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Families of Measurable Conic Sections in the Projective Space $\boldsymbol{P}^4$

Gioia Failla - Giovanni Molica Bisci

Sunto. – *In questo lavoro si dimostra la misurabilità della famiglia delle sezioni coniche nello spazio proiettivo* $\boldsymbol{P}^4$*, ottenute intersecando un cono quadratico non degenere con una varietà lineare di dimensione due non contenente il vertice del cono.*

Summary. – *In this note we prove the measurability of the family of non-degenerate conic sections in the projective space* $\boldsymbol{P}^4$*, obtained by cutting a non-degenerate quadratic cone by a linear variety of dimension two not containing the cone vertex .*

1. – Introduction.

Recently M. Stoka in [5] showed that the family of non degenerate conic sections in $\boldsymbol{P}^n$, obtained by cutting a non-degenerarate quadratic cone with an hyperplane not containing the cone vertex, is measurable (1), and gave the invariant integral function (up to a constant factor)

$$\phi = \frac{1}{\left(\sum_{h=1}^{n} A_h x_h + 1\right)^{n+1} |\Delta|^{(n+1)/2}},$$

where Δ is the determinant of the cone directrix in the hyperplane of equation $X_n = 0, x_1, \dots, x_n$ the non homogeneous projective coordinates of the vertex cone and $\sum_{h=1}^{n} A_h X_h + 1 = 0$ is the equation of the hyperplane in the coordinates $X_1, \dots, X_n$. Other results for low dimensions are investigated in previous papers, [2] for $n = 2$ and [3] for $n = 3$.

We consider a linear variety $\boldsymbol{V}_2$ of dimension two in the projective space $\boldsymbol{P}^4$ of non homogeneous projective coordinates $X_1, \dots, X_4$, not containing the cone

(1) For the notion of measurability of a family of varieties and for all the notions of integral geometry used in this paper, see [4].

vertex

$$X_t = \sum_{s=3}^{4} \beta_s^t X_s + \gamma_t, \tag{1}$$

for $t = 1, 2$, where the coefficients β_s^t and γ_t are the essential parameters for $\boldsymbol{V}_2$. We show that the family of non-degenerate conic sections, obtained by cutting a non degenerate quadratic cone with $\boldsymbol{V}_2$, is measurable.

The result is attained by the Deltheil's system associated to the family of section varieties we deduce the existence of the unique (not trivial) integral invariant function Φ.

2. – The theorem.

Let $\boldsymbol{P}^4$ be the projective space of dimension four with non homogeneous projective coordinates X_1, X_2, X_3, X_4.

Let us consider a quadratic non degenerate cone in $\boldsymbol{P}^4$. We denote by x_1, x_2, x_3, x_4 the non homogeneous projective coordinates of the cone vertex and by

$$\sum_{i=1}^{3} a_{ii} X_i^2 + 2 \sum_{i<j}^{1,3} a_{ij} X_i X_j + 2 \sum_{i=1}^{3} a_{i4} X_i + 1 = 0,$$

the equation of the cone directrix in the hyperplane $X_4 = 0$. Then the cone equation is

$$\sum_{i=1}^{3} a_{ii} \left(x_i - x_4 \frac{X_i - x_i}{X_4 - x_4} \right)^2 + 2 \sum_{i<j}^{1,3} a_{ij} \left(x_i - x_4 \frac{X_i - x_i}{X_4 - x_4} \right) \left(x_j - x_4 \frac{X_j - x_j}{X_4 - x_4} \right) +$$

$$+ 2 \sum_{i=1}^{3} a_{i4} \left(x_i - x_4 \frac{X_i - x_i}{X_4 - x_4} \right) + 1 = 0,$$

that is

$$\sum_{i=1}^{3} a_{ii} X_i^2 + 2 \sum_{i<j}^{1,3} a_{ij} X_i X_j - \frac{2}{x_4} \sum_{i=1}^{3} \left(\sum_{j=1}^{3} a_{ij} x_j + a_{i4} \right) X_i X_4 +$$

$$+ \frac{1}{x_4^2} \left(\sum_{i=1}^{3} a_{ii} x_i^2 + 2 \sum_{i<j}^{1,3} a_{ij} x_i x_j + 2 \sum_{i=1}^{3} a_{i4} x_i + 1 \right) X_4^2 + 2 \sum_{i=1}^{4} a_{i4} X_i +$$

$$- \frac{2}{x_4} \left(\sum_{i=1}^{3} a_{i4} x_i + 1 \right) X_4 + 1 = 0, \qquad (x_4 \neq 0),$$

with the conditions $\varDelta = \det \| a_{hk} \| \neq 0$, where $h, k = 1, \ldots, 4, a_{hk} = a_{kh}$ and $a_{44} = 1$.

We consider a linear variety of dimension two, not containing the cone vertex, given by the following equations:

$$X_t = \sum_{s=3}^{4} \beta_s^t X_s + \gamma_t, \tag{2}$$

with

$$x_t - \sum_{s=3}^{4} \beta_s^t x_s - \gamma_t \neq 0, \tag{3}$$

where $t = 1, 2$.

From now on, we call $\boldsymbol{V}_2$ the previous variety.

THEOREM. – *The family of non degenerate conic sections, obtained by cutting a non degenerate quadratic cone by the linear variety $\boldsymbol{V}_2$, is measurable in the projective space $\boldsymbol{P}^4$.*

PROOF. – The family of non degenerate conic sections in the projective space $\boldsymbol{P}^4$ is defined by $\varGamma$ and (2) together with conditions $\varDelta \neq 0$ and (3). So this family is characterized by the following parameters: $x_1, x_2, x_3, x_4, \beta_3^1, \beta_4^1, \beta_3^2, \beta_4^2, \gamma_1, \gamma_2$ and the coefficients a_{ij} with $i \leq j$, $i, j = 1, \ldots, 4$. We denote this family by $\mathcal{F}_{19}$.

The maximum invariance group of the family is the projective group $\boldsymbol{G}_{24}$, given by:

$$X_i = \frac{\sum_{j=1}^{4} a_{ij} X_j' + a_i}{\sum_{j=1}^{4} a_{5j} X_j' + 1}, \qquad (i = 1, 2, 3, 4), \tag{4}$$

with

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_4 \\ a_{51} & a_{52} & a_{53} & a_{54} & 1 \end{pmatrix} \neq 0.$$

Under the action of the group $\boldsymbol{G}_{24}$, the family defined by $\varGamma$ and (2) in transformed into:

$$\varGamma' : \sum_{i=1}^{3} a_{ii}' X_i'^2 + 2 \sum_{i<j}^{1,3} a_{ij}' X_i' X_j' - \frac{2}{x_4'} \sum_{i=1}^{3} \left(\sum_{j=1}^{3} a_{ij}' x_j' + a_{i4}' \right) X_i' X_4' +$$

$$+\frac{1}{{x'_4}^2}\left(\sum_{i=1}^{3} a'_{ii}{x'_i}^2 + 2\sum_{i<j}^{1,3} a'_{ij}x'_ix'_j + 2\sum_{i=1}^{3} a'_{i4}x'_i + 1\right){X'_4}^2 + 2\sum_{i=1}^{3} a'_{i4}X'_i +$$

$$-\frac{2}{x'_4}\left(\sum_{i=1}^{3} a_{i4}x'_i + 1\right)X'_4 + 1 = 0, \qquad (x'_4 \neq 0),$$

(5)
$$X'_t = \sum_{s=3}^{4} \beta'^t_s X_s + \gamma'_t,$$

where

$$x_i = \frac{\sum_{j=1}^4 a_{ij}x'_j + a_i}{\sum_{j=1}^4 a_{5j}x'_j + 1}, \qquad (i = 1, \ldots, 4),$$

(6)
$$\beta'^t_l = \frac{(\sum_{s=3}^4 \beta^t_s + \gamma_t)a_{5l} - a_{tl}}{a_{tt} - (\sum_{s=3}^4 \beta^t_s + \gamma_t)a_{5t}},$$

(7)
$$\gamma'_l = \frac{(\sum_{s=3}^4 \beta^t_s + \gamma_t) - a_t}{a_{tt} - (\sum_{s=3}^4 \beta^t_s + \gamma_t)a_{5t}},$$

$t = 1, 2$ and $l = 3, 4$.

The group $\boldsymbol{H}_{24}$, associated to the group $\boldsymbol{G}_{24}$ with respect to the family of conic sections in $\boldsymbol{P}^4$, has equations (5), (6), (7). Moreover, we need relations that can be found in the paper [5] ((6), (7), (8)), for $n = 4$. The identity I of $\boldsymbol{G}_{24}$ and consequently of $\boldsymbol{H}_{24}$, is given by

$$I: \ a_{11} = \ldots = a_{44} = 1, \qquad a_{12} = \ldots = a_{34} = a_1 = \ldots = a_4 = 0.$$

Moreover the coefficients of the infinitesimal transformations of the group $\boldsymbol{H}_{24}$ are:

$$\xi^i_{hk} = \left(\frac{\partial x'_i}{\partial a_{hk}}\right)_I \qquad \xi^i_{5k} = \left(\frac{\partial x'_i}{\partial a_{5k}}\right)_I \qquad \xi^i_h = \left(\frac{\partial x'_i}{\partial a_h}\right)_I$$

$$\rho^{tl}_{hk} = \left(\frac{\partial \beta'^t_l}{\partial a_{hk}}\right)_I \qquad \rho^{tl}_{5k} = \left(\frac{\partial \beta'^t_l}{\partial a_{5k}}\right)_I \qquad \rho^{tl}_h = \left(\frac{\partial \beta'^t_l}{\partial a_h}\right)_I$$

$$\eta^t_{hk} = \left(\frac{\partial \gamma'_t}{\partial a_{hk}}\right)_I \qquad \eta^t_{5k} = \left(\frac{\partial \gamma'_t}{\partial a_{5k}}\right)_I \qquad \eta^t_h = \left(\frac{\partial \gamma'_t}{\partial a_h}\right)_I$$

$$\zeta_{hk}^{uu} = \left(\frac{\partial a'_{uu}}{\partial a_{hk}}\right)_I \qquad \zeta_{5h}^{uu} = \left(\frac{\partial a'_{uu}}{\partial a_{5h}}\right)_I \qquad \zeta_{h}^{uu} = \left(\frac{\partial a'_{uu}}{\partial a_{h}}\right)_I$$

$$\zeta_{hk}^{uv} = \left(\frac{\partial a'_{uv}}{\partial a_{hk}}\right)_I \qquad \zeta_{5h}^{uu} = \left(\frac{\partial a'_{uv}}{\partial a_{5,h}}\right)_I \qquad \zeta_{h}^{uv} = \left(\frac{\partial a'_{uv}}{\partial a_{h}}\right)_I$$

$h, k = 1, 2, 3, 4; \quad u = 1, 2, 3; \quad v = 1, 2, 3, 4; \quad u < v.$

Finally, by using the results contained in [5], for $n = 4$ we obtain:

$$\xi_{hk}^{i} = -\delta_{ih} x_k, \qquad \xi_{5k}^{i} = x_i x_k, \qquad \xi_{h}^{i} = -\delta_{ih},$$

$$\rho_{hk}^{tl} = -\delta_{th}\delta_{lk}, \qquad \rho_{5k}^{tl} = \left(\sum_{s=3}^{4} \beta_s^t + \gamma_t\right)\delta_{lk}, \qquad \rho_h^{tl} = 0,$$

$$\eta_{hk}^{t} = \left(\sum_{s=3}^{4} \beta_s^t + \gamma_t\right)(-\,\delta_{th}\delta_{tk}), \qquad \eta_{5k}^{t} = \left(\sum_{s=3}^{4} \beta_s^t + \gamma_t\right)^2 \delta_{tk},$$

$$\eta_h^t = -\delta_{th},$$

and the non identically zero ζ functions:

$$\zeta_{vu}^{uu} = 2a_{uv}, \qquad \zeta_{uv}^{vw} = a_{uw}, \qquad \zeta_{vw}^{vw} = a_{vv},$$

$$\zeta_{ww}^{vw} = a_{vw}, \qquad \zeta_{vu}^{u4} = a_{v4}, \qquad \zeta_{4v}^{vw} = -\frac{1}{x_4}\left(\sum_{j=1}^{3} a_{vj}x_j + a_{v4}\right),$$

$$\zeta_{4u}^{u4} = -\frac{1}{x_4}\left(\sum_{i=1}^{3} a_{i4}x_i + 1\right), \qquad \zeta_{4w}^{vw} = -\frac{1}{x_4}\left(\sum_{j=1}^{3} a_{vj}x_j + a_{v4}\right),$$

$$\zeta_{4u}^{uu} = -\frac{2}{x_4}\left(\sum_{i=1}^{3} a_{ui}x_i + a_{u4}\right),$$

$$\zeta_{uw}^{vw} = a_{uv}, \qquad \zeta_{uu}^{vw} = a_{uv},$$

$$\zeta_{5u}^{uu} = 2a_{u4}, \qquad \zeta_{5,w}^{vw} = a_{v4}, \qquad \zeta_{5u}^{u4} = 1,$$

$$\zeta_{5v}^{vw} = a_{w4}, \qquad \zeta_{v}^{uu} = -2a_{uu}a_{v4}, \qquad \zeta_{4}^{uu} = \frac{2a_{uu}}{x_4}\left(\sum_{i=1}^{3} a_{i4}x_i + 1\right),$$

$$\zeta_u^{vw} = -2a_{vw}a_{u4}, \qquad \zeta_4^{vw} = \frac{2a_{vw}}{x_4}\left(\sum_{i=1}^{3} a_{i4}x_i + 1\right),$$

$$\zeta_v^{u4} = a_{uv} - 2a_{u4}a_{v4}, \qquad \zeta_4^{u4} = -\frac{1}{x_4}\left(\sum_{i=1}^{3} a_{ui}x_i + a_{u4}\right) + \frac{2a_{u4}}{x_4}\left(\sum_{i=1}^{3} a_{i4}x_i + 1\right),$$

where $u, v, w = 1, 2, 3$ and $v < w$.

Now we can write the Deltheil's system for the integral invariant function $\phi \neq 0$ of the group $\boldsymbol{H}_{24}$. Since the system consists of 24 equations, ϕ is a differentiable map in the variables $x_1, x_2, x_3, x_4, \beta_3^1, \beta_4^1, \beta_3^2, \beta_4^2, \gamma_1, \gamma_2$ and coefficients a_{ij} with $i \leq j$, $i, j = 1, \dots, 4$. We consider the following sets of equations, denote by G_1, G_2 G_3

G_1 (given by the 16 equations):

$$-x_1\frac{\partial\phi}{\partial x_1} - \Big(\sum_{s=3}^{4}\beta_s^1 + \gamma_1\Big)\frac{\partial\phi}{\partial\gamma_1} + a_{11}\frac{\partial\phi}{\partial a_{11}} + \sum_{j=1}^{4} a_{1j}\frac{\partial\phi}{\partial a_{1j}} = -3\phi,$$

$$-x_2\frac{\partial\phi}{\partial x_1} + a_{12}\frac{\partial\phi}{\partial a_{22}} + \sum_{j=1}^{4} a_{1j}\frac{\partial\phi}{\partial a_{2j}} = 0,$$

$$-x_3\frac{\partial\phi}{\partial x_1} - \frac{\partial\phi}{\partial\beta_3^1} + a_{13}\frac{\partial\phi}{\partial a_{33}} + \sum_{j=1}^{4} a_{1j}\frac{\partial\phi}{\partial a_{3j}} = 0,$$

$$-x_4\frac{\partial\phi}{\partial x_1} - \frac{\partial\phi}{\partial\beta_4^1} = 0,$$

$$-x_1\frac{\partial\phi}{\partial x_2} + a_{12}\frac{\partial\phi}{\partial a_{11}} + \sum_{j=1}^{4} a_{2j}\frac{\partial\phi}{\partial a_{1j}} = 0,$$

$$-x_2\frac{\partial\phi}{\partial x_2} - \Big(\sum_{s=3}^{4}\beta_s^2 + \gamma_2\Big)\frac{\partial\phi}{\partial\gamma_2} + a_{22}\frac{\partial\phi}{\partial a_{22}} + \sum_{j=1}^{4} a_{2j}\frac{\partial\phi}{\partial a_{2j}} = -3\phi,$$

$$-x_3\frac{\partial\phi}{\partial x_2} - \frac{\partial\phi}{\partial\beta_3^2} + a_{23}\frac{\partial\phi}{\partial a_{33}} + \sum_{j=1}^{4} a_{2j}\frac{\partial\phi}{\partial a_{3j}} = 0,$$

$$-x_4\frac{\partial\phi}{\partial x_2} - \frac{\partial\phi}{\partial\beta_4^2} = 0,$$

$$-x_1\frac{\partial\phi}{\partial x_3}+a_{13}\frac{\partial\phi}{\partial a_{11}}+\sum_{j=1}^{4}a_{3j}\frac{\partial\phi}{\partial a_{1j}}=0,$$

$$-x_2\frac{\partial\phi}{\partial x_3}+a_{23}\frac{\partial\phi}{\partial a_{22}}+\sum_{j=1}^{4}a_{2j}\frac{\partial\phi}{\partial a_{3j}}=0,$$

$$-x_3\frac{\partial\phi}{\partial x_3}+a_{33}\frac{\partial\phi}{\partial a_{33}}+\sum_{j=1}^{4}a_{3j}\frac{\partial\phi}{\partial a_{3j}}=-4\phi,$$

$$\frac{\partial\phi}{\partial x_3}=0,$$

$$-x_1\frac{\partial\phi}{\partial x_4}-\frac{1}{x_4}\Big(\sum_{j=1}^{3}a_{j4}x_j+1\Big)\frac{\partial\phi}{\partial a_{14}}-\frac{1}{x_4}\sum_{h=1}^{3}\Big(\sum_{j=1}^{3}a_{hj}x_j+a_{h4}\Big)\frac{\partial\phi}{\partial a_{1h}}=5\frac{x_1}{x_4}\phi,$$

$$-x_2\frac{\partial\phi}{\partial x_4}-\frac{1}{x_4}\Big(\sum_{j=1}^{3}a_{j4}x_j+1\Big)\frac{\partial\phi}{\partial a_{24}}-\frac{1}{x_4}\sum_{h=1}^{3}\Big(\sum_{j=1}^{3}a_{hj}x_j+a_{h4}\Big)\frac{\partial\phi}{\partial a_{2h}}=5\frac{x_2}{x_4}\phi,$$

$$-x_3\frac{\partial\phi}{\partial x_4}-\frac{1}{x_4}\Big(\sum_{j=1}^{3}a_{j4}x_j+1\Big)\frac{\partial\phi}{\partial a_{34}}-\frac{1}{x_4}\sum_{h=1}^{3}\Big(\sum_{j=1}^{3}a_{hj}x_j+a_{h4}\Big)\frac{\partial\phi}{\partial a_{3h}}=5\frac{x_3}{x_4}\phi,$$

$$-x_4\frac{\partial\phi}{\partial x_4}=\phi.$$

G_2 (given by four equations):

$$\sum_{i=1}^{4}x_ix_1\frac{\partial\phi}{\partial x_i}+\Big(\sum_{s=3}^{4}\beta_s^1+\gamma_1\Big)^2\frac{\partial\phi}{\partial\gamma_1}+a_{14}\frac{\partial\phi}{\partial a_{11}}+\sum_{j=1}^{3}a_{j4}\frac{\partial\phi}{\partial a_{1j}}+\frac{\partial\phi}{\partial a_{14}}=-(5x_1+1)\phi,$$

$$\sum_{i=1}^{4}x_ix_2\frac{\partial\phi}{\partial x_i}+\Big(\sum_{s=3}^{4}\beta_s^2+\gamma_2\Big)^2\frac{\partial\phi}{\partial\gamma_2}+a_{24}\frac{\partial\phi}{\partial a_{22}}+\sum_{j=1}^{3}a_{j4}\frac{\partial\phi}{\partial a_{2j}}+\frac{\partial\phi}{\partial a_{24}}=-(5x_2+1)\phi,$$

$$\sum_{i=1}^{4}x_ix_3\frac{\partial\phi}{\partial x_i}+\sum_{t=1}^{2}\Big(\sum_{s=3}^{4}\beta_s^t+\gamma_t\Big)\frac{\partial\phi}{\partial\beta_3^t}+a_{34}\frac{\partial\phi}{\partial a_{33}}+\sum_{j=1}^{3}a_{j4}\frac{\partial\phi}{\partial a_{3j}}+\frac{\partial\phi}{\partial a_{34}}=-(5x_3+1)\phi,$$

$$\sum_{i=1}^{4}x_ix_4\frac{\partial\phi}{\partial x_i}+\sum_{t=1}^{2}\Big(\sum_{s=3}^{4}\beta_s^t+\gamma_t\Big)\frac{\partial\phi}{\partial\beta_4^t}=-(5x_4+1)\phi,$$

G_3 (given by the four equations):

$$-\frac{\partial\phi}{\partial x_1}-\frac{\partial\phi}{\partial\gamma_1}-\sum_{u=1}^{3}(2a_{uu}a_{14})\frac{\partial\phi}{\partial a_{uu}}-\sum_{u<v}2a_{uv}a_{14}\frac{\partial\phi}{\partial a_{uv}}+$$

$$+\sum_{u=1}^{3}(a_{u1}-2a_{u4}a_{14})\frac{\partial\phi}{\partial a_{u4}}-20a_{14}\phi=0,$$

$$-\frac{\partial\phi}{\partial x_2}-\frac{\partial\phi}{\partial\gamma_2}-\sum_{u=1}^{3}(2a_{uu}a_{24})\frac{\partial\phi}{\partial a_{uu}}-\sum_{u<v}2a_{uv}a_{24}\frac{\partial\phi}{\partial a_{uv}}+$$

$$+\sum_{u=1}^{3}(a_{u2}-2a_{u4}a_{24})\frac{\partial\phi}{\partial a_{u4}}-20a_{24}\phi=0,$$

$$-\frac{\partial\phi}{\partial x_3}-\sum_{u=1}^{3}(2a_{uu}a_{34})\frac{\partial\phi}{\partial a_{uu}}-\sum_{u<v}2a_{uv}a_{34}\frac{\partial\phi}{\partial a_{uv}}+$$

$$+\sum_{u=1}^{3}(a_{u3}-2a_{u4}a_{34})\frac{\partial\phi}{\partial a_{u4}}-20a_{34}\phi=0,$$

and in conclusion

$$-\frac{\partial\phi}{\partial x_4}+\sum_{u=1}^{3}\frac{2a_{uu}}{x_4}\Big(\sum_{i=1}^{3}a_{i4}x_i+1\Big)\frac{\partial\phi}{\partial a_{uu}}+\sum_{u<v}\frac{2a_{uv}}{x_4}\Big(\sum_{i=1}^{3}a_{i4}x_i+1\Big)\frac{\partial\phi}{\partial a_{uv}}+$$

$$+\sum_{u=1}^{3}\Big(\frac{2a_{u4}}{x_4}\Big(\sum_{i=1}^{3}a_{i4}x_i+1\Big)-\frac{1}{x_4}\Big(\sum_{i=1}^{3}a_{ui}x_i+a_{u4}\Big)\Big)\frac{\partial\phi}{\partial a_{u4}}+$$

$$+\Big[\frac{18}{x_4}\Big(\sum_{i=1}^{3}a_{i4}x_i+1\Big)-\frac{3}{x_4}\Big]\phi=0.$$

Let us denote by $G_i(j)$ the i-th equation in the set G_i.

Now we consider the sub-system given by the equations $G_1(12)$ and $G_1(16)$:

$$\frac{\partial\phi}{\partial x_3}=0,\qquad -x_4\frac{\partial\phi}{\partial x_4}=\phi,$$

with $t=1,2$. We obtain

$$\phi=\frac{f}{x_4},\qquad (f\neq 0),$$

where

$$f=f(x_1,x_2,\beta_3^1,\beta_4^1,\beta_3^2,\beta_4^2,\gamma_1,\gamma_2,a_{ij})$$

and $i \leq j$ with $i, j = 1, \ldots, 4$. Hence the aim is to find the differentiable non-zero function f in the 17 variables $x_1, x_2, \beta_3^1, \beta_4^1, \beta_3^2, \beta_4^2, \gamma_1, \gamma_2$ and a_{ij}, with $i \leq j$ and $i, j = 1, \ldots, 4$.

We put $G_1(13)$ in $G_2(1)$, $G_1(14)$ in $G_2(2)$, $G_1(15)$ in $G_2(3)$, $G_1(16)$ in $G_2(4)$, we get four equations $E_1, \ldots, E_4$:

E_1:

$$x_1^2 \frac{\partial \log f}{\partial x_1} + x_1 x_2 \frac{\partial \log f}{\partial x_2} + \Big(\sum_{s=3}^{4} \beta_s^1 + \gamma_1\Big)^2 \frac{\partial \log f}{\partial \gamma_1} +$$

$$+ \left(\frac{\sum_{j=1}^{3} a_{1j}x_j + 9a_{14}}{4}\right) \frac{\partial \log f}{\partial a_{11}} + \left(\frac{\sum_{j=1}^{3} a_{2j}x_j + 5a_{24}}{4}\right) \frac{\partial \log f}{\partial a_{12}} +$$

$$+ \left(\frac{\sum_{j=1}^{3} a_{3j}x_j + 5a_{34}}{4}\right) \frac{\partial \log f}{\partial a_{13}} + \left(\frac{\sum_{j=1}^{3} a_{j4}x_j + 5}{4}\right) \frac{\partial \log f}{\partial a_{14}} = -(5x_1 + 1),$$

E_2:

$$x_1 x_2 \frac{\partial \log f}{\partial x_1} + x_2^2 \frac{\partial \log f}{\partial x_2} + \Big(\sum_{s=3}^{4} \beta_s^2 + \gamma_2\Big)^2 \frac{\partial \log f}{\partial \gamma_2} +$$

$$+ \left(\frac{\sum_{j=1}^{3} a_{1j}x_j + 5a_{14}}{4}\right) \frac{\partial \log f}{\partial a_{12}} + \left(\frac{\sum_{j=1}^{3} a_{2j}x_j + 9a_{24}}{4}\right) \frac{\partial \log f}{\partial a_{22}} +$$

$$+ \left(\frac{\sum_{j=1}^{3} a_{3j}x_j + 5a_{34}}{4}\right) \frac{\partial \log f}{\partial a_{23}} + \left(\frac{\sum_{j=1}^{3} a_{j4}x_j + 5}{4}\right) \frac{\partial \log f}{\partial a_{24}} = -(5x_2 + 1),$$

E_3:

$$x_1 x_3 \frac{\partial \log f}{\partial x_1} + x_2 x_3 \frac{\partial \log f}{\partial x_2} + \sum_{t=1}^{2} \Big(\sum_{s=3}^{4} \beta_s^t + \gamma_t\Big) \frac{\partial \log f}{\partial \beta_3^t} +$$

$$+ \left(\frac{\sum_{j=1}^{3} a_{1j}x_j + 5a_{14}}{4}\right) \frac{\partial \log f}{\partial a_{13}} + \left(\frac{\sum_{j=1}^{3} a_{2j}x_j + 5a_{24}}{4}\right) \frac{\partial \log f}{\partial a_{23}} +$$

$$+ \left(\frac{\sum_{j=1}^{3} a_{3j}x_j + 9a_{34}}{4}\right) \frac{\partial \log f}{\partial a_{33}} + \left(\frac{\sum_{j=1}^{3} a_{j4}x_j + 5}{4}\right) \frac{\partial \log f}{\partial a_{34}} = -(5x_3 + 1),$$

E_4:

$$x_1 x_4 \frac{\partial \log f}{\partial x_1} + x_2 x_4 \frac{\partial \log f}{\partial x_2} + \sum_{t=1}^{2} \Big(\sum_{s=3}^{4} \beta_s^t + \gamma_t\Big) \frac{\partial \log f}{\partial \beta_4^t} = -(5x_4 + 1).$$

We substitute ϕ in $G_3(1)$, $G_3(2)$ and we have

$$\frac{\partial \log f}{\partial x_1}+\frac{\partial \log f}{\partial \gamma_1}+\sum_{u=1}^{3}2a_{uu}a_{14}\frac{\partial \log f}{\partial a_{uu}}+\sum_{u<v}2a_{uv}a_{14}\frac{\partial \log f}{\partial a_{uv}}+$$

$$-\sum_{u=1}^{3}(a_{u1}-2a_{u4}a_{14})\frac{\partial \log f}{\partial a_{u4}}=-20a_{14},$$

$$\frac{\partial \log f}{\partial x_2}+\frac{\partial \log f}{\partial \gamma_2}+\sum_{u=1}^{3}2a_{uu}a_{24}\frac{\partial \log f}{\partial a_{uu}}+\sum_{u<v}2a_{uv}a_{24}\frac{\partial \log f}{\partial a_{uv}}+$$

$$-\sum_{u=1}^{3}(a_{u2}-2a_{u4}a_{24})\frac{\partial \log f}{\partial a_{u4}}=-20a_{24}.$$

By the previous equations, we obtain the equations

A_1:

$$\Big(\Big(\sum_{s=3}^{4}\beta_s^1+\gamma_1\Big)^2-x_1^2\Big)\frac{\partial \log f}{\partial \gamma_1}-x_1x_2\frac{\partial \log f}{\partial \gamma_2}+E_{11}^1\frac{\partial \log f}{\partial a_{11}}+E_{12}^1\frac{\partial \log f}{\partial a_{12}}+$$

$$+E_{13}^1\frac{\partial \log f}{\partial a_{13}}+E_{14}^1\frac{\partial \log f}{\partial a_{14}}+E_{22}^1\frac{\partial \log f}{\partial a_{22}}+E_{23}^1\frac{\partial \log f}{\partial a_{23}}+$$

$$+E_{24}^1\frac{\partial \log f}{\partial a_{24}}+E_{33}^1\frac{\partial \log f}{\partial a_{33}}+E_{34}^1\frac{\partial \log f}{\partial a_{34}}=-[(5x_1+1)-20a_{14}x_1^2-20a_{24}x_1x_2],$$

where

$$E_{11}^1:=-2(a_{11}a_{14}x_1^2+a_{11}a_{24}x_1x_2),\quad E_{12}^1:=-2(x_1^2a_{12}a_{14}+a_{12}a_{24}x_1x_2),$$

$$E_{13}^1:=-2(a_{13}x_1^2a_{14}+a_{13}a_{24}x_1x_2),\quad E_{14}^1:=((a_{11}-2a_{14}^2)x_1^2+(a_{12}-2a_{14}a_{24})x_1x_2),$$

$$E_{22}^1:=-2(a_{22}a_{14}x_1^2+a_{22}a_{24}x_1x_2),\quad E_{23}^1:=-2(a_{23}x_1^2a_{14}+a_{23}a_{24}x_1x_2),$$

$$E_{24}^1:=((a_{21}-2a_{24}a_{14})x_1^2+(a_{22}-2a_{24}^2)x_1x_2),\quad E_{33}^1:=-2(a_{33}a_{14}x_1^2+a_{33}a_{24}x_1x_2),$$

$$E_{34}^1:=((a_{31}-2a_{34}a_{14})x_1^2+(a_{32}-2a_{34}a_{24})x_1x_2).$$

In the same way we obtain

A_2:

$$-x_1x_2\frac{\partial\log f}{\partial\gamma_1}+\Big(\Big(\sum_{s=3}^4\beta_s^2+\gamma_2\Big)^2-x_2^2\Big)\frac{\partial\log f}{\partial\gamma_2}+E_{11}^2\frac{\partial\log f}{\partial a_{11}}+E_{12}^2\frac{\partial\log f}{\partial a_{12}}+$$

$$+E_{13}^2\frac{\partial\log f}{\partial a_{13}}+E_{14}^2\frac{\partial\log f}{\partial a_{14}}+E_{22}^2\frac{\partial\log f}{\partial a_{22}}+E_{23}^2\frac{\partial\log f}{\partial a_{23}}+$$

$$+E_{24}^2\frac{\partial\log f}{\partial a_{24}}+E_{33}^2\frac{\partial\log f}{\partial a_{33}}+E_{34}^2\frac{\partial\log f}{\partial a_{34}}=-[(5x_2+1)-20a_{14}x_1x_2-20a_{24}x_2^2],$$

$$E_{11}^2:=-2(a_{11}a_{14}x_1x_2+a_{11}a_{24}x_2^2),$$

$$E_{12}^2:=\frac{\sum_{j=1}^3a_{1j}x_j+5a_{14}}{4}-2(a_{12}a_{14}x_1x_2+a_{12}a_{24}x_2^2),$$

$$E_{13}^2:=-2(a_{13}a_{14}x_1x_2+a_{13}a_{24}x_2^2),$$

$$E_{14}^2:=((a_{11}-2a_{14}^2)x_1x_2+(a_{12}-2a_{14}a_{24})x_2^2),$$

$$E_{22}^2:=\frac{\sum_{j=1}^3a_{2j}x_j+9a_{24}}{4}-2(a_{22}a_{14}x_1x_2+a_{22}a_{24}x_2^2),$$

$$E_{23}^2:=\frac{\sum_{j=1}^3a_{3j}x_j+5a_{34}}{4}-2(a_{23}a_{14}x_1x_2+a_{23}a_{24}x_2^2),$$

$$E_{24}^2:=\frac{\sum_{j=1}^3a_{j4}x_j+5}{4}+((a_{21}-2a_{24}a_{14})x_1x_2+(a_{22}-2a_{24}^2)x_2^2),$$

$$E_{33}^2:=-2(a_{33}a_{14}x_1x_2+a_{33}a_{24}x_2^2),$$

$$E_{34}^2:=((a_{31}-2a_{34}a_{14})x_1x_2+(a_{32}-2a_{34}a_{24})x_2^2),$$

A_3:

$$\Big(\sum_{s=3}^4\beta_s^1+\gamma_1\Big)\frac{\partial\log f}{\partial\beta_4^1}+\Big(\sum_{s=3}^4\beta_s^2+\gamma_2\Big)\frac{\partial\log f}{\partial\beta_4^2}+$$

$$-x_1x_3\frac{\partial\log f}{\partial\gamma_1}-x_2x_3\frac{\partial\log f}{\partial\gamma_2}+E_{11}^3\frac{\partial\log f}{\partial a_{11}}+E_{12}^3\frac{\partial\log f}{\partial a_{12}}+$$

$$+E^3_{13}\frac{\partial \log f}{\partial a_{13}}+E^3_{14}\frac{\partial \log f}{\partial a_{14}}+E^3_{22}\frac{\partial \log f}{\partial a_{22}}+E^3_{23}\frac{\partial \log f}{\partial a_{23}}+$$

$$+E^3_{24}\frac{\partial \log f}{\partial a_{24}}+E^3_{33}\frac{\partial \log f}{\partial a_{33}}+E^3_{34}\frac{\partial \log f}{\partial a_{34}}=-[(5x_3+1)-20a_{14}x_1x_3-20a_{24}x_2x_3],$$

with

$$E^3_{11}:=-2(a_{11}a_{14}x_1x_3+a_{11}a_{24}x_2x_3),\quad E^3_{12}:=-2(a_{12}a_{14}x_1x_3+a_{12}a_{24}x_2x_3),$$

$$E^3_{13}:=\frac{\sum_{j=1}^3 a_{1j}x_j+5a_{14}}{4}-2(a_{13}a_{14}x_1x_3+a_{13}a_{24}x_2x_3),$$

$$E^3_{14}:=((a_{11}-2a^2_{14})x_1x_3+(a_{12}-2a_{14}a_{24})x_2x_3),$$

$$E^3_{22}:=-2(a_{22}a_{14}x_1x_3+a_{22}a_{24}x_2x_3),$$

$$E^3_{23}:=\frac{\sum_{j=1}^3 a_{2j}x_j+5a_{24}}{4}-2(a_{23}a_{14}x_1x_3+a_{23}a_{24}x_2x_3),$$

$$E^3_{24}:=((a_{21}-2a_{24}a_{14})x_1x_3+(a_{22}-2a^2_{24})x_2x_3),$$

$$E^3_{33}:=\frac{\sum_{j=1}^3 a_{3j}x_j+9a_{34}}{4}-2(a_{33}a_{14}x_1x_3+a_{33}a_{24}x_2x_3),$$

$$E^3_{34}:=\frac{\sum_{j=1}^3 a_{j4}x_j+5}{4}+((a_{31}-2a_{34}a_{14})x_1x_3+(a_{32}-2a_{34}a_{24})x_2x_3).$$

A_4:

$$\Big(\sum_{s=3}^4\beta^1_s+\gamma_1\Big)\frac{\partial \log f}{\partial \beta^1_4}+\Big(\sum_{s=3}^4\beta^2_s+\gamma_2\Big)\frac{\partial \log f}{\partial \beta^2_4}-x_1x_4\frac{\partial \log f}{\partial \gamma_1}-x_2x_4\frac{\partial \log f}{\partial \gamma_2}+$$

$$+E^4_{11}\frac{\partial \log f}{\partial a_{11}}+E^4_{12}\frac{\partial \log f}{\partial a_{12}}+$$

$$+\dot{E}^4_{13}\frac{\partial \log f}{\partial a_{13}}+E^4_{14}\frac{\partial \log f}{\partial a_{14}}+E^4_{22}\frac{\partial \log f}{\partial a_{22}}+E^4_{23}\frac{\partial \log f}{\partial a_{23}}+$$

$$+E^4_{24}\frac{\partial \log f}{\partial a_{24}}+E^4_{33}\frac{\partial \log f}{\partial a_{33}}+E^4_{34}\frac{\partial \log f}{\partial a_{34}}=-4x_4+20a_{14}x_1x_4+20a_{24}x_2x_4,$$

where the coefficients are

$$E^4_{11} := -2(a_{11}a_{14}x_1x_4 + a_{11}a_{24}x_2x_4), \quad E^4_{12} := -2(x_1x_4a_{12}a_{14} + a_{12}a_{24}x_2x_4),$$

$$E^4_{13} := -2(a_{13}x_1x_4a_{14} + a_{13}a_{24}x_2x_4), \quad E^4_{14} := ((a_{11} - 2a^2_{14})x_1x_4 + (a_{12} - 2a_{14}a_{24})x_2x_4),$$

$$E^4_{22} := -2(a_{22}a_{14}x_1x_4 + a_{22}a_{24}x_2x_4), \quad E^4_{23} := -2(a_{23}x_1x_4a_{14} + a_{23}a_{24}x_2x_4),$$

$$E^4_{24} := ((a_{21} - 2a_{24}a_{14})x_1x_4 + (a_{22} - 2a^2_{24})x_2x_4), \quad E^4_{33} := -2(a_{33}a_{14}x_1x_4 + a_{33}a_{24}x_2x_4),$$

$$E^4_{34} := ((a_{31} - 2a_{34}a_{14})x_1x_4 + (a_{32} - 2a_{34}a_{24})x_2x_4),$$

We want to give an explicit description of $G_3(3)$ and $G_3(4)$. Precisely $G_3(3)$ is

$$E^5_{11}\frac{\partial \log f}{\partial a_{11}} + E^5_{12}\frac{\partial \log f}{\partial a_{12}} + E^5_{13}\frac{\partial \log f}{\partial a_{13}} + E^5_{14}\frac{\partial \log f}{\partial a_{14}} + $$
$$+ E^5_{22}\frac{\partial \log f}{\partial a_{22}} + E^5_{23}\frac{\partial \log f}{\partial a_{23}} + E^5_{24}\frac{\partial \log f}{\partial a_{24}} + E^5_{33}\frac{\partial \log f}{\partial a_{33}} + E^5_{34}\frac{\partial \log f}{\partial a_{34}} = 20a_{34},$$

where

$$E^5_{11} := -2a_{11}a_{34}, \qquad E^5_{12} := -2a_{12}a_{34}, \qquad E^5_{13} := -2a_{13}a_{34},$$

$$E^5_{14} := (a_{13} - 2a_{14}a_{34}), \qquad E^5_{22} := -2a_{22}a_{34}, \qquad E^5_{23} := -2a_{23}a_{34},$$

$$E^5_{24} := (a_{23} - 2a_{24}a_{34}), \qquad E^5_{33} := -2a_{33}a_{34}, \qquad E^5_{34} := (a_{33} - 2a^2_{34}).$$

and $G_3(4)$ is

$$E^6_{11}\frac{\partial \log f}{\partial a_{11}} + E^6_{12}\frac{\partial \log f}{\partial a_{12}} + E^6_{13}\frac{\partial \log f}{\partial a_{13}} + E^6_{14}\frac{\partial \log f}{\partial a_{14}} + $$
$$+ E^6_{22}\frac{\partial \log f}{\partial a_{22}} + E^6_{23}\frac{\partial \log f}{\partial a_{23}} + E^6_{24}\frac{\partial \log f}{\partial a_{24}} + $$
$$+ E^6_{33}\frac{\partial \log f}{\partial a_{33}} + E^6_{34}\frac{\partial \log f}{\partial a_{34}} = -\left[18\Big(\sum_{i=1}^{3} a_{i4}x_i + 1\Big) - 2\right],$$

where

$$E^6_{11} := 2a_{11}\Big(\sum_{i=1}^{3} a_{i4}x_i + 1\Big), \quad E^6_{12} := 2a_{12}\Big(\sum_{i=1}^{3} a_{i4}x_i + 1\Big), \quad E^6_{13} := 2a_{13}\Big(\sum_{i=1}^{3} a_{i4}x_i + 1\Big),$$

$$E^6_{14} := \left[2a_{14}\Big(\sum_{i=1}^{3} a_{i4}x_i + 1\Big) - \Big(\sum_{i=1}^{3} a_{1i}x_i + a_{14}\Big)\right], \quad E^6_{22} := 2a_{22}\Big(\sum_{i=1}^{3} a_{i4}x_i + 1\Big),$$

$$E^6_{23} := 2a_{23}\Big(\sum_{i=1}^{3} a_{i4}x_i + 1\Big),$$

$$E^6_{24} := \left[2a_{24}\Big(\sum_{i=1}^{3} a_{i4}x_i + 1\Big) - \Big(\sum_{i=1}^{3} a_{2i}x_i + a_{24}\Big)\right], \quad E^6_{33} := 2a_{33}\Big(\sum_{i=1}^{3} a_{i4}x_i + 1\Big),$$

$$E^6_{34} := \left[2a_{34}\Big(\sum_{i=1}^{3} a_{i4}x_i + 1\Big) - \Big(\sum_{i=1}^{3} a_{3i}x_i + a_{34}\Big)\right].$$

Hence we have a system of 17 equations $G_1(i)$ with $i = 1, \dots, 11$, $A_1, \dots, A_4$ and $G_3(3)$ e $G_3(4)$ in 17 variables. We give the system in the following form

$$\boldsymbol{MX} = \boldsymbol{N},$$

where

$$\boldsymbol{M} := \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix},$$

and

$$\boldsymbol{A} := \begin{pmatrix} -x_1 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 2a_{11} \\ -x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_4 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 2a_{12} \\ 0 & -x_2 & 0 & 0 & 0 & 0 & 0 & c_2 & 0 \\ 0 & -x_3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -x_4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

(where $c_1 = -(\sum_{s=3}^{4} \beta^1_s + \gamma_1)$ and $c_2 = -(\sum_{s=3}^{4} \beta^2_s + \gamma_1)$)

$$\boldsymbol{B} := \begin{pmatrix} a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 & 0 \\ a_{11} & 0 & 0 & 2a_{12} & a_{13} & a_{14} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 & 2a_{13} & a_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 & 0 \\ a_{12} & 0 & 0 & 2a_{22} & a_{23} & a_{24} & 0 & 0 \\ 0 & a_{12} & 0 & 0 & a_{22} & 0 & 2a_{23} & a_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$
\boldsymbol{C} := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2a_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_1^2 - x_1^2 & -x_1 x_2 & E_{11}^1 \\
0 & 0 & 0 & 0 & 0 & 0 & -x_1 x_2 & c_2^2 - x_2^2 & E_{11}^2 \\
0 & 0 & 0 & c_1 & 0 & c_2 & -x_1 x_3 & -x_2 x_3 & E_{11}^3 \\
0 & 0 & 0 & c_1 & 0 & c_2 & -x_1 x_4 & -x_2 x_4 & E_{11}^4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_{11}^5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_{11}^6
\end{pmatrix},
$$

and

$$
\boldsymbol{D} := \begin{pmatrix}
a_{23} & a_{33} & a_{34} & 0 & 0 & 0 & 0 & 0 \\
a_{13} & 0 & 0 & 2a_{23} & a_{33} & a_{34} & 0 & 0 \\
0 & a_{13} & 0 & 0 & a_{23} & 0 & 2a_{33} & a_{34} \\
E_{12}^1 & E_{13}^1 & E_{14}^1 & E_{22}^1 & E_{23}^1 & E_{24}^1 & E_{33}^1 & E_{34}^1 \\
E_{12}^2 & E_{13}^2 & E_{14}^2 & E_{22}^2 & E_{23}^2 & E_{24}^2 & E_{33}^2 & E_{34}^2 \\
E_{12}^3 & E_{13}^3 & E_{14}^3 & E_{22}^3 & E_{23}^3 & E_{24}^3 & E_{33}^3 & E_{34}^3 \\
E_{12}^4 & E_{13}^4 & E_{14}^4 & E_{22}^4 & E_{23}^4 & E_{24}^4 & E_{33}^4 & E_{34}^4 \\
E_{12}^5 & E_{13}^5 & E_{14}^5 & E_{22}^5 & E_{23}^5 & E_{24}^5 & E_{33}^5 & E_{34}^5 \\
E_{12}^6 & E_{13}^6 & E_{14}^6 & E_{22}^6 & E_{23}^6 & E_{24}^6 & E_{33}^6 & E_{34}^6
\end{pmatrix}.
$$

Hence, taking into account that

$$
\boldsymbol{X} := \left(\frac{\partial \log f}{\partial x_1}, \frac{\partial \log f}{\partial x_2}, \frac{\partial \log f}{\partial \beta_3^1}, \frac{\partial \log f}{\partial \beta_4^1}, \frac{\partial \log f}{\partial \beta_3^2}, \frac{\partial \log f}{\partial \beta_4^2}, \frac{\partial \log f}{\partial \gamma_1}, \frac{\partial \log f}{\partial \gamma_2}, \frac{\partial \log f}{\partial a_{ij}} \right)^t,
$$

with $i \leq j$ and $i, j = 1, \ldots, 4$, we have:

$$
\boldsymbol{N} := \Bigg(-3, 0, 0, 0, 0, -3, 0, 0, 0, 0, -4, -[(5x_1 + 1) - 20a_{14}x_1^2 - 20a_{24}x_1x_2]
$$

$$
-[(5x_2 + 1) - 20a_{14}x_1x_2 - 20a_{24}x_2^2], -[(5x_3 + 1) - 20a_{14}x_1x_3 - 20a_{24}x_2x_3],
$$

$$
-4(x_4 + 1) + 20a_{14}x_1x_4 + 20a_{24}x_2x_4,
$$

$$
20a_{34}, -\left[18\Big(\sum_{i=1}^{3} a_{i4}x_i + 1 \Big) - 2 \right] \Bigg)^t.
$$

Since $\det(\boldsymbol{M})$ is not identically zero the system has one unique non trivial solution $\log f$. It follows (by integration) that there exists only one non-zero function ϕ that is the solution of the Deltheil's system. This shows the measurability of the family of the non degenerate conic sections.

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