^{m-\beta_1-\beta_2+l},$$

where

$$\binom{m-\beta_1}{\beta_2-l}=0,$$

if $m-\beta_1<\beta_2-l$ and D is union of sets (disjoint sets) of type

$$\big\{\{\mathbf{K}_1,\dots,\mathbf{K}_m\}/\mathbf{K}_{j_1},\dots,\mathbf{K}_{j_{r_1}}\ni P_1;$$

$$P_1\notin\mathbf{K}_{j_{ri+1}},\dots,\mathbf{K}_{j_m}; P_2\in\mathbf{K}_{h_1},\dots,\mathbf{K}_{h_{r_2}}; P_2\notin\mathbf{K}_{hr_{1+1}},\dots,\mathbf{K}_{h_m}\big\},$$

where $\{j_1,\dots,j_m\}$ and $\{h_1,\dots,h_m\}$ are permutations of the indexes $\{l,\dots,m\}$.

Using Santaló's formulas and the fact that

$$\int_{\{\mathbf{K}_0\cap\mathbf{K}\neq\emptyset; P_1,P_2\in\mathbf{K}\}} dK=\mu\big(\mathbf{K},\overline{P_1P_2}\big),\tag{27}$$

we obtain

$$\int_D d\mathbf{K}_1\dots d\mathbf{K}_m=\binom{m}{\beta_1}\sum_{l=0}^{\beta_1}\binom{\beta_1}{l}\binom{m-\beta_1}{\beta_2-l}\left[\mu\big(\mathbf{K},\overline{P_1,P_2}\big)\right]^l\tag{28}$$

$$\times\left[8\pi^2V-\mu\big(\mathbf{K},\overline{P_1P_2}\big)\right]^{\beta_1+\beta_2-2l}\cdot\big[8\pi^2V_0+V+2\pi\big(S_0\overline{\mathbf{H}}+S\overline{\mathbf{H}}_0\big)$$

$$+\mu\big(\mathbf{K},\overline{P_1P_2}\,\big)\big]^{m-\beta_1-\beta_2+l},$$

then

$$I=\binom{m}{\beta_1}\sum_{l=0}\binom{\beta_1}{l}\binom{m-\beta_1}{\beta_2-l}\int_{\{P_1,P_2\in\mathbf{K}_0\}}\mu\big(\mathbf{K},\overline{P_1P_2}\big)\tag{29}$$

$$\times\left[8\pi^2V-\mu\big(\mathbf{K},\overline{P_1P_2}\big)\right]^{\beta_1+\beta_2-2l}\cdot\big[8\pi^2\big(V_0+V\big)$$

$$+2\pi\big(S_0\overline{\mathbf{H}}+S\overline{\mathbf{H}}_0\big)+\mu\big(\mathbf{K},\overline{P_1P_2}\big)\big]^{m-\beta_1-\beta_2-l}dP_1dP_2,$$

or

$$I=2\binom{m}{\beta_1}\sum_{l=0}^{\beta_1}\binom{\beta_1}{l}\binom{m-\beta_1}{\beta_2-l}\int_{\{\mathbf{G}\cap\mathbf{K}_0\neq\emptyset\}}\phi_l(\mathbf{K};m,\lambda)d\mathbf{G},\tag{30}$$

where

$$(31)\quad \phi_l(\mathbf{K};m\lambda)=\int_0^{\lambda}\int_0^{v}\mu(\mathbf{K},u)\big[8\pi^2V-\mu(\mathbf{K},u)\big]^{\beta_1+\beta_2-2l}$$

$$\times\big[8\pi^2(V_0-V)+2\pi\big(S_0\overline{\mathbf{H}}+S\overline{\mathbf{H}}_0\big)+\mu(\mathbf{K},u)\big]^{m-\beta_1-\beta_2+l}u^2dudv.\quad\square$$

The structure of the correlation coefficient is

$$\rho(\eta(\beta_1,m),\eta(\beta_2,m)):=\frac{E(\eta(\beta_1,m)\eta(\beta_2,m))-E(\eta(\beta_1,m))E(\eta(\beta_2,m))}{D(\eta(\beta_1,m))D(\eta(\beta_2,m))},$$

where

$$D(\eta(\beta_1,m)):=\sqrt{\sigma^2(\eta(\beta_1,m))}\quad\text{and}\quad D(\eta(\beta_2,m)):=\sqrt{\sigma^2(\eta(\beta_2,m))}.$$

3. – Application.

We assume now that the convex body $\mathbf{K}_0$ is a sphere Σ_0 of constat radius δ_0 and we take the following integral

$$J_n:=\int_{\{\mathbf{G}\cap\Sigma_0\neq\emptyset\}}\lambda^n d\mathbf{G},$$

where λ is the length of the chord obtained as intersection between the line $\mathbf{G}$ and $\mathbf{K}_0$ and where $n\in\mathbf{N}$.

Then

$$J_n=\frac{\pi^2(2\delta_0)^{n+2}}{(n+2)},$$

see Stoka's result in [6]. We take a family $\mathcal{F}=\{\mathbf{C}_1,\dots,\mathbf{C}_m\}$ of m independent, uniformly distributed cubes and $\mathbf{C}_i=\mathbf{C}$, where $\mathbf{C}$ is a cube of side a and such that $\mathbf{C}_i\cap\Sigma_0\neq\emptyset$ for $i=1,\dots,m$.

We want to compute

$$\int_{\{\mathbf{G}\cap\Sigma_0\neq\emptyset\}}\phi_j(\mathbf{C},m,r;\lambda)\,d\mathbf{G}.$$

By Buffon's problem we have that

$$\mu(\Sigma_0,l)=4\pi^2a^2-6\pi^2a^2l+8\pi al^2-\frac{\pi}{2}l^3.$$

Putting $\mathbf{l} := (l_1, l_2, l_3, l_4, l_5)$, $\mathbf{j} := (j_1, j_2, j_3, j_4)$ and $\mathbf{k} := (k_1, k_2, k_3, k_4, k_5, k_6)$ and fixing a positive integer $r \le m$ we have

$$J = \sum_{|\mathbf{j}|=j} \sum_{|\mathbf{l}|=2(r-j)} \sum_{|\mathbf{k}|=m-2r+j} \frac{\varepsilon(\mathbf{j}, \mathbf{l}, \mathbf{k}, a, r, \delta_0, m) v^s}{s},$$

where $s = j_2 + 2j_3 + 3j_4 + l_3 + 2l_4 + 3l_5 + k_4 + 2k_5 + 3k_6 + 3$,

$$\varepsilon(\mathbf{j}, \mathbf{l}, \mathbf{k}, a, r, \delta, m) := \frac{j!(2(r-j))!(m-2r+j)!}{j_1!j_2!j_3!j_4!l_1!l_2!l_3!l_4!l_5!k_1!k_2!k_3!k_4!k_5!k_6!} \Psi(a, \delta_0),$$

and

$$\Psi(a, \delta_0) := 2^\alpha 3^\beta \pi^\gamma a^\omega \left(\frac{4}{3}\pi\delta_0^3 - a^3\right)^{k_1} \left(4\pi\delta_0 a(3\pi\delta_0 + a^2)\right)^{k_2},$$

obtained putting

$$\begin{aligned} \alpha := -(j_4 + l_5 + k_6) + 3j_3 + j_2 + 2j_1 + 3l_1 + 2l_2 \\ + l_3 + 3l_4 + 3k_1 + k_2 + 2k_3 + k_4 + 3k_5, \end{aligned}$$

$$\beta := j_2 + l_3 + k_4,$$

$$\gamma := 2j_1 + 2j_2 + j_3 + j_4 + 2l_1 + 2l_2 + 2l_3 +$$

$$l_4 + l_5 + 2k_1 + k_2 + 2k_3 + 2k_4 + k_5 + k_6,$$

$$\omega := 2j_1 + 2j_2 + j_3 + 3l_1 + 2l_2 + 2l_3 + l_4 + 2k_3 + 2k_4 + k_5.$$

Then

$$\phi_j(\mathbf{C}; m, r, \lambda) = \sum_{|\mathbf{j}|=j} \sum_{|\mathbf{l}|=2(r-j)} \sum_{|\mathbf{k}|=m-2r+j} \frac{\varepsilon(\mathbf{j}, \mathbf{l}, \mathbf{k}, a, r, m)}{s(s+1)} \lambda^{s+1},$$

Finally we have the following

$$\int_{\{\mathbf{G} \cap \Sigma_0 \neq \emptyset\}} \phi_j(\mathbf{C}, m, r; \lambda) d\mathbf{G} = \sum_{|\mathbf{j}|=j} \sum_{|\mathbf{l}|=2(r-j)} \sum_{|\mathbf{k}|=m-2r+j} \frac{\varepsilon(\mathbf{j}, \mathbf{l}, \mathbf{k}, a, r, m)}{s(s+1)} \times \frac{\pi^2 (2\delta_0)^{s+3}}{(s+3)}.$$

From this computation we get, using the same notation of before, the following result

THEOREM 3. – *Let Σ_0 be a fixed sphere in $\mathbf{E}_3$ of constant radius δ_0. We denote with $\mathcal{F} = \{\mathbf{C}_1, \dots, \mathbf{C}_m\}$ a family of m independent, uniformly distributed cubes such that $\mathbf{C}_i \cap \Sigma_0 \neq \emptyset$, for $i = 1, \dots, m$ and where $\mathbf{C}$ is a cube of side a. If $\eta(r, m)$ is the random volume $\mathbf{V}ol(\mathbf{C} \cap \dots \cap \mathbf{C}_{i_r})$ obtained intersecting*

$r \leq m$ cubes in $\mathcal{F}$, the variance of this random variable is

$$\sigma^2(\eta(r,m)) = \binom{m}{r} \frac{\Theta(\Sigma_0, \mathbf{C}; r, m)}{8\left[\pi^2\left(\frac{4}{3}\pi\delta_0^3 + a^3\right) + 3\pi^2\delta_0(\delta_0\pi + 2a)a\right]^m}, \tag{32}$$

with

$$\begin{aligned}\Theta(\Sigma_0, \mathbf{C}_j; r, m) := {} & 2\sum_{j=0}\binom{r}{j}\binom{m-r}{r-j} \\ & \times\left(\sum_{|\mathbf{j}|=j}\ \sum_{|\mathbf{l}|=2(r-j)}\ \sum_{|\mathbf{k}|=m-2r+j} \frac{\varepsilon(\mathbf{j}, \mathbf{l}, \mathbf{k}, a, r, m)}{s(s+1)} \frac{\pi^2(2\delta_0)^{s+3}}{(s+3)}\right) \\ & - \binom{m}{r} \frac{(8\pi^2 a^3)^{2r}\left(\frac{32}{3}\pi^3\delta_0^3 + 24\pi^2(\pi\delta_0 + 2a)\delta_0 a\right)^{2m-2r}}{8\left[\pi^2\left(\frac{4}{3}\pi\delta_0^3 + a^3\right) + 3\pi^2\delta_0\left(\delta_0\pi + 2a\right)a\right]^m}\left(\frac{4}{3}\pi\delta_0^3\right)^2.\end{aligned}$$

Other results about random variables arising from intersection problems between convex bodies in the 3-dimensional Euclidean Space are investigated in [3].

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