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Sunto. – *In questa nota vengono costruiti i funtori omologia trasversa, associati ciascuno ad una perversità, e viene provata la loro invarianza rispetto ad una opportuna definizione di omotopia.*

Summary. – *We construct, here, transverse homology functors, and we prove their invariance with respect to a suitable definition of homotopy.*

Introduction.

In [4] M. Goreski and R. MacPherson developed a theory of intersection of homology cycles on a pseudomanifold X, generalization of the Poincaré-Lefschetz theory. In their paper the authors attached to an oriented stratified pseudomanifold X a collection of groups $\{IH_n^{\bar{p}}(X)\}_{n\geq 0}$, called intersection homology groups, by using cycles and homologies, as in classical simplicial theory, such that their supports meet the strata of X according to the perversity $\bar{p}$. The groups $IH_n^{\bar{p}}(X)$ are not invariant with respect to homotopy.

In the note [3] we given a different geometrical approach to this theory by showing that the objects may be defined starting from singular geometric cycles.

More in detail, given a stratified pseudomanifold X, we constructed for each perversity $\bar{p}$ a collection of groups $\{H_n^{\bar{p}}(X)\}_{n\geq 0}$, called $\bar{p}$-transverse homology groups of X, and we proved that for each $n \geq 0$ the group $H_n^{\bar{p}}(X)$ coincides, up to an isomorphism, with the homology intersection group $IH_n^{\bar{p}}(X)$.

In this note we construct, for each perversity $\bar{p}$, a category $S_{\bar{p}}$ of stratified pseudomanifolds, whose morphisms are appropriate maps, called $\bar{p}$ maps (section 2). Moreover the above construction of the groups of $\bar{p}$-transverse homology can be seen as construction of a functor $H^{\bar{p}}$ from the category $S_{\bar{p}}$ to the category $\mathcal{A}$ of graded abelian groups.

This construction provides a graduate transfer from the homology intersection functor associate to the largest perversity, which many in cases coincides with ordinary homology functor (normal spaces), to homology intersection functor corresponding to the smallest perversity, which many in cases coincides with ordinary coomology functor (normal spaces again).

In the section 3 we show that our different approach allows us to prove that each functor $H^{\bar{p}}$ is invariant with respect to a suitable definition of homotopy, that is the $\bar{p}$-homotopy, defined and studied in the mentioned section.

For the reader's convenience we include a brief section that provides the definitions of stratified pseudomanifold and $\bar{p}$-transverse homology groups.

More details can be found in [3], [4], [5].

1. – Preliminaries.

A stratified pseudomanifold X of dimension m is a geometric m-cycle (i.e. a compact polyhedron of dimension m devoid of singularities of codimension one) with a filtration by subpolyhedra

$$X = X_m \supseteq X_{m-1} = X_{m-2} \supseteq \dots \supseteq X_1 \supseteq X_0$$

such that for each point $x \in X_i - X_{i-1}$ there is a filtered space

$$V = V_m \supseteq V_{m-1} \supseteq \dots \supseteq V_1 \supseteq V_0 = \text{a point}$$

and a map $g : V \times D^i \to X$ which for each j takes $V_j \times D^i$ PL-homeomorphically to a neighborhood of x in X_j. (Here, D^i is the PL-disk of dimension i).

If $X_i - X_{i-1}$ is not empty, it is a PL manifold of dimension i, called the i-dimensional stratum of the stratification.

For example, the filtration of X by the skeletons of a fixed triangulation is a stratification. At the other extreme, every cycle X has an *intrinsic stratification* $I_m(X), \dots, I_0(X)$ characterized by the property that, if $X = X_m \supseteq X_{m-1} = X_{m-2} \supseteq \dots \supseteq X_1 \supseteq X_0$ is any stratification of X, then $I_h(X) \subseteq X_h$ for all h.

The intrinsic stratification of a PL manifold M has obviously $I_h(M) = \emptyset$ if $h < \dim M$.

A perversity $\bar{p}$ is a sequence of integers $(p_j)_{j \geq 2}$ such that $p_2 = 0$ and $p_{j+1} = p_j$ or $p_{j+1} = p_j + 1$.

A $\bar{p}$-transverse n-cycle without boundary of X is a pair (C,f), where C is an oriented geometric n-cycle without boundary and $f : C \to X$ is a simplicial map such that the polyhedron $f(C)$ is $(\bar{p}, n)$-allowable to X, that is

$$\dim (f(C) \cap X_i) \leq i + n - m + p_{m-i} \quad \text{for each } 0 \leq i \leq m-2$$

for each polyhedron X_i of the filtration of X which meets $f(C)$.

A $\bar{p}$-transverse n-cycle with boundary of X is a pair (C,f), where C is an oriented geometric n-cycle with boundary ∂C, and $f : C \to X$ is a simplicial map such that $f(C)$ is $(\bar{p}, n)$-allowable to X and $f(\partial C)$ is $(\bar{p}, n-1)$-allowable to X, that is

$$\dim (f(C) \cap X_i) \leq i + n - m + p_{m-i}$$

for each polyhedron X_i of the filtration of X which meets $f(C)$, and

$$\dim(f(\partial C) \cap X_i) \leq i + n - m - 1 + p_{m-i}$$

for each polyhedron X_i of the filtration of X which meets $f(\partial C)$.

Clearly, given a $\bar{p}$-transverse (n+1)-cycle with boundary (C,f) of X, the singular n-cycle $\partial(C,f) = \left(\partial C, f/_{\partial C}\right)$ is a $\bar{p}$-transverse n-cycle without boundary of X, called boundary of (C,f).

Two $\bar{p}$-transverse n-cycles without boundary of X (C,f) and (C',f') are said $\bar{p}$-transverse homologous if there exists a $\bar{p}$-transverse (n+1)-cycle with boundary (W,F) such that

1) $\partial W = C \ \dot{\cup} - C'$
2) $F/_C = f, \quad F/_{C'} = f'$

where $-C'$ is the cycle obtained from C' by reversing the orientation.

(W,F) is said a $\bar{p}$-transverse homology between (C,f) and (C',f').

The $\bar{p}$-transverse homology relation is an equivalence relation.

Denote by $H_n^{\bar{p}}(X)$ $(n \geq 0)$ the set of $\bar{p}$-transverse homology classes of $\bar{p}$-transverse n-cycles without boundary of X. In $H_n^{\bar{p}}(X)$ we can define an addition by disjoint union of the first coordinates of the representative elements of two classes.

$\left(H_n^{\bar{p}}(X), +\right)$ is an abelian group for each $n \geq 0$.

The groups $H_n^{\bar{p}}(X)$ are isomorphic to the homology-intersection groups $IH_n^{\bar{p}}(X)$, introduced by Goreski and MacPherson.

2. – The category $S_{\bar{p}}$.

Let

$$X = X_n \supseteq X_{n-1} = X_{n-2} \supseteq \ldots \supseteq X_1 \supseteq X_0$$

$$Y = Y_m \supseteq Y_{m-1} = Y_{m-2} \supseteq \ldots \supseteq Y_1 \supseteq Y_0$$

be stratified pseudomanifolds of respective dimensions n and m, and let $\bar{p}$ a perversity.

DEFINITION 2.1. – *A $\bar{p}$ map $\varphi : X \to Y$ is a PL map such that*

$$\varphi^{-1}(Y_{m-i}) \subseteq X_{n-i+p_i} \quad \textit{for each } 2 \leq i \leq m$$

(*set $X_h = \emptyset$ if $h < 0$*).

REMARK 2.2. – A PL map $\varphi : X \to Y$ is a $\bar{p}$ map if, and only if

$$\varphi(X - X_{n-i+p_i}) \subseteq Y - Y_{m-i} \quad \text{for each } 2 \leq i \leq m.$$

Roughly speaking the image of a stratum belongs to strata of codimension not lower, in other words it is “cleaner”, according to the perversity $\bar{p}$.

REMARK 2.3. – It is important to note that, if $\varphi : X_n \to Y_m$ is a $\bar{p}$ map, then (X_n, φ) is a $\bar{p}$-transverse n-cycle of Y_m, for we have $\dim(\varphi(X_n) \cap Y_{m-i}) \leq \dim \varphi^{-1}(\varphi(X_n) \cap Y_{m-i}) \leq \dim \varphi^{-1}(Y_{m-i}) \leq \dim X_{n-i+p_i} \leq n-i+p_i$. Conversely, if (X_n, φ) is a transverse n-cycle of Y, φ need not be a $\bar{p}$ map, as the following example shows:

$$Y = X = X_2 = S^2 \underset{v}{\cup} S^2 \supset X_1 = X_0 = v, \quad \bar{p} = (0,0,0,...) = \bar{0}$$

then the constant map c_v is not a $\bar{0}$ map although (X_2, c_v) is a $\bar{0}$-transverse cycle of X.

Observe that, if Y is a PL manifold, each PL map $\varphi : X \to Y$ is a $\bar{p}$ map. On the other hand, if X is a PL manifold, φ is a $\bar{p}$ map if, and only if, $\varphi(X) \subseteq Y - Y_{m-1}$.

Moreover note that, if $\varphi : X \to Y$ is a $\bar{p}$ map with respect to same stratification of Y, then φ is a $\bar{p}$ map with respect to intrinsic stratification of Y; on the other hand, if φ is a $\bar{p}$ map with respect to intrinsic stratification of X, then φ is a $\bar{p}$ map with respect to any stratification of X.

The $\bar{p}$ maps are the morphisms of a category whose objects are the stratified pseudomanifolds, because the identity is a $\bar{p}$ map and the composition of $\bar{p}$ maps is so. We denote by $S_{\bar{p}}$ this category.

In order to construct a functor from $S_{\bar{p}}$ to the category $\mathcal{A}$ of graded abelian groups, we show that the $\bar{p}$ maps preserve the transversality, as stated in the following:

PROPOSITION 2.4. – *Let $\varphi : X \to Y$ be a $\bar{p}$ map and let (C_h, f) be a $\bar{p}$-transverse h-cycle of X. Then $(C_h, \varphi \circ f)$ is a $\bar{p}$-transverse h-cycle of Y. Moreover if (C, f), (C', f') are $\bar{p}$-transverse homologous cycles, then $(C, \varphi \circ f)$, $(C', \varphi \circ f')$ are so.*

PROOF. – Let $\varphi : X \to Y$ be a $\bar{p}$ map and let (C_h, f) be a $\bar{p}$-transverse h-cycle of X. Hence we have $\dim(f(C_h) \cap X_{n-i}) \leq h - i + p_i$.

If σ_l is a top dimensional simplex of $\varphi \circ f(C_h) \cap Y_{m-i}$, there exists a simplex τ_s of a subdivision of X such that $\varphi(\tau_s) = \sigma_l$, $\tau_s \subset f(C_h) \cap X_{n-i+p_i}$ (obviously $s \geq l$).

Then we have $\dim(\varphi \circ f(C_h) \cap Y_{m-i}) = l \leq s \leq \dim(f(C_h) \cap X_{n-i+p_i}) \leq h - i + p_i$. So $(C_h, \varphi \circ f)$ is a $\bar{p}$-transverse h-cycle of Y.

The last assertion of the proposition follows easily by observing that if (W, F) is a $\bar{p}$-transverse homology between (C, f) and (C', f'), then $(W, \varphi \circ F)$ is a $\bar{p}$-transverse homology between $(C, \varphi \circ f)$ and $(C', \varphi \circ f')$. □

PROPOSITION 2.5. – *A $\bar{p}$ map $\varphi : X \to Y$ determines a homomorphism $\varphi_* : H_r^{\bar{p}}(X) \to H_r^{\bar{p}}(Y)$ for each $r \geq 0$. Moreover $id_* = id$ and $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$ where $\psi : Y \to Z$ is a $\bar{p}$ map.*

PROOF. – Let a be an element of $H_r^{\bar{p}}(X)$ and let (C,f) be a representative cycle of a. The Prop. 2.4 assures that the $\bar{p}$-transverse homology class of $[C, \varphi \circ f]$ does not depend on the representative element (C,f) of a. So it makes sense to define

$$\varphi_* a = [C, \varphi \circ f]$$

φ_* is a homomorphism because the addition in $H_r^{\bar{p}}(X)$ is defined by disjoint union of the first coordinates of the representative elements.

The remainder of the proposition is straightforward. □

The process above described allows us to build a covariant functor $H^{\bar{p}}$ from the category $S_{\bar{p}}$ of stratified pseudomanifolds to the category $\mathcal{A}$ of graded abelian groups, called $\bar{p}$-transverse homology functor.

3. – $\bar{p}$-homotopy.

As proved in [3], the groups $H_r^{\bar{p}}(X)$ coincide with the groups $IH_r^{\bar{p}}(X)$ attached to a stratified pseudomanifold X, introduced by M. Goreski and R. MacPherson in [4]. The authors observe that these groups are not homotopy invariants.

For example let $X = X_3$ the cone on a torus T of vertex v (more in general on a PL manifold having the first homology group not trivial) and let S_1 a meridian of T. X is stratified as follows

$$X = X_3 \supset X_2 = X_1 = X_0 = v$$

The 1-cycle $d = (S_1,\ \text{inclusion})$ is $\bar{0}$-transverse to X, but it is not $\bar{0}$-transverse homologous to zero, because it is not $\bar{0}$-transverse homologous to zero in T (neither homologous to zero). d is homologous to zero in X, because it is the boundary of the cycle $d' = (vS_1,\ \text{inclusion})$, but this cycle is not a $\bar{0}$-transverse cycle. Hence $H_1^{\bar{0}}(X) \neq 0$ although X is contractible.

Our alternative construction of the groups $IH_r^{\bar{p}}(X)$ allows to find an invariance property with respect to a suitable, different definition of homotopy.

Let

$$X = X_n \supseteq X_{n-1} = X_{n-2} \supseteq \dots \supseteq X_1 \supseteq X_0$$

be a stratified pseudomanifold of dimension n and let $L = X \times I$ the cylinder on X.

In order to give the new definition of homotopy, we need a standard filtration of L. Set $L_i = X_{i-1} \times I$ for each $0 \leq i \leq n+1$.

Each stratum $L_i - L_{i-1} = (X_{i-1} - X_{i-2}) \times I$, if not empty, is a manifold of dimension $(i-1)+1 = i$. In particular $L_1 = X_0 \times I$ is a manifold of dimension 1, if $X_0 \neq \emptyset$.

If

$$V = V_m \supseteq V_{m-1} \supseteq \ldots \supseteq V_1 \supseteq V_0 = \text{a point}$$

is the filtered space related to a point $x \in X$ and $g : V \times D^i \to X$ is the map which for each j takes $V_j \times D^i$ PL-homeomorphically to a neighborhood of x in X_j, it suffices to take the filtered space $V \times I$ and the map $g \times Id$ for the point $(x, t) \in L$.

DEFINITION 3.1. – *Let*

$$X = X_n \supseteq X_{n-1} = X_{n-2} \supseteq \ldots \supseteq X_1 \supseteq X_0$$
$$Y = Y_m \supseteq Y_{m-1} = Y_{m-2} \supseteq \ldots \supseteq Y_1 \supseteq Y_0$$

be stratified pseudomanifolds of respective dimensions n and m and let $\bar{p}$ be a perversity. Two $\bar{p}$ maps φ, $\psi : X \to Y$ are called $\bar{p}$-homotopic (written $\varphi \sim \psi$) if there exists a $\bar{p}$ map $\Omega : X \times I \to Y$ such that $\Omega(x,0) = \varphi(x)$ and $\Omega(x,1) = \psi(x)$ for each $x \in X$.

As in classical homotopy theory, a $\bar{p}$-homotopy Ω is a deformation of a $\bar{p}$ map to the another $\bar{p}$ map. Each stage $\Omega/X \times t$ is a $\bar{p}$ map $\Omega_t : X \to Y$. Indeed for each $t \in I$,

$$\Omega_t(X - X_{n-i+p_i}) \subseteq \Omega\big((X - X_{n-i+p_i}) \times I\big) = \Omega\big(X \times I - (X \times I)_{n+1-i+p_i}\big) \subseteq Y - Y_{m-i}.$$

Analogously for each $x \in X - X_{n-1}$ we have

$$\Omega_x(I) = \Omega(x \times I) \subseteq \Omega((X - X_{n-1}) \times I) = \Omega((X \times I) - (X \times I)_{n+1-1}) \subseteq Y - Y_{m-1}$$

That is Ω_x is a $\bar{p}$ map, Ω_x is a $\bar{p}$ path from $\varphi(x)$ to $\psi(x)$.

REMARK 3.2. – It is important to observe that, if $\Omega : X \times I \to Y$ is a $\bar{p}$-homotopy, the restriction of Ω to $(X - X_h) \times I$ is a homotopy, for each polyhedron X_h of the filtration of X.

The relation of $\bar{p}$-homotopy is an equivalence relation in the set of the $\bar{p}$ maps of X in Y, satisfying the same elementary properties of the classical homotopy.

PROPOSITION 3.3. – *Let $\varphi, \psi : X \to Y$ two $\bar{p}$ maps, $\varphi_*, \psi_* : H^{\bar{p}}(X) \to H^{\bar{p}}(Y)$ the induced homomorphisms. Then $\varphi \sim \psi \Longrightarrow \varphi_* = \psi_*$.*

PROOF. – Let $\varphi \sim \psi$ and let $\Omega : X \times I \to Y$ be a $\bar{p}$-homotopy between φ and ψ. If $a = [(C,f)]$ is an element of $H_r^{\bar{p}}(X)$, then $((C \times I), f \times id)$ is a $\bar{p}$-transverse cycle of $(X \times I)$. By Prop. 2.4 $(C \times I, \Omega \circ (f \times id))$ is a $\bar{p}$-transverse cycle of Y, and it is a $\bar{p}$-transverse homology between $(C, \varphi \circ f)$ and $(C, \psi \circ f)$, so $\varphi_* = \psi_*$. $\square$

DEFINITION 3.4. – *Let X, Y be two stratified pseudomanifolds. We say that X is $\bar{p}$-homotopic to Y (written $X \sim Y$) if there exist two $\bar{p}$ maps $\varphi : X \to Y$ and $\psi : Y \to X$ such that $\psi \circ \varphi \sim id_X$ and $\varphi \circ \psi \sim id_Y$. φ is said $\bar{p}$-homotopy equivalence, ψ is said $\bar{p}$-homotopic inverse of φ.*

As in the classical homotopy theory the $\bar{p}$-homotopy between spaces is an equivalence relation. In particular a stratified pseudomanifold $\bar{p}$-homotopic to a point is said $\bar{p}$-contractible.

PROPOSITION 3.5. – *Let X, Y be stratified pseudomanifolds. If X is $\bar{p}$-homotopic to Y, then $H_r^{\bar{p}}(X)$ is isomorphic to $H_r^{\bar{p}}(Y)$ for each $r \geq 0$.*

PROOF. – If $\varphi : X \to Y$ is a $\bar{p}$-homotopy equivalence, by Prop. 3.4 $\varphi_* : H^{\bar{p}}(X) \to H^{\bar{p}}(Y)$ is an isomorphism. $\square$

Now come back to examine the example of the cone X on the torus T, failure of the homotopy invariance of the $\bar{p}$-transverse homology. By Remark 3.2 X is not $\bar{p}$-contractible because $X - v$ is not contractible.

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