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# On Relative $\gamma_k$-Sets.

M. Bonanzinga - F. Cammaroto - B.A. Pansera

**Sunto.** – *In questo articolo viene presentata una versione relativa del* $\gamma_k$*-insieme introdotto in* [12]. *Vengono date varie caratterizzazioni di questa proprietà; in particolare una delle caratterizzazioni riguarda la teoria di Ramsey. Inoltre viene fornito un risultato che coinvolge una proprietá della corrispondente funzione tra spazi di funzioni.*

**Summary.** – *In this note we show a relative version of* $\gamma_k$*-set introduced and studied in* [12]. *We give several characterizations of this property; in particular one of the characterizations is Ramsey theoretical. Also we give a result involving a property of the corresponding mapping between function spaces.*

## 1. – Introduction.

In several papers it was demonstrated that $k$-covers has a great importance in selection principles theory and fields related to it (see [4], [5]) and in theory of function spaces with the compact-open topology (see [10], [16], [17], [19]). In [14] (see also [2], [18]) it was studied a relative version of the selection principle $S_1(\Omega, \Gamma)$, introduced in [7]; the spaces which satisfy $S_1(\Omega, \Gamma)$ are called *$\gamma$-set*. In this paper we introduce a relative version of the selection principle $S_1(\mathcal{K}, \Gamma_k)$, introduced in [12] where it was called $\gamma_k'$-set. We denote this relative selection principle by $S_1(\mathcal{K}_X, (\Gamma_k)_Y)$, where $Y$ is a subspace of the space $X$. In particular, we give characterizations of it in terms of games and Ramsey theory. Also we give a characterization of the previous relative selection principle in terms of mappings. The reason for this is that several results in the literature(see [13], [8] and [14]) show that there is a nice duality between relative covering properties of a subspace $Y$ of a Tychonoff space $X$ and the closure-type properties of the mapping $\pi$ (introduced in [1]) between function spaces with the pointwise topology.

The notation and terminology we follow are standard as in [6].

An open cover $\mathcal{U}$ of a space $X$ is called:

- a *k-cover* [17], [10] if each compact subset $C$ of $X$ is contained in an element of $\mathcal{U}$ and $X \notin \mathcal{U}$ (i.e. $\mathcal{U}$ is a non-trivial cover);

- a $\gamma_k$-cover [12] if $\mathcal{U}$ is infinite, $X \notin \mathcal{U}$, and for each compact subset $C$ of $X$ the set $\{U \in \mathcal{U} : C \nsubseteq U\}$ is finite.

Because of these definitions we consider only **Hausdorff non-compact spaces**.

Let us mention that any $k$-cover is infinite and large (i.e. each point of the space belongs to infinitely many elements of the cover), and that any infinite subfamily of a $\gamma_k$-cover is also a $\gamma_k$-cover.

Recall that spaces whose each $k$-cover contains a countable subset that is a $k$-cover are called *k-Lindelöf.*

Let $X$ be a topological space and $Y$ be a subspace of $X$. We denote:

1 $\mathcal{K}_X$ - the collection of $k$-covers of $X$;
2 $\mathcal{K}_Y$ the collection of $k$-cover of $Y$ by sets open in $X$;
3 $(\Gamma_k)_X$ - the collection of $\gamma_k$-covers of $X$;
4 $(\Gamma_k)_Y$ - the collection of $\gamma_k$-covers of $Y$ by open sets in $X$.

In what follows, $\mathbb{K}(X)$ denotes the family of all non-empty compact subsets of a space $X$.

### 1.1 – *Function space topologies.*

For a Tychonoff space $X$, $C(X)$ is the set of all continuous real-valued functions on $X$. $C_p(X)$ (resp., $C_k(X)$) denotes the space $C(X)$ with the pointwise topology (resp., with the compact-open topology). For a function $f \in C(X)$, basic open neighbourhoods at $f$ in $C_p(X)$ (resp., $C_k(X)$) are of the form

$$W(f; C; \varepsilon) = \{g \in C_p(X) : |f(x) - g(x)| < \varepsilon, \forall x \in C\},$$

with $C \in \mathbb{F}(X)$ (resp., $C \in \mathbb{K}(X)$) and $\varepsilon > 0$.

The symbol $\underline{0}$ denotes the constantly zero function in $C_p(X)$ and in $C_k(X)$. Since $C_p(X)$ and $C_k(X)$ are homogenous spaces we may consider the point $\underline{0}$ when studying local properties of them.

In [1] Arhangel'skii introduced the mapping $\pi$ from $C_p(X)$ (resp. $C_k(X)$) into $C_p(Y)$ (resp. $C_k(Y)$) defined by $\pi(f) = f_{|Y}$, for each $f \in C_p(X)$ (resp. $f \in C_k(X)$).

### 1.2 – *Selection principles, games and partition relations.*

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set $X$. Then:

a) the symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(b_n : n \in \mathbb{N})$ such that for each $n$, $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$;

b) the symbol $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection principle: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(B_n : n \in \mathbb{N})$ such that for each $n$, $B_n$ is a finite subset of $A_n$ and $\bigcup_{n\in\mathbb{N}} B_n$ is an element of $\mathcal{B}$;

c) the symbol $G_1(\mathcal{A}, \mathcal{B})$ [20] denotes an infinitely long game for two players, ONE and TWO, which play a round for each positive integer. In the $n$-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \cdots; A_n, b_n; \cdots)$ if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

If ONE does not have a winning strategy in the game $G_1(\mathcal{A}, \mathcal{B})$, then the selection hypothesis $S_1(\mathcal{A}, \mathcal{B})$ is true, but the converse need not be always true. In many cases the game characterizes the corresponding selection principle.

For positive integers $n$ and $m$ the symbol $\mathcal{A} \to (\mathcal{B})^n_m$ denotes the statement:

> For each $A \in \mathcal{A}$ and for each function $f : [A]^n \to \{1, \cdots, m\}$ there are a set $B \subset A$, $B \in \mathcal{B}$, and an $i \in \{1, \cdots, m\}$ such that for each $Y \in [B]^n$, $f(Y) = i$.

Here $[A]^n$ denotes the set of $n$-element subsets of $A$. We call $f$ a "coloring" and say that "$B$ is homogeneous of color $i$ for $f$".

This symbol is called the *ordinary partition symbol* [20]. Several selection principles of the form $S_1(\mathcal{A}, \mathcal{B})$ have been characterized by the ordinary partition relation (see [20], [9], [15], [11], [2], [3]).

## 2. – The selection principle $S_1(\mathcal{K}_X, (\Gamma_k)_Y)$.

DEFINITION 2.1 [12]. – *A space $X$ is said to be a $\gamma_k$-set if each k-cover $\mathcal{U}$ of $X$ contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a $\gamma_k$-cover of $X$. $X$ is said to be a $\gamma_k'$-set if it satisfies the selection hypothesis $S_1(\mathcal{K}, \Gamma_k)$.*

In [4] it was shown that these two notions coincide and the term *$\gamma_k$-set* is used for spaces having this property.

THEOREM 2.1 [4]. – *Let $X$ be a space. Then the following are equivalent:*

(1) *$X$ satisfies $S_1(\mathcal{K}, \Gamma_k)$;*
(2) *Each k-cover of $X$ contains a sequence which is a $\gamma_k$-cover of $X$.*

Now we introduce the relative version of $\gamma_k$-sets.

DEFINITION 2.2. – *Let $Y$ be a subset of a space $X$. We say that $Y$ is a $\gamma_k$-set in $X$ (or a relative $\gamma_k$-set) if the selection hypothesis $S_1(\mathcal{K}_X, (\Gamma_k)_Y)$ holds.*

Now we introduce the following relative version of hemicompactness.

DEFINITION 2.3. – *Let $Y$ be a subset of a space $X$. We say that $Y$ is hemicompact in $X$ if there exists a sequence $(C_n : n \in \mathbb{N})$ of compact subsets of $X$ which is cofinal in $\mathbb{K}(Y)$, i.e. each compact set $K \subset Y$ is contained in $C_n$, for some $n \in \mathbb{N}$.*

THEOREM 2.2. – *Let $Y$ be a subspace of a space $X$. If $Y$ is hemicompact in $X$, then $S_1(\mathcal{K}_X, (\Gamma_k)_Y)$ holds.*

PROOF. – Let $(C_n : n \in \mathbb{N})$ be an increasing sequence of compact subsets of $X$ which is cofinal with respect to $\mathbb{K}(Y)$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $k$-covers of $X$. For each $n \in \mathbb{N}$, pick $U_n \in \mathcal{U}_n$ such that $C_n \subset U_n$. We claim that the set $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_k$-cover of $Y$. Let $K \in \mathbb{K}(Y)$. Then there exists $n_0 \in \mathbb{N}$ such that $K \subset C_{n_0}$ and thus $K \subset C_n$, for all $n \geq n_0$. Since $C_n \subset U_n$ we have $K \subset U_n$ for all $n \geq n_0$.

COROLLARY 2.1 (see [17]). – *If $X$ is a hemicompact space, then $X$ satisfies $S_1(\mathcal{K}, \Gamma_k)$.*

Now we give the following characterization of relative $\gamma_k$-sets.

THEOREM 2.3. – *For a k-Lindelöf space $X$ and a subset $Y$ of $X$, the following are equivalent:*

(*a*) *$S_{fin}(\mathcal{K}_X, (\Gamma_k)_Y)$ holds;*
(*b*) *$S_1(\mathcal{K}_X, (\Gamma_k)_Y)$ is satisfied, i.e. $Y$ is a $\gamma_k$-set in $X$;*
(*c*) *ONE does not have a winning strategy in the game $G_1(\mathcal{K}_X, (\Gamma_k)_Y)$;*
(*d*) *For all $n, m \in \mathbb{N}$, it holds $\mathcal{K}_X \to ((\Gamma_k)_Y)^n_m$.*

PROOF. – $(a) \Rightarrow (b)$: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of countable $k$-covers of $X$; suppose that, for each $n$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. For each $n$, let $\mathcal{V}_n$ denote the family of sets of the form $U_{1,k_1} \cap U_{2,k_2} \cap \ldots \cap U_{n,k_n}$. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of $k$-covers of $X$. Since $S_{fin}(\mathcal{K}_X, (\Gamma_k)_Y)$ holds, choose for each $n$ a finite subset $\mathcal{W}_n$ of $\mathcal{V}_n$ such that $\bigcup_{n \in \mathbb{N}} \mathcal{W}_n$ is a $\gamma_k$-cover of $Y$.

The set $\bigcup_{n \in \mathbb{N}} \mathcal{W}_n$ is infinite and all $\mathcal{W}_n$'s are finite, so that there exists a sequence $m_1 < m_2 < \cdots < m_p < \cdots$ in $\mathbb{N}$ such that for each $i \in \mathbb{N}$ we have $\mathcal{W}_{m_i} \setminus \bigcup_{j<i} \mathcal{W}_{m_j} \neq \emptyset$. Choose an element $W_{m_i} \in \mathcal{W}_{m_i} \setminus \bigcup_{j<i} \mathcal{W}_{m_j}$, $i \in \mathbb{N}$, and fix its representation $W_{m_i} = U_{1,k_1} \cap U_{2,k_2} \cap \cdots \cap U_{m_i,k_{m_i}}$ as above.

Using the fact that each infinite subset of a $\gamma_k$-cover is also a $\gamma_k$-cover, we have that the set $\{W_{m_i} : i \in \mathbb{N}\}$ is a $\gamma_k$-cover of $Y$. For each $n \leq m_1$ let $U_n \in \mathcal{U}_n$ be the $n$-th coordinate of $W_{m_1}$ in the chosen representation of $W_{m_1}$, and for each

$n \in (m_i, m_{i+1}]$, $i \geq 1$, let $U_n \in \mathcal{U}_n$ be the $n$-th coordinate of $W_{m_{i+1}}$ in the above representation of $W_{m_{i+1}}$. Observe that each $U_n \supset W_{m_{i+1}}$. Therefore, we obtain a sequence $(U_n : n \in \mathbb{N})$ of elements, one from each $\mathcal{U}_n$, which form a $\gamma_k$-cover of $Y$ and show that $S_1(\mathcal{K}_X, (\Gamma_k)_Y)$ holds.

$(b) \Rightarrow (c)$: Let $\sigma$ be a strategy for ONE in $G_1(\mathcal{K}_X, (\Gamma_k)_Y)$ and let the first move of ONE be a $k$-cover $\sigma(\emptyset) = \{U_{(1)}, U_{(2)} ..., U_{(n)}, ...\}$. Suppose that for each finite sequence $s$ of natural numbers of length $\leq m$, $U_s$ has been already defined. Then define $\{U_{(n_1,...,n_m,k)} : k \in \mathbb{N}\}$ to be the set

$$\sigma(U_{(n_1)}, U_{(n_1,n_2)} ..., U_{(n_1,n_2,...,n_m)}) \setminus \{U_{(n_1)}, U_{(n_1,n_2)} ..., U_{(n_1,n_2,...,n_m)}\}.$$

Because each compact subset of $X$ belongs to infinitely many elements of a $k$-cover of $X$, we have that for each $s$ a finite sequence of natural numbers, the set $\{U_{s\frown(n)} : n \in \mathbb{N}\}$ is a $k$-cover of $X$. Apply (2) and for each $s$ choose $n_s \in \mathbb{N}$ such that $\{U_{s\frown(n_s)} : s \text{ a finite sequence of natural numbers}\}$ is a $\gamma_k$-cover of $Y$.

Inductively define a sequence $n_1 = n_\emptyset, n_{k+1} = n_{(n_1,...n_k)}$ for $k \geq 1$. Then

$$U_{(n_1)}, U_{(n_1,n_2)} ..., U_{(n_1,n_2,...,n_k)}, ...$$

is a $\gamma_k$-cover of $Y$, and because it is actually a sequence of moves of TWO in the game $G_1(\mathcal{K}_X, (\Gamma_k)_Y)$, $\sigma$ is not a winning strategy for ONE.

$(c) \Rightarrow (d)$: We consider the case $n = m = 2$, because the general case can be easily obtained from it by standard induction arguments. Suppose $\mathcal{U} = \{U_1, U_2, \cdots\}$ is a $k$-cover of $X$ and let $f : [\mathcal{U}]^2 \to \{1,2\}$ be a coloring. For $j \in \{1,2\}$ let $\mathcal{H}_j = \{V \in \mathcal{U} : f(\{U_1, V\}) = j\}$. Then at least one of the sets $\mathcal{H}_1$ and $\mathcal{H}_2$ is a $k$-cover of $X$. Denote such a set by $\mathcal{U}_1$ and the corresponding $j$ by $i_1$. In a similar way define inductively sets $\mathcal{U}_n$ of $k$-covers of $X$ and elements $i_n$ from $\{1,2\}$ such that

$$\mathcal{U}_n = \{V \in \mathcal{U}_{n-1} : f(\{U_n, V\}) = i_n\}.$$

So we can define a strategy $\sigma$ for ONE in the game $G_1(\mathcal{K}_X, (\Gamma_k)_Y)$. In the first round ONE plays $\sigma(\emptyset) = \mathcal{U}$. Then choose $i_n \in \{1,2\}$, $n \in \mathbb{N}$, such that $\sigma(U_n) = \{V \in \mathcal{U} : f(\{U_n, V\}) = i_n\}$ is a $k$-cover of $X$. Let us write $\sigma(U_n) = \{U_{n,m} : m \in \mathbb{N}\}$. Suppose for each finite sequence $(n_1, \cdots, n_p)$ of natural numbers we have defined sets $U_{n_1,\cdots,n_p}$ and $i_{n_1,\cdots,n_{p-1}} \in \{1,2\}$ satisfying the condition $\{U_{n_1,\cdots,n_p,m} : m \in \mathbb{N}\}$ is a $k$-cover of $X$ which is equal to the set

$$\{V \in \sigma(U_{n_1}, U_{n_1,n_2}, \cdots, U_{n_1,n_2,\cdots,n_p}) : f(\{U_{n_1,n_2,\cdots,n_p}, V\}) = i_{n_1,n_2,\cdots,n_p}\}.$$

In this way one defines a strategy $\sigma$ for ONE in $G_1(\mathcal{K}_X, (\Gamma_k)_Y)$. As ONE has no winning strategy, there is a play (for TWO)

$$U_{n_1}, U_{n_1,n_2,}, \cdots U_{n_1,n_2,\cdots,n_m}$$

which defeats this strategy. The set $\{U_{n_1}, \cdots U_{n_1,n_2,\cdots,n_m}\}$ is a $\gamma_k$-cover of $Y$. Besides, if $p < q$, then

$$f(\{U_{n_1,n_2,\cdots,n_p}, U_{n_1,n_2,\cdots,n_q}\}) = i_{n_1,n_2,\cdots,n_p}.$$

We may choose $i \in \{1,2\}$ such that for infinitely many $m$ we have $i_{n_1,n_2,\cdots,n_m} = i$. Then define

$$\mathcal{V} = \{U_{n_1,n_2,\cdots,n_m} : i_{n_1,n_2,\cdots,n_m} = i\} \subset \mathcal{U}.$$

This set is a $\gamma_k$-cover of $Y$ (because an infinite subset of a $\gamma_k$-cover is also a $\gamma_k$-cover) and, by construction, is homogeneous for $f$ of color $i$.

$(d) \Rightarrow (a)$: We show that $\mathcal{K}_X \to ((\Gamma_k)_Y)^2_2$ implies $(a)$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $k$-covers of $X$ and suppose that for each $n$, $\mathcal{U}_n = \{U_{n;m} : m \in \mathbb{N}\}$. Consider now the set $\mathcal{V}$ of all nonempty sets of the form $U_{1;m} \cap U_{m;k}$, $n, k \in \mathbb{N}$. Clearly, $\mathcal{V}$ is a $k$-cover of $X$. Define $f : [\mathcal{V}]^2 \to \{1,2\}$ by

$$f(\{U_{1;n_1} \cap U_{n_1;k}, U_{1;n_2} \cap U_{n_2;m}\}) = \begin{cases} 1, & \text{if } n_1 = n_2, \\ 2, & \text{otherwise.} \end{cases}$$

Since $\mathcal{K}_X \to ((\Gamma_k)_Y)^2_2$ holds there are $j \in \{1,2\}$ and a $\mathcal{W} \subset \mathcal{V}$ homogeneous for $f$ of color $j$ such that $\mathcal{W} \in (\Gamma_k)_Y$. Consider two possibilities:

(i) $j = 1$: Then there is some $n$ such that for each $W \in \mathcal{W}$ we have $W \subset U_{1,n}$. However, this means that $\mathcal{W}$ is not a $\gamma_k$-cover of $Y$ and we have a contradiction which shows that this case is impossible.

(ii) $j = 2$: For each $W \in \mathcal{W}$ choose, when it is possible, $U_{n;k_n}$ to be the second term in the chosen representation of $W$; otherwise let $U_{n;k_n} = \emptyset$. Let $\mathcal{V}'$ be the set of all $U_{n,k_n}$'s chosen in this way. Then $\mathcal{V}'$ is a $\gamma_k$-cover of $Y$ witnessing for $(\mathcal{U}_n : n \in \mathbb{N})$ that $S_{fin}(\mathcal{K}_X, (\Gamma_k)_Y)$ is satisfied.

The following lemma will be used in what follows.

LEMMA 2.1. – *For space $X$ and $Y$ and $n \in \mathbb{N}$ the following hold:*

$(a)$ *If $\mathcal{U}$ is a $k$-cover of $X^n$, then there exists a $k$-cover $\mathcal{V}$ of $X$ such that $\{V^n : V \in \mathcal{V}\}$ refines $\mathcal{U}$* [5];

$(b)$ *If $Y$ is compact and $\mathcal{U}$ is a $k$-cover of $X \times Y$, then there is a $k$-cover $\mathcal{V}$ of $X$ such that $\{V \times Y : V \in \mathcal{V}\}$ refines $\mathcal{U}$* [4].

THEOREM 2.4. – *For a space $X$ and a subset $Y$ of $X$ the following are equivalent:*

$(a)$ *$Y$ is a $\gamma_k$-set in $X$;*

$(b)$ *$Y^2$ is a $\gamma_k$-set in $X^2$ (and thus for each positive integer $n$, $Y^n$ is a $\gamma_k$-set in $X^n$).*

PROOF. – $(a) \Rightarrow (b)$: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $k$-covers of $X^2$. For each $n$ let, by (a) in Lemma 2.1, $\mathcal{V}_n$ a $k$-cover of $X$ such that $\{V^2 : V \in \mathcal{V}_n\}$ refines $\mathcal{U}_n$. Since $Y$ is a $\gamma_k$-set in $X$, one can find a sequence $(V_n : n \in \mathbb{N})$ such that for each $n$, $V_n \in \mathcal{V}_n$ and for every compact set $K$ of $Y$, there exists $n_0 \in \mathbb{N}$ such that $K \subset V_n$ for any $n > n_0$. For each $n$, we let $U_n$ denote an element in $\mathcal{U}_n$ satisfying $V_n^2 \subset U_n$.

We claim that $(U_n : n \in \mathbb{N})$ witnesses that $Y^2$ is a $\gamma_k$-set in $X^2$. Let $T$ be a compact subset of $Y^2$. Then the union $\bigcup_{i=1,2} p_i(T) = M$ of the projections of $T$ into $X$ is a compact subset of $Y$ and thus there exists $n_0 \in \mathbb{N}$ such that $M \subset V_n$ for any $n > n_0$. For each $n$ take the corresponding $U_n \in \mathcal{U}_n$ with $V_n^2 \subset U_n$. Then for each $n > n_0$, we have $T \subset M^2 \subset V_n^2 \subset U_n$, i.e. $Y^2$ is a $\gamma_k$-set in $X^2$.

$(b) \Rightarrow (a)$: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $k$-covers of $X$. For each $n$ let $\mathcal{V}_n = \{U^2 : U \in \mathcal{U}_n\}$. It easy to show that $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of $k$-covers of $X^2$. By assumption, for each $n$, we can choose a $U_n \in \mathcal{U}_n$ such that, for every compact set $K$ of $Y^2$, there exists $n_0 \in \mathbb{N}$ such that $K \subset U_n^2$, for any $n > n_0$. We verify that the sequence $(U_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that $Y$ is a $\gamma_k$-set in $X$.

Let $T$ be a compact subset of $Y$. Since $T^2$ is a compact subset of $Y^2$ there is some $s \in \mathbb{N}$ such that $T^2 \subset U_n^2$ for all $n > s$. It is implies $T \subset U_n$ for all $n > s$.

In [4] it was shown the following result:

THEOREM 2.5 ([4]). – *The product $X \times Y$ of a space $X$ satisfying $S_1(\mathcal{K}, \mathcal{K})$ and a hemicompact space $Y$ belongs to the class $S_1(\mathcal{K}, \mathcal{K})$.*

We prove:

THEOREM 2.6. – *Let $X$ and $Y$ be spaces and $Z \subseteq X$, $T \subseteq Y$. If $Z$ satisfies $S_1(\mathcal{K}_X, (\Gamma_k)_Z)$ and $T$ is hemicompact in $Y$, then $S_1(\mathcal{K}_{X\times Y}, (\Gamma_k)_{Z\times T})$ holds.*

PROOF. – Let $\{K_n : n \in \mathbb{N}\}$ be an increasing sequence of compact subsets of $Y$ such that each compact subset of $T$ is contained in some $K_m$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $k$-covers of $X \times Y$. By (b) in Lemma 2.1, for each $n \in \mathbb{N}$, there is a $k$-cover $\mathcal{V}_n$ of $X$ such that $\{V \times K_n : V \in \mathcal{V}_n\}$ refines $\mathcal{U}_n$. Since $S_1(\mathcal{K}_X, (\Gamma_k)_Z)$ is satisfied, select for each $n \in \mathbb{N}$ an element $V_n \in \mathcal{V}_n$ such that $\{V_n : n \in \mathbb{N}\}$ is a $\gamma_k$-cover of $Z$. Choose for each $n \in \mathbb{N}$ some $U_n \in \mathcal{U}_n$ such that $V_n \times K_n \subset U_n$. We claim that the sequence $(U_n : n \in \mathbb{N})$ is a $\gamma_k$-cover of $Z \times T$. Let $C$ be a compact subset of $Z \times T$. Then $p_Y(C)$, the projection of $C$ into $Y$, is a compact subset of $T$ and thus is a subset of $K_n$ for all $n \in \mathbb{N}$ greater than some $n_0$. The set $p_X(C)$, the projection of $C$ into $X$, is a compact subset of $Z$. Thus there is $n_1 \in \mathbb{N}$, such that $p_X(C)$ is contained in $V_n$ for all $n \in \mathbb{N}$ with $n \geq n_1$. Thus $C$ is contained in $U_n$, for all $n \in \mathbb{N}$, with $n \geq \max\{n_0, n_1\}$.

COROLLARY 2.2 ([4]). – *The product $X \times Y$ of a space $X$ satisfying $S_1(\mathcal{K}, \Gamma_k)$ and a hemicompact space $Y$ belongs to the class $S_1(\mathcal{K}, \Gamma_k)$.*

## 3. – $S_1(\mathcal{K}_X, (\Gamma_k)_Y)$ and mappings.

In this section we characterize relative $\gamma_k$-sets in terms of the restriction mapping between function spaces with the compact-open topology.

DEFINITION 3.1. – ([14]) *A continuous mapping $f : X \to Y$ is said to be strongly Fréchet if for each sequence $(A_n : n \in \mathbb{N})$ of subsets of $X$ and each $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, there is a sequence $(a_n : n \in \mathbb{N})$ such that $a_n \in A_n$ for each $n$, such that the sequence $(f(a_n) : n \in \mathbb{N})$ converges to $f(x)$.*

Now we prove the following result (compare with [14]).

THEOREM 3.1. – *For a Tychonoff space $X$ and its subspace $Y$, the following are equivalent:*

(1) *$Y$ is a $\gamma_k$-set in $X$;*
(2) *The mapping $\pi : C_k(X) \to C_k(Y)$ is strongly Fréchet.*

PROOF. – (1) $\Rightarrow$ (2): Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $C_k(X)$, such that $\underline{0} \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. Fix $n$. For every compact subset $K$ of $X$, the basic neighborhood $W = W\left(\underline{0}, K, \frac{1}{n}\right)$ of $\underline{0}$ intersects $A_n$; pick a function $f_{K,n} \in A_n$ such that $|f_{K,n}(x)| < \frac{1}{n}$ for each $x \in K$. Since $f_{K,n}$ is a continuous mapping, there are neighborhoods $U_x$ of $x$, $x \in K$, such that for $U_{K,n} = \bigcup_{x\in K} U_x \supset K$ it holds $f_{K,n}(U_{K,n}) \subset \subset \left(-\frac{1}{n}, \frac{1}{n}\right)$. If $\mathcal{U}_n = \{U_{K,n} : K$ compact subset of $X\}$, then $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of $k$-covers of $X$. By assumption, one can find a sequence $(U_{K_n,n} : n \in \mathbb{N})$ such that for each $n$, $U_{K_n,n} \in \mathcal{U}_n$ and each compact subset of $Y$ is contained in all elements of $U_{K_s,s}$, for all $s$ bigger than some $n_0 \in \mathbb{N}$. For each $n$ consider the corresponding function $f_{K_n,n} \in A_n$. We verify that the sequence $(f_{K_n,n} : n \in \mathbb{N})$ witnesses for $(A_n : n \in \mathbb{N})$ that $\pi$ is strongly Fréchet.

Let $W = W(\pi(\underline{0}), K, \varepsilon))$ be a neighborhood of $\pi(\underline{0})$ in $C_k(Y)$ and suppose that $m$ is a positive integer such that $\frac{1}{m} < \varepsilon$. Since $K$ is a compact subset of $Y$ and $Y$ is a $\gamma_k$-set in $X$, there is $n_0 \in \mathbb{N}$ such that $K \subset U_{K_s,s}$ for each $s > n_0$. This means that for each $s > n_0$, it holds $(\pi(f_{K_s,s}))(K) \subset \left(-\frac{1}{s}, \frac{1}{s}\right)$. For all $n >$max$\{n_0, m\}$

we have

$$\pi(f_{K_n,n})(K) = f_{K_n,n}(K) \subset f_{K_n,n}(U_{K_n,n}) \subset \left(-\frac{1}{n}, \frac{1}{n}\right) \subset (-\varepsilon, \varepsilon),$$

i.e. $\pi(f_{K_n,n}) \in W$ for each $n >$max$\{n_0, m\}$.

(2) $\Rightarrow$ (1) : Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $k$-covers of $X$. For each $n \in \mathbb{N}$ and each compact subset $K$ of $X$ we denote by $\mathcal{U}_{n,K}$ the set $\{U \in \mathcal{U}_n : K \subset U\}$. If $U \in \mathcal{U}_{n,K}$, let $f_{U,K} : X \to [0,1]$ be a continuous function satisfying $f_{U,K}(K) = 0$, $f_{U,K}(X \setminus U) = 1$. Let $A_n = \{f_{U,K} : K$ compact subset of $X$, $U \in \mathcal{U}_{n,K}\}$. Then $\underline{0} \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$: if $W(\underline{0}, H, \varepsilon)$ is a basic neighborhood of $\underline{0}$ and $U \in \mathcal{U}_{n,H}$, then the function $f_{U,H}$ belongs to $A_n \cap W(\underline{0}, H, \varepsilon)$, for each $n$.

Since $\pi$ is strongly Fréchet there exists a sequence $(f_{U_n,K_n} : n \in \mathbb{N})$ such that for each $n$, $f_{U_n,K_n} \in A_n$ and $(\pi(f_{U_n,K_n}) : n \in \mathbb{N})$ converges to $\pi(\underline{0})$. Consider the corresponding sets $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$ and prove that the sequence $(U_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that $Y$ is a $\gamma_k$-set in $X$.

Let $T$ be a compact subset of $Y$. Then there exists $n_0 \in \mathbb{N}$ such that the neighborhood $W = W(\pi(\underline{0}), T, 1)$ of $\pi(\underline{0}) \in C_k(Y)$ contains all $\pi(f_{U_n,K_n})$ with $n > n_0$, i.e. $\pi(f_{U_n,K_n}) \in W$ for each $n > n_0$. This implies $T \subset U_n$ for each $n > n_0$, i.e. (1) holds.

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