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On the Information Dimensions.

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Sunto. – *Si studiano i legami fra la dimensione informatica (information dimension) e la dimensione media (average dimension) della misura. Inoltre si dimostra che la dimensione media è positivamente lineare e continua rispetto della norma supremum nello spazio delle misure.*

Summary. – *A relationship between the information dimension and the average dimension of a measure is given. Properties of the average dimension are studied.*

1. – Introduction.

Let μ be a probability Borel measure on $\mathbb{R}^m$. The *upper information dimension*, the *upper local dimension* at the point $x \in \operatorname{supp}\mu$ (see [14]) and the *upper average dimension* (see [2]) of μ are respectively defined by

$$\overline{I}_\mu = \limsup_{r\to 0^+} \int f_r(x)\, d\mu(x), \quad \overline{d}_\mu(x) = \limsup_{r\to 0^+} f_r(x), \quad \overline{d}_\mu = \int \overline{d}_\mu(x)\, d\mu(x),$$

where

$$f_r(x) = \frac{\log \mu(B(x,r))}{\log r}$$

and $B(x,r)$ is the open ball centred at x with radius $r > 0$. The *lower information dimensions* $\underline{I}_\mu$, the *lower local dimension* $\underline{d}_\mu(x)$ and the *lower average dimension* $\underline{d}_\mu$ are defined in the same way but replacing $\limsup$ by $\liminf$. If $\underline{I}_\mu = \overline{I}_\mu$ then this common value is denoted by I_μ and it is called the *information dimension* of μ. Analogously we define the *local dimension* $d_\mu(x)$ and the *average dimension* d_μ.

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Note that the information dimension defined above is slightly different from the traditional one that uses the Renyi entropy. The information dimensions are special cases of generalized (Rényi) dimensions introduced by Hentschel and Procaccia (see [6]) and were intensively studied among others in [1, 2, 5, 8, 9, 10, 13].

Pointwise dimensions satisfies the following inequalities $\underline{d}_\mu(x) \le \dim \operatorname{supp} \mu$ and $\overline{d}_\mu(x) \le \operatorname{Dim} \operatorname{supp} \mu$ for μ-a.e. x, where dim and Dim denote the Hausdorff and the packing dimensions. Consequently $\underline{d}_\mu \le \overline{d}_\mu \le m$. Numbers $\underline{d}_\mu$ and $\overline{d}_\mu$ are also called the average Hausdorff and packing dimensions of μ (see [2] for the justification of these names).

The relation between the numbers $\underline{I}_\mu$, $\underline{d}_\mu$, $\overline{I}_\mu$, $\overline{d}_\mu$ was studied by Cutler [2]. She proved that if the measure μ has a compact support then $\underline{d}_\mu \le \underline{I}_\mu \le \overline{I}_\mu \le \overline{d}_\mu$. In this note we show that these inequalities hold also when the measure μ has some finite positive moment. Our proof is quite different from that of Cutler which strongly based on some compactness arguments. We also show that the lower average dimension is a linear function of μ. From this result it follows that the lower average dimension is continuous with respect to the strong convergence in the space of measures.

2. -- Information and average dimensions.

In this section we prove the following

THEOREM 1. – *Let μ be a probability Borel measure on $\mathbb{R}^m$. Assume that there exist positive constants M and γ such that*

$$\int \Big(log\big(1 + \|x\|\big)\Big)^{1+\gamma} \mu(dx) \le M. \tag{1}$$

Then $\underline{d}_\mu \le \underline{I}_\mu \le \overline{I}_\mu \le \overline{d}_\mu$.

LEMMA 1. – *Let r and β be positive constants and*

$$A \subset \Big\{x \in \mathbb{R}^m : \mu\big(B(x,r)\big) \le \beta\Big\}.$$

Let

$$A^r = \big\{x \in \mathbb{R}^m : \rho(x,A) \le r\big\} \quad and \quad C = \lambda\big(B(0,1/2)\big),$$

where λ is the Lebesgue measure. Then

$$\lambda(A^r) \ge C\mu(A)r^m\beta^{-1}.$$

□

PROOF. – Let N be the greatest possible numbers of disjoint balls of radius $r/2$ that can be found with centres in A. If $N = \infty$ then $\lambda(A^r) = \infty$. If $N < \infty$ we assume that balls $B(x_1, r/2), \ldots, B(x_N, r/2)$ have the above property. Then the balls $B(x_1, r), \ldots, B(x_N, r)$ cover the set A. Since $\mu\big(B(x_i, r)\big) \le \beta$ for $i = 1, \ldots, N$ we get $N \ge \mu(A)\beta^{-1}$. As $\bigcup_{i=1}^{N} B(x_i, r/2) \subset A^r$ we have

$$\lambda(A^r) \ge N\lambda\big(B(0, r/2)\big) \ge C\mu(A)r^m\beta^{-1}.$$

LEMMA 2. – *Assume that the measure μ satisfies the condition* (1). *Then there exists a constant $\overline{C}$ which depends only on M, γ and m such that*

$$\mu\Big(\Big\{x \in \mathbb{R}^m : \mu(B(x, r)) \le r^k\Big\}\Big) \le \overline{C}(k - m)^{-1-\gamma}|\log r|^{-1} \tag{2}$$

for every $k > m$ and $r \in (0, 1/e)$.

PROOF. – From (1) and the Markov inequality it follows that

$$\mu\Big(\Big\{x \in \mathbb{R}^m : \log^{1+\gamma}(1 + \|x\|) \ge s\Big\}\Big) \le Ms^{-1}$$

for every $s > 0$. This implies that

$$\mu\Big(\Big\{x \in \mathbb{R}^m : \ \|x\| \ge R\Big\}\Big) \le M\Big(\log\,(1 + R)\Big)^{-1-\gamma}$$

for $R > 0$ and, consequently,

$$\mu\Big(\Big\{x \in \mathbb{R}^m : \ \|x\| \ge R\Big\}\Big) \le M(\log\, R)^{-1-\gamma} \tag{3}$$

for every $R > 1$.

Fix $r \in (0, 1/2)$. For $k \in \mathbb{N}$ and $R > 1$ we define

$$A_k = \Big\{x \in \mathbb{R}^m : \mu\big(B(x, r)\big) \le r^k\Big\}, \quad A_{k,R} = \Big\{x \in A_k : \ \|x\| \le R\Big\}.$$

Then from Lemma 1 we obtain

$$\lambda(A_{k,R}^r) \ge Cr^{m-k}\mu(A_{k,R}). \tag{4}$$

Since $A_{k,R}^r \subset B(0, R + r)$ we have

$$\lambda(A_{k,R}^r) \le C(2R + 2r)^m. \tag{5}$$

From inequalities (4) and (5) it follows that

$$\mu(A_{k,R}) \le (2R + 2r)^m r^{k-m} \le (3R)^m r^{k-m}. \tag{6}$$

Using (3) and (6) we get

$$\mu(A_k) \le \mu(A_{k,R}) + \mu\big(\{x \in \mathbb{R}^m : \|x\| \ge R\}\big)$$
$$\le (3R)^m r^{k-m} + M(\log R)^{-1-\gamma}.$$

Let $R = r^{(m-k)/(2m)}$. Then $R > 1$ and from the last inequality we obtain

$$\mu(A_k) \le 3^m r^{(k-m)/2} + M(2m)^{1+\gamma}(k-m)^{-1-\gamma}|\log r|^{-1-\gamma}.$$

Since there exists a constant C_1 such that $r^{s/2} \le C_1 s^{-1-\gamma}|\log r|^{-1}$ for all $r \in (0, 1/e)$ and $s > 0$, we have

$$\mu(A_k) \le \overline{C}(k-m)^{-1-\gamma}|\log r|^{-1} \quad \text{for } r \in (0,\ 1/e) \text{ and } k > m,$$

where $\overline{C} = 3^m C_1 + M(2m)^{1+\gamma}$. □

Proof of Theorem 1. – Let $f_r(x) = \log \mu(B(x,r))/\log r$. Then

$$\int\limits_{\{f_r \ge m+1\}} f_r(x)\,\mu(dx) \le \sum_{k=m+1}^{\infty} (k+1)\mu\big(\{x : k \le f_r(x) < k+1\}\big)$$
$$= \sum_{k=m+1}^{\infty} (k+1)\big(\mu(A_k) - \mu(A_{k+1})\big)$$
$$= (m+2)\mu(A_{m+1}) + \sum_{k=m+2}^{\infty} \mu(A_k),$$

where the sets A_k are defined in the proof of Lemma 2. From (2) it follows that

$$\int\limits_{\{f_r \ge m+1\}} f_r(x)\,\mu(dx) \le \widetilde{C}|\log r|^{-1}, \tag{7}$$

where $\widetilde{C} = \overline{C}\left(m + 2 + \sum_{n=2}^{\infty} n^{-1-\gamma}\right)$. Inequality (7) implies that

$$\overline{I}_\mu = \limsup_{r\to 0^+} \int f_r(x)\,\mu(dx) < \infty.$$

From (7) and from the Fatou lemma applied to $m + 1 - f_r\mathbf{1}_{\{f_r < m+1\}}$ we obtain

$$\limsup_{r\to 0^+} \int f_r(x)\,\mu(dx) \le \int \limsup_{r\to 0^+} f_r(x)\,\mu(dx)$$

which implies that $\overline{I}_\mu \le \overline{d}_\mu$. The inequality $\underline{d}_\mu \le \underline{I}_\mu$ follows directly from the Fatou's lemma. The inequality $\underline{I}_\mu \le \overline{I}_\mu$ is trivial. □

3. – Remark.

REMARK 1. – Observe that if for some $a > 0$ a measure μ has a finite a-moment then it satisfies inequality (1). But generaly we cannot omit this assumption in the formulation of Theorem 1. Indeed, there exist probability measures with infinite information dimension and with zero average dimension. That is the case of $\mu = \sum_{n=2}^{\infty} c_n \delta_n$, where $c_n = c/(n\log^2 n)$ and δ_x denotes the Dirac measure supported at the point x.

REMARK 2. – The existence of the information dimension and condition (1) does not imply the existence of average dimension. We construct a Cantor-like measure μ on $\mathbb{R}$ such that $I_\mu = 1/2$ and $\overline{d}_\mu(x) = 1$, $\underline{d}_\mu(x) = 0$ for μ-a.e. $x \in \mathbb{R}$.

Let (k_n) be a strictly increasing sequence of positive integers. We start with the definition of a Cantor-like measure corresponding to the sequence (k_n). Let $k_0 = 0$ and $h_n = 2^{-k_n}$ for non-negative integer n. We define a sequence of measures (μ_n) by induction. Let $a_1^0 = 0$. For $n \geq 0$ and $i = 1, \dots, 2^n$ we put

$$a_{2i-1}^{n+1} = a_i^n, \quad a_{2i}^{n+1} = a_i^n + h_n - h_{n+1}.$$

Let $\mu_n = 2^{-n} \sum_{i=1}^{2^n} \delta_{a_i^n}$ and let μ be the weak limit of the sequence (μ_n). Then we say that the measure μ corresponds to the sequence (k_n). Let X be the support of μ. Fix $x \in X$ and $r \in [h_{n+1}, h_n]$. Since $x \in X$, for each $s \in \mathbb{N}$ there exists i_s such that $a_{i_s}^s \leq x \leq a_{i_s}^s + h_s$. This implies that $(a_{i_{n+1}}^{n+1}, a_{i_{n+1}}^{n+1} + h_{n+1}) \subset B(x, r)$. From this it follows that for $m \geq n+1$ we have

$$2^{m-n-1} - 1 \leq \#\{i : a_i^m \in B(x, r)\},$$

where $\#A$ denotes the number of the elements of the set A. Moreover since $2r \leq h_{n-1}$ there exists a constant λ such that $B(x, r) \subset (\lambda, \lambda + h_{n-1})$. In the interval $(\lambda, \lambda + h_{n-1})$ we have at most 2^{m-n+1} different points a_i^m. Thus for each $m \geq n+1$ we have

$$2^{m-n-1} - 1 \leq \#\{i : a_i^m \in B(x, r)\} \leq 2^{m-n+1}.$$

From these inequalities and from the definition of the measure μ we obtain

$$2^{-1-n} \leq \mu\big(B(x, r)\big) \leq 2^{1-n}.$$

This implies that

$$\frac{(n-1)\log 2}{|\log r|} \leq \frac{\log \mu\big(B(x,r)\big)}{\log r} \leq \frac{(n+1)\log 2}{|\log r|} \tag{8}$$

for $r \in [h_{n+1}, h_n]$ and $x \in X$. Let

$$\beta_\mu(s) = -\int s^{-1}(\log 2)^{-1}\log \mu\big(B(x, 2^{-s})\big)\, \mu(dx)$$

for $s > 0$ and define a function $n : (0, \infty) \to \mathbb{N}$ by $n(s) = n$ if $s \in [k_n, k_{n+1})$. Then

$$\overline{I}_\mu = \limsup_{s\to\infty} \beta_\mu(s), \quad \underline{I}_\mu = \liminf_{s\to\infty} \beta_\mu(s)$$

and from (8) we receive

$$\frac{n(s)-1}{s} \le \beta_\mu(s) \le \frac{n(s)+1}{s} \tag{9}$$

for $s > 0$. From (8) also follows that $\overline{d}_\mu(x) = \overline{I}_\mu$, $\underline{d}_\mu(x) = \underline{I}_\mu$ for μ-a.e. x. Now let (k'_n) and (k''_n) be two strictly increasing sequences of positive integers such that

$$\limsup_{n\to\infty} \frac{k'_n}{n} = \infty, \quad \limsup_{n\to\infty} \frac{k''_n}{n} = \infty, \tag{10}$$

and

$$\lim_{s\to\infty} \frac{n'(s) + n''(s)}{s} = 1, \tag{11}$$

where $n'(s)$ and $n''(s)$ are functions which corresponds to the sequences (k'_n) and (k''_n), respectively, and defined as above. Let μ' and μ'' be the measures corresponding to the sequences (k'_n) and (k''_n), respectively. Let $\mu(A) = \frac{1}{2}\big[\mu'(A) + \mu''(A-2)\big]$, where $A - 2 = \{x - 2 :\, x \in A\}$. Let $\beta_\mu(s)$, $\beta_{\mu'}(s)$, $\beta_{\mu''}(s)$ be the functions which corresponds to the measures μ, μ', and μ'', respectively, and defined as above. Since

$$\beta_\mu(s) = \beta_{\mu'}(s)/2 + \beta_{\mu''}(s)/2 + s^{-1}$$

from (9) and (11) it follows that $I_\mu = 1/2$. On the other hand, from (10) and (11) it follows that $\overline{d}_{\mu'}(x) = 1$, $\underline{d}_{\mu'}(x) = 0$ for μ'-a.e. x and $\overline{d}_{\mu''}(x) = 1$, $\underline{d}_{\mu''}(x) = 0$ for μ''-a.e. x. Consequently $\overline{d}_\mu = (\overline{d}_{\mu'} + \overline{d}_{\mu''})/2 = 1$ and $\underline{d}_\mu = (\underline{d}_{\mu'} + \underline{d}_{\mu''})/2 = 0$.

THEOREM 2. – *Let μ_1 and μ_2 be a probability measure on $\mathbb{R}^m$ and a and β be positive numbers such that $a + \beta = 1$. Then*

$$\underline{d}_{a\mu_1+\beta\mu_2} = a\underline{d}_{\mu_1} + \beta\underline{d}_{\mu_2}.$$

PROOF. – Let $\mu = a\mu_1 + \beta\mu_2$. From the definition of the lower average dimension it follows that

$$\underline{d}_\mu(x) = \min\big\{\underline{d}_{\mu_1}(x),\, \underline{d}_{\mu_2}(x)\big\}. \tag{12}$$

for $x \in \operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2$. For simplicity we set $\underline{d}_\mu(x) = \infty$ if $x \notin \operatorname{supp}\mu$. Then still

$$\underline{d}_\mu = \int_X \underline{d}_\mu(x)\,\mu(dx)$$

but (12) holds for every $x \in \mathbb{R}^m$. Let

$$A_1 = \{x \in \mathbb{R}^m\colon \underline{d}_{\mu_1}(x) < \underline{d}_{\mu_2}(x)\}, \quad A_2 = \{x \in \mathbb{R}^m\colon \underline{d}_{\mu_1}(x) > \underline{d}_{\mu_2}(x)\},$$

and $A_0 = \mathbb{R}^m \setminus (A_1 \cup A_2)$. Observe that

$$\liminf_{r\to 0^+} \frac{\mu_2(B(x,r))}{\mu_1(B(x,r))} = 0$$

for $x \in A_1$. From the well known results concerning the derivative of Radon measures on $\mathbb{R}^m$ (see [3, Ch. 1.6]) it follows that $\mu_2(A_1) = 0$. Analogously $\mu_1(A_2) = 0$. Finally, from (12) we obtain

$$\begin{aligned}\underline{d}_\mu &= \int_X \underline{d}_\mu(x)\,(\alpha\mu_1 + \beta\mu_2)(dx) = \int_{A_1} \alpha\underline{d}_{\mu_1}(x)\,\mu_1(dx)\\ &+ \int_{A_2} \beta\underline{d}_{\mu_2}(x)\,\mu_2(dx) + \int_{A_0} \alpha\underline{d}_{\mu_1}(x)\,\mu_1(dx) + \int_{A_0} \beta\underline{d}_{\mu_2}(x)\,\mu_2(dx)\\ &= \alpha\underline{d}_{\mu_1} + \beta\underline{d}_{\mu_2}. \qquad \square\end{aligned}$$

An interesting application of Theorem 2 concerns the strong convergence of measures. This convergence is given by the *supremum norm* $\|\mu\| = \mu^+(\mathbb{R}^m) + \mu^-(\mathbb{R}^m)$ for any signed finite measure μ.

COROLLARY 1. – *Let (μ_n) be a sequence of probability measures on $\mathbb{R}^m$ convergent strongly to μ. Then*

$$\lim_{n\to\infty} \underline{d}_{\mu_n} = \underline{d}_\mu.$$

REMARK 3. – As far as we know the lower average dimension is the only measure dimension which is linear and is continuous with respect to the strong convergence of measures. For example, the measures μ' and μ'' defined in Remark 2 are such that $\overline{I}_{\mu'} = \overline{I}_{\mu''} = \overline{d}_{\mu'} = \overline{d}_{\mu'} = 1$ and $\underline{I}_{\mu'} = \underline{I}_{\mu''} = 0$ but $\overline{I}_\mu = \overline{d}_\mu = 0$ for $\mu = (\mu' + \mu'')/2$ and $I_\mu = 1/2$ for $\mu(A) = [\mu'(A) + \mu''(A-2)]/2$. Using the measure μ_c defined on the interval $[0,1]$ by $\mu_c = c\delta_0 + (1-c)\lambda$, where $0 \le c \le 1$ and λ is the Lebesgue measure on $\mathbb{R}$, one can check that the Hausdorff, the packing, the box, the generalized dimensions and their lower and upper versions are neither linear nor continuous.

REMARK 4. – The properties of linearity and continuity of the lower average dimension is useful in studying stochastically perturbed dynamical systems. Measures describing the state of such a system often strongly converges to some invariant measure μ_* when time goes to infinity. More precisely, consider a dynamical system on $X \subset \mathbb{R}^m$ with random jumps or a randomly control dynamical system. Then their evolution is described by a semigroup $\{P(t)\}_{t\geq 0}$ defined on the space of probabilistic measures (see [7, 11]). Such a semigroup is of the form $P(t) = K(t) + S(t)$, where $K(t)$ are kernels operators: $K(t)\mu(A) = \int_A \int_X k(t,x,y)\,\mu(dy)\,dx$ and $S(t)$ are "singular" operators such that $S(t)\mu(X) \to 0$ as $t \to \infty$. Under mild assumptions on $\{P(t)\}_{t\geq 0}$ (for example: that there exists only one invariant measure μ_* with support X) $P(t)\mu$ is strongly convergent to μ_* for any probability measure μ. By Corollary 1 the lower average dimension also converges to $\underline{d}_{\mu_*}$ as $t \to \infty$.

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