# Bollettino Unione Matematica Italiana



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# Interior $C^{1,a}$-Regularity of Weak Solutions to the Equations of Stationary Motions of Certain Non-Newtonian Fluids in Two Dimensions.

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**Sunto.** – *Si dimostra l'hölderianitá del gradiente di ogni soluzione debole di un sistema di equazioni degenerate, che descrivono il moto di un fluido incomprimibile non-newtoniano in due dimensioni, sotto condizioni usuali di monotonia e di andamento all'infinito di ordine* $q-1\,(1<q<2)$.

**Summary.** – *In the present work we prove the interior Hölder continuity of the gradient matrix of any weak solution of equations, which describes the motion of non-Newtonian fluid in two dimensions, restricting ourself to the shear thinning case* $1<q<2$.

## 1. – Introduction. Statement of the main result.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. The stationary motion of an incompressible fluid through $\Omega$ is governed by the following two equations

$$-\operatorname{div} S + u\cdot\nabla u = -\nabla p + f \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

where

$$\begin{aligned} S &= \{S_{ij}\} = \text{deviatoric stress tensor } (^1), \\ p &= \text{pressure}, \\ u &= \{u_1, u_2\} = \text{velocity}, \\ f &= \{f_1, f_2\} = \text{external force}. \end{aligned}$$

(1) Throughout Latin subscripts take the values 1, 2. Repeated subscripts imply summation over 1, 2.

Assuming the condition of adherence on the boundary of $\Omega$ we have

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

In addition $S$ may depend on the «rate of strain tensor» $D = \{D_{ij}\}$, which is defined by

$$D_{ij} = D_{ij}(u) := \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i,j = 1,2.$$

(cf. [2], [3], [10]).

In the present paper we consider constitutive laws of the following both types

$$S = \upsilon(D_{II})^{(q-2)/2} D, \quad 1 < q < 2$$
$$S = \upsilon(1 + D_{II})^{(q-2)/2} D, \quad 1 < q < 2 \quad (\upsilon = \text{const} > 0)$$

as special cases, where

$$D_{II} = \frac{1}{2} D_{ij} D_{ij} = \text{second invariant of D}$$

(cf. [2], [4], [14]) . A fluid which is determined by the first of these constitutive laws is said «pseudoplastic» or «shear thinning». This motivates us to impose the following conditions on $S$:

$$S_{ij} \in C(M^4_{\text{sym}}) \cap C^1(\mathbf{M}^4_{\text{sym}} \setminus \{0\})\,(^2),$$

$$\sum_{i,j,k,l=1}^{2} \left| \frac{\partial S_{ij}}{\partial \xi_{kl}}(\xi) \right| \le c_0(\mu + \|\xi\|)^{q-2} \quad \forall\, \xi \in \mathbf{M}^4_{\text{sym}} \setminus \{0\}; \tag{1.4}$$

$$\frac{\partial S_{ij}}{\partial \xi_{kl}}(\xi)\eta_{kl}\eta_{ij} \ge \upsilon_0(\mu + \|\xi\|)^{q-2}\|\eta\|^2 \quad \forall\, \xi, \eta \in \mathbf{M}^4_{\text{sym}} \setminus \{0\}, \tag{1.5}$$

($c_0 > 0, \upsilon_0 > 0$). Here $1 < q < 2$ and $\mu \ge 0$ are fixed numbers.

Clearly, (1.4) implies

$$\|S(\xi)\| \le \frac{c}{q-1}\,(\mu + \|\xi\|)^{q-1} + \|S(0)\| \quad \forall\, \xi \in M^4_{\text{sym}}. \tag{1.6}$$

*Weak solution to* (1.1), (1.2). By $W^{1,\sigma}(\Omega)$ and $W^{1,\sigma}_0(\Omega)\,(1 \le \sigma < +\infty)$ we denote the usual Sololev spaces.

DEFINITION 1.1. – *Assume* (1.6). *Let* $f \in [L^2(\Omega)]^2$. *A vector-valued function* $u \in [W^{1,q}(\Omega)]^2$ *with* $\operatorname{div} u = 0$ *is called a weak solution to* (1.1), (1.2) *if the fol-*

($^2$) $\mathbf{M}^4_{\text{sym}}$ = vector space of all symmetric $2 \times 2$ matrices $\xi = \{\xi_{ij}\}$; $\|\xi\| := (\xi_{ij}\,\xi_{ij})^{1/2}$. By $|a|$ we denote the Euclidean norm of $a \in \mathbb{R}^2$.

*lowing integral identity is fulfilled for all* $\varphi \in [C_c^\infty(\Omega]^2$ *with* $\operatorname{div}\varphi = 0$:

$$\int_\Omega S_{ij}(D(u))D_{ij}(\varphi)\,\mathrm{d}x + \int_\Omega u_i \frac{\partial u_j}{\partial x_i}\varphi_j\,\mathrm{d}x = \int_\Omega f_j\varphi_j\,\mathrm{d}x. \tag{1.7}$$

Our main result is the following

THEOREM. *Assume* (1.4), (1.5). *Let* $u \in [W^{1,q}(\Omega)]^2$ *be a weak solution to* (1.1), (1.2). *Suppose*

$$\text{there exists } \sigma > 2: \quad \tilde f = f - u\cdot\nabla u \in [L^\sigma_{\mathrm{loc}}(\Omega)]^2. \tag{1.8}$$

*Then there exists a number* $0 < \alpha < 1$ *such that*

$$u \in [C^{1,\alpha}(\Omega)]^2. \tag{1.9}$$

□

In the case $\frac{3}{2} < q < 2$ the existence of the second weak derivatives in $L^s_{\mathrm{loc}}(\Omega)\,(q \le s < 2)$ of any weak solution $u \in [W^{1,q}(\Omega)]^2$ to (1.1), (1.2) has been proved in [13]. Thus, by means of Sobolev's imbedding theorem it is readily seen that the assumption (1.8) in the Theorem is always fulfilled in this particular case. Therefore replacing this condition by

$$\frac{3}{2} < q < 2, \quad f \in [L^\sigma_{\mathrm{loc}}(\Omega)]^2 \quad (\sigma > 2) \tag{1.8$'$}$$

implies the

COROLLARY I. *Assume* (1.4), (1.5) *and* (1.8′). *Let* $u \in [W^{1,q}(\Omega)]^2$ *be a weak solution to* (1.1), (1.2). *Then* $\frac{\partial u_j}{\partial x_i}$ $(i,j = 1,2)$ *are* (*locally*) *Hölder continuous functions.*

□

In addition neglecting the convective term and making use of [11], where the author has proved the existence of the second derivatives for those weak solutions we also have

COROLLARY II. *Assume* (1.4), (1.5) *and* $f \in [L^\sigma_{\mathrm{loc}}(\Omega)]^2$ $(\sigma > 2)$. *Let* $u \in [W^{1,q}(\Omega)]^2$ *satisfy the identity*

$$\int_\Omega S_{ij}(D(u))D_{ij}(\varphi)\,\mathrm{d}x = \int_\Omega f_j\varphi_j\,\mathrm{d}x \tag{1.7$'$}$$

*for all* $\varphi \in [C_c^\infty(\Omega]^2$ *with* $\operatorname{div}\varphi = 0$. *Then* $\frac{\partial u_j}{\partial x_i}$ $(i,j = 1,2)$ *are* (*locally*) *Hölder continuous functions.* □

REMARK 1. – The existence of a strong solution $u \in [C^{1,a}(\Omega)]^2$ $\left(q > \frac{6}{5}\right)$ resp. $u \in [C^{1,a}(\overline{\Omega})]^2$ $\left(q > \frac{3}{2}\right)$ to the system (1.1)-(1.3) has been proved by Kaplický, Málek, Stará in [9] imposing conditions on the constitutive law which are slightly more restrictive then ours. In contrast to [9] here the boundary condition (1.3) does not play an essential role for the proof of the Hölder continuity of $\nabla u$. Therefore our result is applicable to weak solutions to (1.1), (1.2) fulfilling, instead of (1.3), any boundary condition.

2. The Hölder continuity of $\nabla u$, where $u$ is any weak solution of an elliptic system with coefficients satisfying conditions similar to (1.4), (1.5) has been proved in [12] via higher integrability of the function $(1 + |\nabla u|)^{(q-2)/2}\nabla^2 u$ (3). In order to achieve the necessary reverse Hölder inequality in the appendix of this paper, based on the well known Poincaré inequality, the authors show that there exists $\Lambda \in \mathbb{R}^4$ such that

$$\int_{B_r} (1 + |\nabla u| + |\nabla u - \Lambda|)^{q-2} |\nabla u - \Lambda|^2 \,\mathrm{d}x \;\le\; c\,r^2 \int_{B_r} (1 + |\nabla u|)^{q-2} |\nabla^2 u|^2 \,\mathrm{d}x,$$

where $c = \text{const} > 0$ depends only on $q$.

Unfortunately such an argument seems not to work equally if one replaces the gradient $\nabla u$ by the symmetric gradient $D(u)$. However in the present paper by an entirely different method we are able to establish an appropriate reverse Hölder inequality which will imply the higher integrability of $(\mu + |D(u)|)^{(q-2)/2}\nabla D(u)$ $(\mu \ge 0)$ (cf. section 4) based on two different Caccioppoli inequalities (cf. Theorem 3.1 resp. Theorem 3.2) and the Poincaré inequality.

In the appendix of our paper, dealing with the special case $\mu = 0$, we prove that $\|D(u)\|^{(q-2)/2}\nabla D(u) \in [L^2(B_r)]^8$ for each $B_r \subset\subset \Omega$. In addition we derive an appropriate estimate of the $L^2(B_r)$- norm of $\|D(u)\|^{(q-2)/2}\nabla D(u)$ which is needed in section 3.

## 2. – Preliminaries.

This section is devoted to some preliminary lemmas which will be used in the following sections.

LEMMA 2.1. – *Let $N$ be an integer $\ge 1$. Let $0 < a < 1$. Then the following inequality holds for all $\xi, \eta \in \mathbb{R}^N \setminus \{0\}$*

$$(\mu + |\xi| + |\eta|)^{-a} \;\le\; \int_0^1 (\mu + |\xi + t\eta|)^{-a} \,\mathrm{d}t \;\le\; \frac{3^a}{1-a} (\mu + |\xi| + |\eta|)^{-a}. \tag{2.1}$$

(3) Here $\nabla^2 u$ denotes the matrix of second derivatives of $u$.

PROOF. – The first inequality of (2.1) is trivially fulfilled. It only remains to prove the second.

First let us consider the case $\max\{|\xi|,|\eta|\} \le \mu$. Here we easily see that

$$\int_0^1 (\mu+|\xi+t\eta|)^{-a}\,\mathrm{d}t \;\le\; 3^a(\mu+\mu+\mu)^{-a} \;\le\; 3^a(\mu+|\xi|+|\eta|)^{-a}.$$

Next, assume that $|\xi| \le |\eta|$ and $|\eta| > \mu$. Set $\tau := \dfrac{|\xi|}{|\eta|}$. Then $0 < \tau \le 1$ and we estimate

$$\begin{aligned}\int_0^1 (\mu+|\xi+t\eta|)^{-a}\,\mathrm{d}t &\le \int_0^\tau (\mu+(\tau-t)|\eta|)^{-a}\,\mathrm{d}t + \int_\tau^1 (\mu+(t-\tau)|\eta|)^{-a}\,\mathrm{d}t\\ &= \frac{1}{(1-a)|\eta|}\Big((\mu+\tau|\eta|)^{1-a} + (\mu+(1-\tau)|\eta|)^{1-a} - 2\mu\Big)\\ &\le \frac{1}{(1-a)}|\eta|^{-a} \;\le\; \frac{3^a}{1-a}(\mu+|\xi|+|\eta|)^{-a}.\end{aligned}$$

Finally, we consider the case $|\eta| \le |\xi|$ and $|\xi| > \mu$. Here we estimate

$$\begin{aligned}\int_0^1 (\mu+|\xi+t\eta|)^{-a}\,\mathrm{d}t &\le \int_0^1 (\mu+(1-t)|\xi|)^{-a}\,\mathrm{d}t\\ &= \frac{1}{(1-a)|\xi|}(\mu+|\xi|)^{1-a} - \mu)\\ &\le \frac{1}{(1-a)}|\xi|^{-a} \;\le\; \frac{3^a}{1-a}(\mu+|\xi|+|\eta|)^{-a}.\end{aligned}$$

LEMMA 2.2. – *Let* $\phi : [a,b] \to [0,+\infty[$ $(-\infty < a < b < +\infty)$ *be bounded. Assume that there are constants* $A, B, a$ *and* $0 < \varepsilon < 1$, *such that*

(2.2) $$\phi(\rho) \le A(R-\rho)^{-a} + B + \varepsilon\phi(R) \quad \forall\, a \le \rho < R \le b.$$

*Then there exists a positive constant* $c = c(a,\varepsilon)$ *such that*

(2.3) $$\phi(\rho) \le c\,(A(R-\rho)^{-a} + B)) \quad \forall\, a \le \rho < R \le b.$$

For the proof of Lemma 2.3 see for instance [6, chap. III]. □

## 3. – Caccioppoli-type inequalities.

The aim of this section is the proof of two Caccioppoli inequalities, which play an essential role for the proof of the Theorem. To begin with, for $\mu \ge 0$ we define

$$\begin{cases} V_\mu(\xi) := (\mu+\|\xi\|)^{(q-2)/2} & \text{for} \quad \xi \in \mathbf{M}^4_{\mathrm{sym}} \setminus \{0\}\\ V_\mu(0) := \mu^{(q-2)/2} & \text{if} \quad \mu > 0,\\ V_\mu(0) := 0 & \text{if} \quad \mu = 0.\end{cases}$$

THEOREM 3.1. – *Let* $u \in [W^{1,q}(\Omega)]^2$ *be a weak solution to the equations* (1.1), (1.2). *Let all assumptions of the Theorem be fulfilled. Then*

$$V_\mu(\|D(u)\|)\frac{\partial^2 u_i}{\partial x_k \partial x_l} \in L^2_{\rm loc}(\Omega) \quad (i,k,l=1,2) \tag{3.1}$$

*and for each* $x_0 \in \Omega, 0 < r < \operatorname{dist}(x_0, \partial\Omega)$ *and* $\lambda \in \mathbb{R}^2 \times \mathbb{R}^2\,(\lambda^+ \neq 0)$ *there holds*

$$\begin{aligned}\sum_{k=1}^{2} \int\limits_{B_{r/2}} V_\mu^2 \Big\| \frac{\partial}{\partial x_k} D(u) \Big\|^2 \mathrm{d}x \\ \le \frac{c}{r^2}\,(\mu + \|\lambda^+\|)^{\frac{q(q-2)}{q-1}} \Bigg( \fint\limits_{B_r} \|D(u)\|^q \, \mathrm{d}x \Bigg)^{\frac{2-q}{q(q-1)}} \int\limits_{B_r} |\nabla u - \lambda|^2 \, \mathrm{d}x \;(^4) \\ + c \int\limits_{B_r} \Big( 1 + \|D(u)\|^{(2-q)\frac{\sigma+2}{\sigma-2}} + |\Delta u|^{(2+\sigma)/\sigma} + |\tilde{f}|^{(2+\sigma)/2} \Big) \, \mathrm{d}x,\end{aligned} \tag{3.2}$$

*where*

$$\lambda^+_{ij} := \frac{1}{2}(\lambda_{ij} + \lambda_{ji}) \quad (i,j=1,2),$$

*and* $c = \mathrm{const} > 0$ *depending on* $c_0/\nu_0$ *and* $q$ *only.*

To prove Theorem 3.1 we will make essential use of the following two lemmas, which can be proved by an elementary calculus.

LEMMA 3.1. – *For each function* $w \in [W^{2,1}(\Omega)]^2$ *there holds*

$$\Big| \frac{\partial^2 w_i}{\partial x_k \partial x_l} \Big| \le \sum_{j=1}^{2} \Big\| \frac{\partial}{\partial x_j} D(w) \Big\| \quad a.e.\ in \quad \Omega \quad (i,k,l=1,2). \tag{3.3}$$

PROOF. – Assertion (3.3) immediately follows from the the following equations:

$$\frac{\partial^2 w_1}{\partial x_1^2} = \frac{\partial}{\partial x_1} D_{11}(w), \frac{\partial^2 w_1}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_2} D_{11}(w),$$

$$\frac{\partial^2 w_2}{\partial x_2^2} = \frac{\partial}{\partial x_2} D_{22}(w), \frac{\partial^2 w_2}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1} D_{22}(w),$$

(4) For $A \subset \mathbb{R}^2$, $A$ being Lebesgue measurable and $\varphi \in L^1(A)$ define.

$$\fint\limits_A \varphi \, \mathrm{d}x = \varphi_A = \frac{1}{\mathrm{mes}(A)} \int\limits_A \varphi \, \mathrm{d}x.$$

$$\frac{\partial^2 w_1}{\partial x_2^2} = \frac{\partial^2 w_1}{\partial x_2^2} + \frac{\partial^2 w_2}{\partial x_2 \partial x_1} - \frac{\partial^2 w_2}{\partial x_2 \partial x_1} = 2\frac{\partial}{\partial x_2} D_{12}(w) - \frac{\partial}{\partial x_1} D_{22}(w),$$

$$\frac{\partial^2 w_2}{\partial x_1^2} = \frac{\partial^2 w_2}{\partial x_1^2} + \frac{\partial^2 w_1}{\partial x_2 \partial x_1} - \frac{\partial^2 w_1}{\partial x_2 \partial x_1} = 2\frac{\partial}{\partial x_1} D_{12}(w) - \frac{\partial}{\partial x_2} D_{11}(w).$$

□

LEMMA 3.2. – *There exists a positive constant c, such that for any given ball $B_R = B_R(x_0) \subset \mathbb{R}^2$ we have*

$$(3.4) \qquad \begin{cases} \textit{for every} \quad v \in W^{1,2}(B_R(x_0)): \\ \left( \fint_{B_R} |v|^{q'} \,\mathrm{d}x \right)^{2/q'} \le c\, \varepsilon^{\frac{q-2}{q-1}} \fint_{B_R} |v|^2 \,\mathrm{d}x + \varepsilon \int_{B_R} |\nabla v|^2 \,\mathrm{d}x \quad \forall \varepsilon > 0. \end{cases}$$

PROOF. – 1) First, let us consider the case $x_0 = 0$ and $R = 1$. By the aid of Sobolev's embedding theorem we find

$$(3.5) \qquad \|v\|_{L^{2q'}(B_1)} \le c(q) \left( \|v\|^2_{L^2(B_1)} + \|\nabla v\|^2_{[L^2(B_1)]^2} \right)^{1/2} \quad \forall\, v \in W^{1,2}(B_1).$$

Set $\theta := 2 - q$. Then $0 < \theta < 1$ and

$$\frac{1-\theta}{2} + \frac{\theta}{2q'} = \frac{1}{q'}.$$

Thus, by interpolation, making use of (3.5) and Young's inequality applied with $\varepsilon > 0$ arbitrarily chosen gives

$$\begin{aligned} \|v\|^2_{L^{q'}(B_1)} &\le \|v\|^{2(1-\theta)}_{L^2(B_1)} \|v\|^{2\theta}_{L^{2q'}(B_1)} \\ &\le c\, \|v\|^{2(1-\theta)}_{L^2(B_1)} \left( \|v\|^2_{L^2(B_1)} + \|\nabla v\|^2_{[L^2(B_1)]^2} \right)^{\theta} \\ &\le c\, \varepsilon^{-\theta/(1-\theta)} \|v\|^2_{L^2(B_1)} + \varepsilon \|\nabla v\|^2_{[L^2(B_1)]^2} \\ &= c\, \varepsilon^{(q-2)/(q-1)} \|v\|^2_{L^2(B_1)} + \varepsilon \|\nabla v\|^2_{[L^2(B_1)]^2}. \end{aligned}$$

2) Second, for any given ball $B_R(x_0) \subset \mathbb{R}^2$ the estimate (3.4) easily follows from 1) using an elementary rescaling argument. □

PROOF OF THEOREM 3.1. The proof is divided into four steps

1° *Interior differentiability of the weak solution.* Let $\lambda \in \mathbb{R}^2 \times \mathbb{R}^2$ ($\lambda^+ \neq 0$) be fixed. Taking into account (1.8) from [11] it follows that the second weak derivatives exist and

$$(1 + \|D(u)\|)^{(q-2)/2} \frac{\partial}{\partial x_k} D(u) \in [L^2_{\mathrm{loc}}(\Omega)]^4 \quad (k = 1, 2).$$

Then by $\lambda^+ \neq 0$,

$$(\mu + \|D(u)\| \; + \; \|D(u) - \lambda^+\|)^{q-2} \frac{\partial}{\partial x_k} D(u) \in [L^2_{\mathrm{loc}}(\Omega)]^4 \quad (k = 1, 2)\ (^5).$$

Using the product- and chain rule gives

$$\begin{aligned} &\frac{\partial}{\partial x_k}(\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q-2}(D(u) - \lambda^+) \\ &= (\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q-2} \frac{\partial}{\partial x_k} D(u) \\ &+ (q-2)\,(\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q-3}(D(u) - \lambda^+) \\ &\times \Big(\frac{\partial}{\partial x_k} D_{ij}(u)\Big)\Big(\frac{D_{ij}(u)}{\|D(u)\|} + \frac{D_{ij}(u) - \lambda^+_{ij}}{\|D(u) - \lambda^+\|}\Big) \end{aligned}$$

a.e. in $\Omega$. Hence

$$(\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q-2}(D(u) - \lambda^+) \in [W^{1,2}_{\mathrm{loc}}(\Omega)]^4, \tag{3.6}$$

and

$$\begin{aligned} &\Big\|\frac{\partial}{\partial x_k}(\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q-2}(D(u) - \lambda^+)\Big\| \\ &\qquad \le 2\,(\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q-2}\Big\|\frac{\partial}{\partial x_k} D(u)\Big\| \end{aligned} \tag{3.7}$$

a.e. in $\Omega$ $(k = 1, 2)$.

2° *Local existence of the pressure and interior pressure estimates.* Let $x_0 \in \Omega, 0 < R < \mathrm{dist}\,(x_0, \partial\Omega)$ and $\lambda \in \mathbb{R}^2 \times \mathbb{R}^2$ with $\lambda^+ \neq 0$ be arbitrarily chosen. Consulting [5; Th. III 3.1, Th. III 5.2] one gets a pressure $\hat{p} \in L^{q'}(B_R)/\mathbb{R}$, such that for any $p \in \hat{p}$,

$$\int_{B_R} (S_{ij}(D(u)) - S_{ij}(\lambda^+))D_{ij}(\varphi)\,\mathrm{d}x = \int_{B_R} \tilde{f}_j \varphi_j \,\mathrm{d}x + \int_{B_R} p \,\mathrm{div}\,\varphi \,\mathrm{d}x \tag{3.8}$$

for all $\varphi \in [\overset{\circ}{W}{}^{1,q}(B_R)]^2$. In addition, observing

$$(q^*)' = \frac{q^*}{q^* - 1} = \frac{2q}{3q - 2} < 2$$

(5) Here we have made use of the inequality

$$\mu + \|D(u)\| + \|D(u) - \lambda^+\| \ge \frac{1}{2}(\|\lambda^+\| + \|D(u)\|).$$

it follows

$$\|p - p_{B_R}\|_{L^{q'}(B_R)} \le c\left(\sum_{i,j=1}^{2}\|S_{ij}(D(u)) - S_{ij}(\lambda^+)\|_{L^{q'}(B_R)} + \|\tilde{f}\|_{L^{2q/(3q-2)}(B_R)}\right) \ \forall p \in \hat{p}\ (^6).$$

Then by the aid of (1.4) applying Lemma 2.1 and Hölder's inequality gives

$$(3.9)\quad \begin{aligned} &\left(\int_{B_R} |p - p_{B_R}|^{q'}\,\mathrm{d}x\right)^{2/q'} \\ &\le c\left(\int_{B_R}\left[(\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{(q-2)}\|D(u) - \lambda^+\|\right]^{q'}\mathrm{d}x\right)^{2/q'} \\ &\quad + cR^{4/q'}\int_{B_R}|\tilde{f}|^2\,\mathrm{d}x \quad \forall p \in \hat{p} \end{aligned}$$

(cf. [5], [11]). Taking into account (3.7), the first integral on the right of the last inequality may be estimated by (3.4) (with $v = (\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q-2}(D_{ij}(u) - \lambda^+_{ij})\,(ij = 1,2)$). Thus for each $\varepsilon > 0$,

$$(3.10)\quad \begin{aligned} &\left(\fint_{B_R} |p - p_{B_R}|^{q'}\,\mathrm{d}x\right)^{2/q'} \\ &\le c\,\varepsilon^{\frac{q-2}{q-1}}\fint_{B_R}(\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{2(q-2)}\|D(u) - \lambda^+\|^2\,\mathrm{d}x \\ &\quad + c\,\varepsilon\sum_{k=1}^{2}\int_{B_R}(\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{2(q-2)}\left\|\frac{\partial}{\partial x_k}D(u)\right\|^2\mathrm{d}x \\ &\quad + c\int_{B_R}|\tilde{f}|^2\,\mathrm{d}x. \end{aligned}$$

On the other hand again appealing to [5], [11] the following estimate is true for every $p \in \hat{p}$

$$\|p - p_{B_R}\|_{L^2(B_R)} \le c\left(\sum_{i,j=1}^{2}\|S_{ij}(D(u)) - S_{ij}(\lambda^+)\|_{L^2(B_R)} + R\|\tilde{f}\|_{L^2(B_R)}\right).$$

($^6$) To this end, $c$ denotes a constant which may change its numerical value from line to line, but depends neither on $R$ nor on $u$.

Then taking into account (1.4) we obtain

$$
\begin{aligned}
&\int\limits_{B_R} (p - p_{B_R})^2 \,\mathrm{d}x \\
(3.11) \qquad &\leq c \int\limits_{B_R} (\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{2(q-2)} \|D(u) - \lambda^+\|^2 \,\mathrm{d}x \\
&+ c R^2 \int\limits_{B_R} |\tilde{f}|^2 \,\mathrm{d}x.
\end{aligned}
$$

3° Let $x_0 \in \Omega$ and $0 < r < \dfrac{1}{2}\operatorname{dist}(x_0, \partial\Omega)$ be fixed. Let $\frac{r}{2} \leq \rho < R \leq r$ be arbitrarily chosen. Let $\zeta \in C_c^\infty(B_R)$ be a cut-off function for the ball $B_\rho$, i.e. $0 \leq \zeta \leq 1$ in $B_R, \zeta \equiv 1$ on $B_\rho$, such that

$$
\left|\frac{\partial \zeta}{\partial x_i}\right| \leq \frac{c}{R-\rho}, \quad \left|\frac{\partial^2 \zeta}{\partial x_i \partial x_j}\right| \leq \frac{c}{(R-\rho)^2} \quad \text{in} \quad B_R \quad (i,j = 1,2)
$$

($c =$ const independent of $r$). It is readily seen that

$$
\varphi_j = \Delta_{k,-h}(\zeta^2((\Delta_{k,h} u_j) - \lambda_{jk} h)) \quad (0 < h < r, j = 1,2)\ (^7)
$$

is an admissible test function in (3.8). Then inserting $\varphi$ into (3.8) and applying the transformation formula of the Lebesgue integral gives

$$
\begin{aligned}
&\int\limits_{B_R} (\Delta_{k,h} S_{ij}(D(u)))(\Delta_{k,h} D_{ij}(u))\zeta^2 \,\mathrm{d}x \\
&= -\int\limits_{B_R} (S_{ij}(D(u)) - S_{ij}(\lambda^+))\Delta_{k,-h}\bigg(\zeta \frac{\partial \zeta}{\partial x_i}((\Delta_{k,h} u_j) - \lambda_{jk} h) \\
(3.12) \qquad &+ \zeta \frac{\partial \zeta}{\partial x_j}((\Delta_{k,h} u_i) - \lambda_{ik} h)\bigg) \,\mathrm{d}x \\
&+ \int\limits_{B_R} \tilde{f}_i \Delta_{k,-h}(\zeta^2((\Delta_{k,h} u_i) - \lambda_{ik} h)) \,\mathrm{d}x \\
&+ 2\int\limits_{B_R} p\, \Delta_{k,-h}\bigg(\zeta \frac{\partial \zeta}{\partial x_i}((\Delta_{k,h} u_i) - \lambda_{ik} h)\bigg) \,\mathrm{d}x\ (^8)
\end{aligned}
$$

for all $p \in \hat{p}$.

($^7$) Here $\Delta_{\lambda,k}\, v$ denotes the *difference* $v(\cdot + \lambda e_k) - v$, where $e_1 = (1,0)$ and $e_2 = (0,1)$.

($^8$) Notice that $\dfrac{\partial u_i}{\partial x_i} = 0$.

To proceed we first note that

$$V_\mu \frac{\partial}{\partial x_k} D(u) \in [L^2_{\mathrm{loc}}(\Omega)]^4 \quad (k = 1, 2), \tag{3.13}$$

and

$$\begin{aligned} &\nu_0 \sum_{k=1}^{2} \int_{B_R} V_\mu^2 \Big\| \frac{\partial}{\partial x_k} D(u) \Big\|^2 \zeta^2 \, \mathrm{d}x \\ &\le 2 \liminf_{h \to 0} \int_{B_R} (\varDelta_{k,h} S_{ij}(D(u)))(\varDelta_{k,h} D_{ij}(u)) \zeta^2 \, \mathrm{d}x. \end{aligned} \tag{3.14}$$

Indeed in case $\mu > 0$ (3.13) and (3.14) can be proved by the aid of (1.5) using a similar reasoning as in [13]), whereas in the case $\mu = 0$ (3.13) and (3.14) will be verified in the appendix below.

Then in (3.12) letting $h$ tend to zero, using chain- and product rule gives

$$\begin{aligned} &\nu_0 \sum_{k=1}^{2} \int_{B_R} V_\mu^2 \Big\| \frac{\partial}{\partial x_k} D(u) \Big\|^2 \zeta^2 \, \mathrm{d}x \\ &\le 2 \int_{B_R} (S_{ij}(D(u)) - S_{ij}(\lambda^+)) \Big( \frac{\partial u_j}{\partial x_k} - \lambda_{jk} \Big) \Big( \zeta \frac{\partial^2 \zeta}{\partial x_i \partial x_k} + \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_k} \Big) \, \mathrm{d}x \\ &+ 2 \int_{B_R} (S_{ij}(D(u)) - S_{ij}(\lambda^+)) \frac{\partial^2 u_j}{\partial x_k \partial x_k} \zeta \frac{\partial \zeta}{\partial x_i} \, \mathrm{d}x \\ &+ 2 \int_{B_R} p \Big( \frac{\partial u_i}{\partial x_k} - \lambda_{ik} \Big) \Big( \zeta \frac{\partial^2 \zeta}{\partial x_i \partial x_k} + \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_k} \Big) \, \mathrm{d}x \\ &+ 2 \int_{B_R} p \, \frac{\partial^2 u_i}{\partial x_k \partial x_k} \zeta \frac{\partial \zeta}{\partial x_i} \, \mathrm{d}x \\ &+ 2 \int_{B_R} \tilde{f}_i \Big( \frac{\partial u_i}{\partial x_k} - \lambda_{ik} \Big) \zeta \frac{\partial \zeta}{\partial x_k} \, \mathrm{d}x + 2 \int_{B_R} \tilde{f}_i \frac{\partial^2 u_i}{\partial x_k \partial x_k} \zeta^2 \, \mathrm{d}x \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \end{aligned} \tag{3.15}$$

where $p \in \hat{p}$ is taken such that $p_{B_R} = 0$.

In order to estimate integrals $I_1, I_2, I_3$ and $I_4$ we will make extensively use of the following inequality

$$\mu + \|D(u)\| + \|D(u) - \lambda^+\| \ge \mu + \|\lambda^+\| \quad \text{a. e. in} \quad \Omega. \tag{3.16}$$

1) First, observing (1.4) the estimation of integral $I_1$ can be easily done by the

aid of Lemma 2.1. This yields

$$I_1 \le \frac{c}{(R-\rho)^2} \int\limits_{B_R} (\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{q-2} |\nabla u - \lambda|^2 \, \mathrm{d}x.$$

Then using (3.16) gives

$$I_1 \le \frac{c}{(R-\rho)^2} (\mu + \|\lambda^+\|)^{q-2} \int\limits_{B_r} |\nabla u - \lambda|^2 \, \mathrm{d}x. \tag{3.17}$$

2) Analogously as above, using Lemma 3.1 and then applying Young's inequality we obtain

$$\begin{aligned} I_2 \le {} & \frac{\upsilon_0}{4} \sum_{k=1}^{2} \int\limits_{B_R} V_\mu^2 \Big\| \frac{\partial}{\partial x_k} D(u) \Big\|^2 \zeta^2 \, \mathrm{d}x \\ & + \frac{c}{(R-\rho)^2} (\mu + \|\lambda^+\|)^{q-2} \int\limits_{B_r} \|\nabla u - \lambda\|^2 \, \mathrm{d}x. \end{aligned} \tag{3.18}$$

3) To estimate integral $I_3$ we apply Cauchy Schwarz's inequality and make use of the estimate (3.11). It follows

$$\begin{aligned} I_3 \le {} & \frac{c}{(R-\rho)^2} \Bigg( \int\limits_{B_R} (\mu + \|D(u)\| + \|D(u) - \lambda^+\|)^{2(q-2)} |\nabla u - \lambda|^2 \, \mathrm{d}x \\ & + R^2 \int\limits_{B_R} |\tilde{f}|^2 \, \mathrm{d}x \Bigg)^{1/2} \Bigg( \int\limits_{B_R} |\nabla u - \lambda|^2 \, \mathrm{d}x \Bigg)^{1/2}. \end{aligned}$$

In addition, applying Young's inequality together with (3.16) shows that

$$I_3 \le \frac{c}{(R-\rho)^2} \Bigg\{ (\mu + \|\lambda^+\|)^{q-2} \int\limits_{B_r} |\nabla u - \lambda|^2 \, \mathrm{d}x + r^2 \int\limits_{B_r} (1 + |\tilde{f}|^{(2+\sigma)/2}) \, \mathrm{d}x \Bigg\}. \tag{3.19}$$

4) In order to estimate $I_4$ we first apply Hölder's and Young's inequality and then making use of (3.10) (with $\varepsilon > 0$ arbitrarily chosen). Hence, together with (3.16) one obtains

$$\begin{aligned} I_4 \le {} & \frac{c}{R-\rho} \Bigg( {}_{B_{\rho_2}} |p|^{q'} \, \mathrm{d}x \Bigg)^{1/q'} \Bigg( \sum_{k=1}^{2} \int\limits_{B_R} \Big\| \zeta \frac{\partial}{\partial x_l} D(u) \Big\|^q \, \mathrm{d}x \Bigg)^{1/q} \\ \le {} & \frac{\upsilon_0}{4} \sum_{k=1}^{2} \int\limits_{B_R} V_\mu^2 \Big\| \frac{\partial}{\partial x_k} D(u) \Big\|^2 \zeta^2 \, \mathrm{d}x \end{aligned}$$

$$+\frac{c\,\rho_2^2}{(R-\rho)^2}\bigg(\fint_{B_R}|p|^{q'}\,\mathrm{d}x\bigg)^{2/q'}\bigg(\fint_{B_R}(\mu+\|D(u)\|)^q\,\mathrm{d}x\bigg)^{(2-q)/q}\ (^9)$$

$$\le\frac{\upsilon_0}{4}\sum_{k=1}^{2}\int_{B_R}V_\mu^2\Big\|\frac{\partial}{\partial x_k}D(u)\Big\|^2\zeta^2\,\mathrm{d}x$$

$$+\frac{c\,\rho_2^2}{(R-\rho)^2}\Bigg\{\varepsilon^{\frac{q-2}{q-1}}(\mu+\|\lambda^+\|)^{2(q-2)}\fint_{B_R}\|D(u)-\lambda^+\|^2\,\mathrm{d}x$$

$$\times\bigg(\fint_{B_R}(\mu+\|D(u)\|)^q\,\mathrm{d}x\bigg)^{(2-q)/q}$$

$$+\varepsilon(\mu+\|\lambda^+\|)^{q-2}\sum_{k=1}^{2}\int_{B_R}V_\mu^2\Big\|\frac{\partial}{\partial x_k}D(u)\Big\|^2\mathrm{d}x\bigg(\fint_{B_R}(\mu+\|D(u)\|)^q\,\mathrm{d}x\bigg)^{(2-q)/q}$$

$$+\int_{B_r}|\tilde f|^2\,\mathrm{d}x\bigg(\fint_{B_R}(\mu+\|D(u)\|)^q\,\mathrm{d}x\bigg)^{(2-q)/q}\Bigg\}.$$

Thus, choosing

$$\varepsilon:=\upsilon_0\frac{(R-\rho)^2}{4c\rho_2^2}(\mu+\|\lambda^+\|)^{2-q}\bigg(\fint_{B_R}(\mu+\|D(u)\|)^q\,\mathrm{d}x\bigg)^{(q-2)/q}$$

one arrives at

$$\begin{aligned}I_4\ &\le\frac{\upsilon_0}{4}\sum_{k=1}^{2}\int_{B_R}V_\mu^2\Big\|\frac{\partial}{\partial x_k}D(u)\Big\|^2\zeta^2\,\mathrm{d}x\\&\quad+\frac{c\,r^{2(2-q)/q}}{(R-\rho)^{2/(q-1)}}(\mu+\|\lambda^+\|)^{q'(q-2)}\bigg(\int_{B_r}(\mu+\|D(u)\|)^q\,\mathrm{d}x\bigg)^{\frac{2-q}{q(q-1)}}\\&\quad\times\int_{B_r}\|D(u)-\lambda^+\|^2\,\mathrm{d}x\\&\quad+\frac{c\,r^2}{(R-\rho)^2}\int_{B_r}|\tilde f|^2\,\mathrm{d}x\bigg(\fint_{B_r}(\mu+\|D(u)\|)^q\,\mathrm{d}x\bigg)^{(2-q)/q}\\&\quad+\frac{\upsilon_0}{4}\sum_{k=1}^{2}\int_{B_R}V_\mu^2\Big\|\frac{\partial}{\partial x_k}D(u)\Big\|^2\mathrm{d}x.\end{aligned}\tag{3.20}$$

(9) Observing $\nabla Du=0$ almost everywhere on $\{y\in\Omega|D(u(y))=0\}$ by Hölder's inequality,

$$\int_{B_R}\Big\|\zeta\frac{\partial}{\partial x_l}D(u)\Big\|^q\mathrm{d}x=\int_{B_R}(\mu+\|Du\|)^{\frac{q(2-q)}{2}}V_\mu^q\Big\|\zeta\frac{\partial}{\partial x_l}D(u)\Big\|^q\mathrm{d}x$$

$$\le\bigg(\int_{B_R}V^2\Big\|\zeta\frac{\partial}{\partial x_l}D(u)\Big\|^q\mathrm{d}x\bigg)^{q/2}\bigg(\int_{B_R}(\mu+\|D(u)\|)^q\,\mathrm{d}x\bigg)^{(2-q)/2}.$$

5) Using Cauchy Schwarz's inequality and Young's inequality we get

$$I_5 \le \frac{c\, r^2}{(R-\rho)} \int\limits_{B_r} |\nabla u - \lambda|^2 \,\mathrm{d}x + c \int\limits_{B_r} (1 + |\tilde{f}|^{(2+\sigma)/2}) \,\mathrm{d}x. \tag{3.21}$$

6) Finally, by the aid of Hölder's inequality and Young's inequality it follows that

$$I_6 \le c \int\limits_{B_r} (1 + |\varDelta u|^{(2+\sigma)/\sigma} + |\tilde{f}|^{(2+\sigma)/2}) \,\mathrm{d}x. \tag{3.22}$$

Now, inserting the estimates (3.17)-(3.22) into (3.15) and taking into account the estimate

$$\begin{aligned}
&\int\limits_{B_r} |\tilde{f}|^2 \,\mathrm{d}x \Big( \fint\limits_{B_r} (\mu + \|D(u)\|)^q \,\mathrm{d}x \Big)^{(2-q)/q} \\
&\le 2 \Big( \int\limits_{B_r} (\mu + \|D(u)\|)^{(2-q)\frac{\sigma+2}{\sigma-2}} \,\mathrm{d}x \Big)^{(\sigma-2)/(\sigma+2)} \Big( \int\limits_{B_r} |f|^{(\sigma+2)/\sigma} \,\mathrm{d}x \Big)^{4/(\sigma+2)} \\
&\le c \int\limits_{B_r} \Big( 1 + \|D(u)\|^{(2-q)\frac{\sigma+2}{\sigma-2}} + |f|^{(\sigma+2)/\sigma} \Big) \,\mathrm{d}x,
\end{aligned}$$

which easily follows by the aid of Hölder's and Young's inequality, implies

$$\begin{aligned}
&\sum_{k=1}^{2} \int\limits_{B_\rho} V_\mu^2 \Big\| \frac{\partial}{\partial x_k} D(u) \Big\|^2 \,\mathrm{d}x \\
&\le \frac{c\, r^{2(2-q)/q}}{(R-\rho)^{2/(q-1)}} (\mu + \|\lambda^+\|)^{q'(q-2)} \Big( \int\limits_{B_r} (\mu + \|D(u)\|)^q \,\mathrm{d}x \Big)^{\frac{2-q}{q(q-1)}} \\
&\times \int\limits_{B_r} \|D(u) - \lambda^+\|^2 \,\mathrm{d}x \\
&+ \frac{c\, r^{2/(q-1)}}{(R-\rho)^{2/(q-1)}} \int\limits_{B_r} \Big( 1 + \|D(u)\|^{(2-q)\frac{\sigma+2}{\sigma-2}} + \|\varDelta u\|^{(2+\sigma)/\sigma} + |\tilde{f}|^{(2+\sigma)/2} \Big) \,\mathrm{d}x \\
&+ \frac{1}{2} \sum_{k=1}^{2} \int\limits_{B_R} V_\mu^2 \Big\| \frac{\partial}{\partial x_k} D(u) \Big\|^2 \,\mathrm{d}x,
\end{aligned}$$

where $c = \mathrm{const} > 0$ depends on $c_0/\nu_0$ and $q$ only. Now the assertion (3.2) immediately follows from the last estimate together with Lemma 2.2. □

Using a similar reasoning which let to (3.2) we have the following alternative Caccioppoli inequality

THEOREM 3.2. – *Let $u \in [W^{1,q}(\Omega)]^2$ be a weak solution to the equations* (1.1), (1.2). *Let all assumptions of the Theorem be fulfilled. Then for each $x_0 \in \Omega, 0 < r < \operatorname{dist}(x_0, \partial\Omega)$ and $\lambda \in \mathbb{R}^2 \times \mathbb{R}^2$ there holds*

$$\sum_{k=1}^{2} \int_{B_r} V_\mu^2 \Big\| \frac{\partial}{\partial x_k} D(u) \Big\|^2 \mathrm{d}x$$

$$\tag{3.23} \leq c \left\{ \fint_{B_r} |\nabla u - \lambda|^2 \, \mathrm{d}x + \fint_{B_r} (\mu + \|D(u)\|)^2 \, \mathrm{d}x \right\}^{q/2}$$

$$+ c \left( \int_{B_r} |\tilde{f}|^2 \, \mathrm{d}x \right)^{q'/2} + c \int_{B_r} \left( 1 + |\varDelta u|^{(2+\sigma)/\sigma} + |\tilde{f}|^{(2+\sigma)2} \right) \mathrm{d}x,$$

*where $c =$ const $> 0$ depending on $c_0/\upsilon_0$ and $q$ only.*

PROOF. – Let $x_0 \in \Omega$ and $0 < r < \dfrac{1}{2} \operatorname{dist}(x_0, \partial\Omega)$ be fixed. Let $\zeta \in C_c^\infty(B_r)$ be a cut-off function for the ball $B_r$, i.e. $0 \leq \zeta \leq 1$ in $B_r, \zeta \equiv 1$ on $B_{r/2}$, such that

$$\Big| \frac{\partial \zeta}{\partial x_i} \Big| \leq \frac{c}{r}, \quad \Big| \frac{\partial^2 \zeta}{\partial x_i \partial x_j} \Big| \leq \frac{c}{r^2} \quad \text{in} \quad B_r \quad (i,j = 1,2)$$

($c =$ const independent of $r$). As in the proof of Theorem 3.1 we insert the admissible test function $\varphi_j = \varDelta_{k,-h}(\zeta^2((\varDelta_{k,h} u_j) - \lambda_{jk} h))$ $(0 < h < r, j = 1,2)$ into (3.8) applying the transformation formula of the Lebesgue integral and passing to the limit $h \to 0$ (cf. also the appendix below). This yields

$$\upsilon_0 \sum_{k=1}^{2} \int_{B_r} V_\mu^2 \Big\| \frac{\partial}{\partial x_k} D(u) \Big\|^2 \zeta^2 \, \mathrm{d}x$$

$$\leq 2 \int_{B_r} (S_{ij}(D(u)) - S_{ij}(0)) \left( \frac{\partial u_j}{\partial x_k} - \lambda_{jk} \right) \left( \zeta \frac{\partial^2 \zeta}{\partial x_i \partial x_k} + \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_k} \right) \mathrm{d}x$$

$$+ 2 \int_{B_r} (S_{ij}(D(u)) - S_{ij}(0)) \frac{\partial^2 u_j}{\partial x_k \partial x_k} \zeta \frac{\partial \zeta}{\partial x_i} \, \mathrm{d}x$$

$$\tag{3.24} + 2 \int_{B_r} p \left( \frac{\partial u_i}{\partial x_k} - \lambda_{ik} \right) \left( \zeta \frac{\partial^2 \zeta}{\partial x_i \partial x_k} + \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_k} \right) \mathrm{d}x$$

$$+ 2 \int_{B_r} p \, \frac{\partial^2 u_i}{\partial x_k \partial x_k} \zeta \frac{\partial \zeta}{\partial x_i} \, \mathrm{d}x$$

$$+ 2 \int_{B_r} \tilde{f}_i \left( \frac{\partial u_i}{\partial x_k} - \lambda_{ik} \right) \zeta \frac{\partial \zeta}{\partial x_k} \, \mathrm{d}x + 2 \int_{B_R} \tilde{f}_i \frac{\partial^2 u_i}{\partial x_k \partial x_k} \zeta^2 \, \mathrm{d}x$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

1) Observing (1.4) and applying Hölder's inequality we get

$$
\begin{aligned}
I_1 &\le \frac{c}{r^2} \int\limits_{B_r} \int\limits_0^1 (\mu + \|tD(u)\|)^{q-2} \|D(u)\| \, |\nabla u - \lambda|^2 \, \mathrm{d}t \, \mathrm{d}x \\
&\le \frac{c}{r^2} \Big( \int\limits_{B_r} (\mu + \|D(u)\|^q \, \mathrm{d}x \Big)^{1/q'} \Big( \int\limits_{B_r} |\nabla u - \lambda|^q \, \mathrm{d}x \Big)^{1/q} \qquad (3.25) \\
&\le c \bigg\{ \fint\limits_{B_r} |\nabla u - \lambda|^2 \, \mathrm{d}x + \fint\limits_{B_r} (\mu + |D(u)|)^2 \, \mathrm{d}x \bigg\}^{q/2}.
\end{aligned}
$$

2) Similarly, making use of Hölder's inequality (cf. footnote ([9])) and Young's inequality gives

$$
\begin{aligned}
I_2 &\le \frac{c}{r} \Big( \int\limits_{B_r} (\mu + \|D(u)\|^q \, \mathrm{d}x \Big)^{1/q'} \Big( \sum_{k=1}^{2} \int\limits_{B_r} \Big\| \frac{\partial}{\partial x_k} D(u) \Big\|^q \zeta^q \, \mathrm{d}x \Big)^{1/q} \qquad (3.26) \\
&\le \frac{\upsilon_0}{4} \sum_{k=1}^{2} \int\limits_{B_r} V_\mu^2 \Big\| \frac{\partial}{\partial x_k} D(u) \Big\|^2 \zeta^2 \, \mathrm{d}x + c \Big( \fint\limits_{B_r} (\mu + \|D(u)\|)^2 \, \mathrm{d}x \Big)^{q/2}.
\end{aligned}
$$

3) In order to estimate $I_3$ we apply Hölder's inequality and make use of (3.9) (with $\lambda^+ = 0$), it follows

$$
\begin{aligned}
I_3 &\le c \Big( \fint\limits_{B_r} |p|^{q'} \, \mathrm{d}x \Big)^{1/q'} \Big( \fint\limits_{B_r} |\nabla u - \lambda|^q \, \mathrm{d}x \Big)^{1/q} \\
&\le c \bigg\{ \Big( \fint\limits_{B_r} (\mu + \|D(u)\|)^q \, \mathrm{d}x \Big)^{1/q'} + \Big( \int\limits_{B_r} |\tilde{f}|^2 \, \mathrm{d}x \Big)^{1/2} \bigg\} \\
&\quad \times \Big( \fint\limits_{B_r} |\nabla u - \lambda|^q \, \mathrm{d}x \Big)^{1/q}.
\end{aligned}
$$

Then applying Young's inequality gives

$$
I_3 \le c \bigg\{ \fint\limits_{B_r} |\nabla u - \lambda|^2 \, \mathrm{d}x + \fint\limits_{B_r} (\mu + \|D(u)\|)^2 \, \mathrm{d}x \bigg\}^{q/2} + c \Big( \int\limits_{B_r} |\tilde{f}|^2 \, \mathrm{d}x \Big)^{q'/2}. \tag{3.27}
$$

4) Applying again Hölder's and Young's inequality we obtain

$$
\begin{aligned}
I_4 &\le \frac{\upsilon_0}{4} \sum_{k=1}^{2} \int\limits_{B_r} V_\mu^2 \Big\| \frac{\partial}{\partial x_k} D(u) \Big\|^2 \zeta^2 \, \mathrm{d}x \\
&\quad + c \Big( \fint\limits_{B_r} |p|^{q'} \, \mathrm{d}x \Big)^{2/q'} \Big( \fint\limits_{B_r} (\mu + \|D(u)\|)^q \, \mathrm{d}x \Big)^{(2-q)/q}.
\end{aligned}
$$

Then estimating the right of the latter inequality from above by (3.9) (with $\lambda^+ = 0$) and then applying Hölder's and Young's inequality gives

$$
\begin{aligned}
I_4 \le\ & \frac{\nu_0}{4}\sum_{k=1}^{2}\int\limits_{B_r} V_\mu^2\Big\|\frac{\partial}{\partial x_k}D(u)\Big\|^2\zeta^2\,\mathrm{d}x \\
&+ c\bigg(\fint\limits_{B_r}(\mu+\|D(u)\|)^2\,\mathrm{d}x\bigg)^{q/2} + c\bigg(\int\limits_{B_r}|\tilde f|^2\,\mathrm{d}x\bigg)^{q'/2}.
\end{aligned}
\tag{3.28}
$$

5) The integrals $I_5, I_6$ may be estimated in the same way as in Theorem 3.1. Thus

$$
\begin{aligned}
I_5 + I_6 \le\ & c\bigg(\fint\limits_{B_r}|\nabla u-\lambda|^2\,\mathrm{d}x\bigg)^{q/2} + c\bigg(\int\limits_{B_r}|\tilde f|^2\,\mathrm{d}x\bigg)^{q'/2} (3.2)\\
&+ c\int\limits_{B_r}(1+|\varDelta u|^{(\sigma+2)/\sigma}+|\tilde f|^{(\sigma+2)/2})\,\mathrm{d}x.
\end{aligned}
\tag{3.29}
$$

Now, inserting (3.25)-(3.29) into (3.24) gives (3.22). □

## 4. – Proof of the Main Result.

By means of Sobolev's embedding theorem to prove the Theorem it suffices to obtain the higher integrability of second order derivative of $u$. This can be achieved with the help the result of Giaquinta and Modica (cf. [7]) based on Gehring's lemma [8] after having established appropriate"reverse Hölder inequalities". For, we consider the following two cases.

**First case**: $\mu + \|(D(u))_{B_r}\| > \bigg(\fint\limits_{B_r}|\nabla u-(\nabla u)_{B_r}|^2\,\mathrm{d}x\bigg)^{1/2}$:

Making use of triangular inequality we obtain

$$
\begin{aligned}
\bigg(\fint\limits_{B_r}(\mu+\|D(u)\|)^2\,\mathrm{d}x\bigg)^{1/2} &\le \mu+\|(D(u))_{B_r}\| + \bigg(\fint\limits_{B_r}|\nabla u-(\nabla u)_{B_r}|^2\,\mathrm{d}x\bigg)^{1/2}\\
&\le 2(\mu+\|(D(u))_{B_r}\|).
\end{aligned}
\tag{4.1}
$$

Next, from (3.2) (with $\lambda := (\nabla u)_{B_r}$) applying Hölder's inequality and making use of (4.1) it follows

$$
\begin{aligned}
&\int\limits_{B_{r/2}} V_\mu^2 \|\nabla D(u)\|^2 \,\mathrm{d}x \\
&\quad \le \frac{c}{r^2} (\mu + \|(D(u))_{B_r}\|)^{\frac{q(q-2)}{q-1}} \left( \fint\limits_{B_r} (\mu + \|D(u)\|)^2 \,\mathrm{d}x \right)^{\frac{2-q}{2(q-1)}} \\
&\quad\quad \times \int\limits_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \,\mathrm{d}x \\
&\quad\quad + c \int\limits_{B_r} \left( 1 + \|D(u)\|^{(2-q)\frac{\sigma+2}{\sigma-2}} + |\varDelta u|^{(\sigma+2)/\sigma} + |\tilde{f}|^{(\sigma+2)/2} \right) \mathrm{d}x \\
&\quad \le c\,(\mu + \|(D(u))_{B_r}\|)^{q-2} \fint\limits_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \,\mathrm{d}x \;+\; c \int\limits_{B_r} \tilde{g} \,\mathrm{d}x,
\end{aligned}
\tag{4.2}
$$

where

$$
\tilde{g}(x) := 1 + \|D(u(x))\|^{(2-q)\frac{\sigma+2}{\sigma-2}} + |\varDelta u(x)|^{(\sigma+2)/\sigma} + |\tilde{f}(x)|^{(\sigma+2)/2}, \quad x \in \Omega.
$$

To proceed we first note that

$$
\fint\limits_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \,\mathrm{d}x \le c\, r^2 \left( \fint\limits_{B_r} \|\nabla D(u)\| \,\mathrm{d}x \right)^2,
$$

which easily follows by the aid of Poincaré's inequality together with (3.3).

Then making use of Hölder's inequality and (4.1) gives

$$
\begin{aligned}
&\fint\limits_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \,\mathrm{d}x \\
&\quad \le c\, r^2 \left( \fint\limits_{B_r} (\mu + \|D(u)\|)^{(2-q)/2} V_\mu \|\nabla D(u)\| \,\mathrm{d}x \right)^2 \\
&\quad \le c\, r^2 \left( \fint\limits_{B_r} (\mu + \|D(u)\|) \,\mathrm{d}x \right)^{2-q} \left( \fint\limits_{B_r} \left[ V_\mu \|\nabla D(u)\| \right]^{2/q} \mathrm{d}x \right)^q \\
&\quad \le c\, r^2 (\mu + \|(D(u))_{B_r}\|)^{2-q} \left( \fint\limits_{B_r} \left[ V_\mu \|\nabla D(u)\| \right]^{2/q} \mathrm{d}x \right)^q .
\end{aligned}
$$

Thus inserting the latter inequality into (4.2) gives

$$
\begin{aligned}
&\fint\limits_{B_{r/2}} \left[ V_\mu \|\nabla D(u)\| \right]^2 \mathrm{d}x \\
&\quad \le c \left( \fint\limits_{B_r} \left[ V_\mu \|\nabla D(u)\| \right]^{2/q} \mathrm{d}x \right)^q + c \fint\limits_{B_r} \tilde{g} \,\mathrm{d}x
\end{aligned}
\tag{4.3}
$$

for all $x_0 \in \Omega$ and $0 < r < \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega)$, where $c = \text{const} > 0$ depending only on $c_0/\upsilon_0, q$.

**Second case:** $\mu + \|(D(u))_{B_r}\| \le \left( \fint_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx \right)^{1/2}$:

First using the triangular inequality we easily get

$$\left( \fint_{B_r} (\mu + \|D(u)\|^2 \, dx \right)^{1/2} \le 2 \left( \fint_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx \right)^{1/2}. \tag{4.4}$$

Then (3.22) (with $\lambda = (\nabla u)_{B_r}$) reads

$$\begin{aligned} \int_{B_{r/2}} V_\mu^2 \|\nabla D(u)\|^2 \, dx \le\ & c \left( \fint_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx \right)^{q/2} \\ & + c \left( \int_{B_r} |\tilde{f}|^2 \, dx \right)^{q'/2} + c \int_{B_r} \left( 1 + |\varDelta u|^{(2+\sigma)/\sigma} + |\tilde{f}|^{(2+\sigma)/2} \right) dx. \end{aligned} \tag{4.5}$$

As above using the Poincaré inequality and Hölder's inequality gives

$$\begin{aligned} \left( \fint_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx \right)^{q/2} \le\ & c\, r^q \left( \fint_{B_r} (\mu + \|D(u)\|)^2 \, dx \right)^{q(2-q)/4)} \\ & \times \left( \fint_{B_r} \left[ V_\mu \|\nabla D(u)\| \right]^{2/q} dx \right)^{q^2/2}. \end{aligned}$$

Then taking into account (4.4) applying Young's inequality we get

$$\left( \fint_{B_r} |\nabla u - (\nabla u)_{B_r}|^2 \, dx \right)^{q/2} \le c\, r^2 \left( \fint_{B_r} \left[ V_\mu \|\nabla D(u)\| \right]^{2/q} dx \right)^{q}.$$

Now, inserting this inequality into (4.5) yields

$$\begin{aligned} & \fint_{B_r} \left[ V_\mu \|\nabla D(u)\| \right]^2 dx \\ & \le c \left( \fint_{B_r} \left[ V_\mu \|\nabla D(u)\| \right]^{2/q} dx \right)^{q} \\ & \quad + c(1 + (o(r; x_0))) \fint_{B_r} \left( 1 + |\varDelta u|^{(2+\sigma)/\sigma} + |\tilde{f}|^{(2+\sigma)/2} \right) dx, \end{aligned} \tag{4.6}$$

where

$$o(r; x_0) := \left( \int_{B_r} |\tilde{f}|^2 \, dx \right)^{\frac{2-q}{2(q-1)}}.$$

Let $\Omega' \subset \Omega$ be an open set with $\overline{\Omega}' \subset \Omega$. By the absolutely continuity of the Lebesgue integral there exists a number $0 < r_0 < \operatorname{dist}(\Omega', \partial\Omega)$ such that

$$o(r; x_0) \leq 1 \quad \forall\, x_0 \in \Omega', \quad \forall\, 0 < r \leq r_0.$$

Then combining (4.3) and (4.6) gives

$$\tag{4.7} \fint_{B_r} \left[V_\mu \|\nabla D(u)\|\right]^2 \mathrm{d}x \leq \left(\fint_{B_r} \left[V_\mu \|\nabla D(u)\|\right]^{2/q} \mathrm{d}x\right)^q + c \fint_{B_r} \tilde{g}\, \mathrm{d}x,$$

for all $x_0 \in \Omega'$ and $0 < r \leq r_0$, where $c = \text{const} > 0$ depends only on $c_0, \upsilon_0, q$ and $\sigma$.

Now we are in a position to apply the following result of higher integrability due to M. Giaquinta and G. Modica (cf. [7]) based on Gehring's lemma (cf. [8]).

LEMMA 4.1. – *Let $F \in L^t_{\mathrm{loc}}(\Omega)$ and $G \in L^d_{\mathrm{loc}}(\Omega)\,(1 < t < d < +\infty)$ be given non-negative functions. Suppose there are constants $K_0 \geq 1$ and $r_0 > 0$ such that*

$$\tag{4.8} \fint_{B_{r/2}(x_0)} F^t \,\mathrm{d}x \leq K_0 \left(\fint_{B_r(x_0)} F\, \mathrm{d}x\right)^t + \fint_{B_r(x_0)} G^t \,\mathrm{d}x$$

*for each $x_0 \in \Omega, 0 < r < \min\{r_0, \operatorname{dist}(x_0, \partial\Omega)\}$. Then there exists $t < \tau_0 \leq d$, such that*

$$\tag{4.9} F \in L^\tau_{\mathrm{loc}}(\Omega) \quad \forall\, \tau \in [1, \tau_0[.$$

□

PROOF OF THEOREM 1.1 continued. Applying Lemma 4.1 with

$$F := \left[\sum_{k=1}^{2} V_\mu \left\| \frac{\partial}{\partial x_k} D(u)\right\|\right]^{2/q}, \quad G := \tilde{g}^{1/q},$$

$$t := q, \quad d := \frac{2q\sigma}{\sigma + 2}$$

gives

$$\sum_{k=1}^{2} V_\mu \left\| \frac{\partial}{\partial x_k} D(u)\right\| \in L^\tau(\Omega') \quad \text{for some} \quad 2 < \tau < \frac{4\sigma}{\sigma + 2}.$$

Then by an analogous reasoning which led to (3.7) we obtain

$$\tag{4.10} V_\mu D(u) \in [W^{1,\tau}(\Omega')]^4$$

and hence applying Sobolev's embedding theorem it follows that $D(u)$ is bounded on $\overline{\Omega}'$. Using (3.3) we obtain

$$\left|\sum_{k,l=1}^{2}\frac{\partial^2 u_i}{\partial x_k \partial x_l}\right| \le 2\Big(\mu + \max_{\Omega'}\|D(u)\|\Big)^{(2-q)/2} V_\mu \left\|\sum_{k=1}^{2}\frac{\partial}{\partial x_k}D(u)\right\|.$$

Thus making use of Sobolev's embedding theorem from (4.10) it follows

$$u|_{\Omega'} \in [C^{1,(\tau-2)/\tau}(\overline{\Omega}')]^2. \tag{4.11}$$

This completes the proof of the theorem. □

## 5. – Appendix.

This appendix is devoted to the proof of (3.13) and (3.14) for the special case $\mu = 0$.

Let $\Omega \in \mathbb{R}^d$ $(d \ge 2)$ be a bounded open set. Let $N \ge 1$ denote an integer. For $v : \Omega \to \mathbb{R}^N$ Lebesgue measurable we define

$$V(v)(x) := \begin{cases} |v(x)|^{(q-2)/2} & \text{if} \quad v(x) \neq 0 \\ 0 & \text{if} \quad v(x) = 0, \end{cases}$$

$x \in \Omega$, and given $\lambda > 0$ we set,

$$V_{\lambda,k}(v)(x) := \begin{cases} (|v(x)| + |v(x+\lambda e_k)|)^{(q-2)/2} & \text{if} \quad |v(x)| + |v(x+\lambda e_k)| > 0 \\ 0 & \text{if} \quad |v(x)| + |v(x+\lambda e_k)| = 0, \end{cases}$$

$x \in \Omega$, $\operatorname{dist}(x, \partial\Omega) \ge \lambda\, (k = 1, \dots, d)$.

LEMMA A.1 *Let $v \in [L^q(\Omega)]^N$. Suppose for every $G \subset\subset \Omega$ there exists $K_G > 0$ such that*

$$\int_G (V_{\lambda,k}(v))^2 |\Delta_{\lambda,k} v|^2 \,\mathrm{d}x \;\le\; K_G\, \lambda^2 \quad \forall\, 0 < \lambda < \operatorname{dist}(G, \partial\Omega) \tag{A.1}$$

$(k = 1, \dots, d)$. *Then* $v \in [W^{1,q}_{\mathrm{loc}}(\Omega)]^N$,

$$V(v)\frac{\partial v}{\partial x_k} \in [L^2_{\mathrm{loc}}(\Omega)]^N \quad (k = 1, \dots, d) \tag{A.2}$$

*and for every* $G \subset\subset \Omega$,

$$\int_G (V(v))^2 \left|\frac{\partial v}{\partial x_k}\right|^2 \mathrm{d}x \;\le\; 2 \liminf_{\lambda \to 0} \int_G (V_{\lambda,k}(v))^2 \left|\frac{1}{\lambda}\Delta_{\lambda,k} v\right|^2 \mathrm{d}x \;\le\; K_G \tag{A.3}$$

$(k = 1, \dots, d)$.

PROOF. – Let $G \subset\subset \Omega$ be fixed. As in [11] (cf. also [12], [13]) from (A.1) one easily deduces that $v \in [W^{1,q}(G)]^N$ and

$$\text{(A.4)} \qquad \int_G (t+2|v|)^{q-2}\left|\frac{\partial v}{\partial x_k}\right|^2 \mathrm{d}x \le \liminf_{\lambda\to 0}\int_G (V_{\lambda,k}(v))^2\left|\frac{1}{\lambda}\Delta_{\lambda,k}v\right|^2 \mathrm{d}x \le K_G$$

for every $t > 0\,(k = 1, \dots, d)$.

For $t > 0$ we set $w_{k,t} := (t+2|v|)^{(q-2)/2}\dfrac{\partial v}{\partial x_k}$ $(k = 1, \dots, d)$.. Then (A4) implies that $\{w_{k,t}\,|\,t > 0\}$ is bounded in $[L^2(G)]^N$. By virtue of reflexivity there exists $w_k \in [L^2(G)]^N$ and a decreasing sequence $t_1 > t_2 > \cdots > t_m \to 0$ such that

$$\text{(A.5)} \qquad w_{k,t_m} \to w_k \quad \text{weakly in} \quad [L^2(G)]^N \quad \text{as} \quad m \to +\infty.$$

In addition taking into account (A.4) by the aid of Banach-Steinhaus' theorem we get

$$\text{(A.6)} \qquad \int_G |w_k|^2 \,\mathrm{d}x \le \liminf_{\lambda\to 0}\int_G (V_{\lambda,k}(v))^2\left|\frac{1}{\lambda}\Delta_{\lambda,k}v\right|^2.$$

Next for each $\varepsilon > 0$ define

$$G_\varepsilon := \{y \in G\,|\,|v(y)| > \varepsilon\}.$$

Using Lebesgue's theorem it is easily seen that

$$w_{k,t} \to (2|v|)^{(q-2)/2}\frac{\partial v}{\partial x_k} \quad \text{in} \quad [L^q(G_\varepsilon)]^N \quad \text{as} \quad t \to 0.$$

Thus by (A.5),

$$\text{(A.7)} \qquad w_k = 2^{(q-2)/2}|v|^{(q-2)/2}\frac{\partial v}{\partial x_k} \quad \text{a. e. in} \quad \{y \in G\,|\,v(y) \ne 0\}.$$

On the other hand, observing

$$\frac{\partial v}{\partial x_k} = 0 \quad \text{a. e. in} \quad \{y \in G\,|\,v(y) = 0\}$$

making use of (A6) and (A7) it follows

$$\int_G (V(v))^2\left|\frac{\partial v}{\partial x_k}\right|^2 \mathrm{d}x = 2^{2-q}\int_{G\cap\{v\ne 0\}} |w_k|^2\,\mathrm{d}x \le 2^{2-q}\liminf_{\lambda\to 0}\int_G (V_{\lambda,k}(v))^2\left|\frac{1}{\lambda}\Delta_{\lambda,k}v\right|^2 \mathrm{d}x.$$

Whence (A.2) and (A.3). □

PROOF OF (3.13) AND (3.14). Based on (3.15) and the fact that $f \in u\cdot\nabla u \in [L^{q'}_{\mathrm{loc}}(\Omega)]^d$ as in [11] we infer

$$\int_{B_r} (\Delta_{k,h} S_{ij}(D(u)))(\Delta_{k,h} D_{ij}(u))\,\mathrm{d}x \le K_r \lambda^2, \tag{A.8}$$

for all $0 < \lambda < \frac{1}{2}\operatorname{dist}(B_r, \partial\Omega)$ (here $K_r > 0$ is independent of $\lambda$).

By the aid of (1.5) with the notation introduced above setting $v := D(u)$ one finds

$$\nu_0 |V_{\lambda,k}(v)\Delta_{\lambda,k} v|^2 \le (\Delta_{\lambda,k} S_{ij}(D(u)))(\Delta_{\lambda,k} D_{ij}(u)) \quad \text{a. e. in} \quad B_r. \tag{A.9}$$

Then integrating both sides of (A.9) over $B_r$ using (A.8) yields

$$\nu_0 \int_{B_r} (V_{\lambda,k}(v))^2 |\Delta_{\lambda,k} v|^2\,\mathrm{d}x \le K_r\,\lambda^2.$$

Thus, the assumptions of Lemma A.1 are satisfied for $v := D(u)$. Now both (3.13) and (3.14) are an immediate consequence of Lemma A.1. □

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