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Remarks on $\mathcal{S}$-Closedness in Topological Spaces.

ZBIGNIEW DUSZYŃSKI

Sunto. – *Relativamente al* [27], *sono provate alcune proprietà dei sottospazi $\mathcal{S}$-chiusi e dei sottoinsiemi $\mathcal{S}$-chiusi di uno spazio topologico. Sono studiate delle condizioni mediante le quali le applicazioni conservano alcuni sottospazi $\mathcal{S}$-chiusi.*

Summary. – *Corresponding to* [27], *some properties of $\mathcal{S}$-closed subspaces and subsets $\mathcal{S}$-closed relative to a topological space are proved. Conditions under which mappings preserve certain $\mathcal{S}$-closed subspaces are investigated.*

1. – Preliminaries.

Topological spaces are denoted by (X, τ). Let S be a subset of a space (X, τ). We denote the interior and the closure of S in this space by $\mathrm{int}\,(S)$ (or $\mathrm{int}_X(S)$) and $\mathrm{cl}\,(S)$ (or $\mathrm{cl}_X(S)$), respectively. The set S is said to be *regular open* (resp. *regular closed*) in (X, τ), if $S = \mathrm{int}\,(\mathrm{cl}\,(S))$ (resp. $S = \mathrm{cl}\,(\mathrm{int}\,(S))$). The S is said to be *a-open* [23] (resp. *semi-open* [19]; *preopen* [20]; *semi-preopen* [1, 2]), if $S \subset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S)))$ (resp. $S \subset \mathrm{cl}\,(\mathrm{int}\,(S))$; $S \subset \mathrm{int}\,(\mathrm{cl}\,(S))$; $S \subset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S)))$). The complement to X of an a-open (resp. semi-open; preopen; semi-preopen) set is said to be an *a-closed* (resp. *semi-closed*; *preclosed*; *semi-preclosed*) set. The intersection of all a-closed (resp. semi-closed; preclosed; semi-preclosed) sets (in (X, τ)) containing S is called the *a-closure* (resp. *semi-closure*; *preclosure*; *semi-preclosure*) of S in (X, τ), and it is denoted respectively by a-$\mathrm{cl}(S)$ (or a-$\mathrm{cl}_X(S)$), $\mathrm{scl}\,(S)$ (or $\mathrm{scl}_X(S)$), $\mathrm{pcl}\,(S)$ (or $\mathrm{pcl}_X(S)$), $\mathrm{spcl}(S)$ (or $\mathrm{spcl}_X(S)$). The set S is a-closed (resp. semi-closed; preclosed; semi-preclosed) if and only if a-$\mathrm{cl}(S) = S$ (resp. $\mathrm{scl}(S) = S$; $\mathrm{pcl}(S) = S$; $\mathrm{spcl}(S) = S$). Each closed subset of a space (X, τ) is a-closed, semi-closed, preclosed, and semi-preclosed. The collection of all a-open (resp. semi-open; preopen; semi-preopen) subsets of a space (X, τ) is denoted by a-$\mathrm{O}(X, \tau)$ or τ^a (resp. $\mathrm{SO}(X, \tau)$, $\mathrm{PO}(X, \tau)$; $\mathrm{SPO}(X, \tau)$). The family of all regular open (resp. regular closed; semi-closed) subsets of (X, τ) is denoted by $\mathrm{RO}(X, \tau)$ (resp. $\mathrm{RC}(X, \tau)$, $\mathrm{SC}(X, \tau)$). Members of the intersection $\mathrm{SR}(X, \tau) = \mathrm{SO}(X, \tau) \cap \mathrm{SC}(X, \tau)$ are called *semi-regular* sets [9]. A space (X, τ) is *extremally disconnected* (briefly *e.d.*) if $\mathrm{cl}\,(V) \in \tau$ for each $V \in \tau$.

T. Thompson [40] has defined an (X, τ) to be *$\mathcal{S}$-closed*, if for every cover $\{V_a : a \in \mathcal{A}\} \subset \mathrm{SO}(X, \tau)$ of X there exists a finite subfamily $\mathcal{A}_1 \subset \mathcal{A}$ such that $X = \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}_X(V_a)$. T. Noiri [27] has defined a subset S of (X, τ) to be *$\mathcal{S}$-closed relative* to (X, τ), if for every cover $\{V_a : a \in \mathcal{A}\} \subset \mathrm{SO}(X, \tau)$ of S there exists a finite subfamily $\mathcal{A}_1 \subset \mathcal{A}$ such that $S \subset \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}_X(V_a)$.

2. – $\mathcal{S}$-closed subspaces.

From [27, Corollary 3.4] we obtain, as a particular case, the following

COROLLARY 2.1. – *If A and B are both $\mathcal{S}$-closed regular open subspaces of a space (X, τ), then $A \cap B$ is an $\mathcal{S}$-closed subspace of (X, τ).*

Utilizing [27, Theorem 3.1], we can reexpress Corollary 2.1 as follows:

COROLLARY 2.1′. – *If sets A and B are both regular open and are $\mathcal{S}$-closed relative to (X, τ), then $A \cap B$ is an $\mathcal{S}$-closed subspace of (X, τ).*

This result we generalize in the following way.

THEOREM 2.2. – *Let $A, B \in \mathrm{SC}(X, \tau)$ and $A \cap B \in \tau$. If A and B are both $\mathcal{S}$-closed relative to (X, τ), then $A \cap B$ is an $\mathcal{S}$-closed subspace of (X, τ).*

PROOF. – Since $A \cap B \in \mathrm{SC}(X, \tau) \cap \tau$, clearly we have $A \cap B \in \mathrm{RO}(X, \tau)$ (see also for instance [11, Lemma 2.2 (2)]). So, it follows from [27, Theorems 3.3 and 3.1] that $A \cap B$ is an $\mathcal{S}$-closed subspace. □

LEMMA 2.3. – *Let A be a subset of (X, τ). Then, the following holds:*

(a) [18, Proposition 2.7]. $A \in \mathrm{PO}(X, \tau)$ *iff* $\mathrm{scl}(A) = \mathrm{int}(\mathrm{cl}(A))$.
(b) [2, Theorem 2.20(a)]. $A \in a\text{-}O(X, \tau)$ *iff* $\mathrm{spcl}(A) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$.
(c) [2, Theorem 2.20(c)]. $A \in \mathrm{SPO}(X, \tau)$ *iff* $a\text{-}\mathrm{cl}(A) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$.
(d) $A \in \mathrm{SO}(X, \tau)$ *iff* $\mathrm{pcl}(A) = \mathrm{cl}(\mathrm{int}(A))$.

PROOF. – To prove the case (d) we use [2, Theorem 1.5 (e)]. □

THEOREM 2.4. – *Let an A be $\mathcal{S}$-closed relative to (X, τ). Then,*

(a) $a\text{-}\mathrm{cl}_X(A)$, $\mathrm{scl}_X(A)$, $\mathrm{pcl}_X(A)$, $\mathrm{spcl}_X(A)$ *are $\mathcal{S}$-closed relative to* (X, τ)*;*
(b_1) *if* $A \in \mathrm{PO}(X, \tau)$, *then* $\mathrm{int}_X(\mathrm{scl}_X(A))$ *is $\mathcal{S}$-closed relative to* (X, τ)*;*
(b_2) *if* $A \in a\text{-}\mathrm{O}(X, \tau)$, *then* $\mathrm{int}_X(\mathrm{spcl}_X(A))$ *is $\mathcal{S}$-closed relative to* (X, τ)*;*
(b_3) *if* $A \in \mathrm{SPO}(X, \tau)$, *then* $\mathrm{int}_X(a\text{-}\mathrm{cl}_X(A))$ *is $\mathcal{S}$-closed relative to* (X, τ)*;*
(b_4) *if* $A \in \mathrm{SO}(X, \tau)$, *then* $\mathrm{int}_X(\mathrm{pcl}_X(A))$ *is $\mathcal{S}$-closed relative to* (X, τ).

PROOF. – (a). Proofs for all kinds of closures are quite similar to that of [27, Theorem 3.4] (for $\mathrm{cl}\,(A)$).

(b). We apply: respective parts of the case (a), Lemma 2.3, and [27, Theorem 3.3]. □

COROLLARY 2.5. – *If* $A \in \mathrm{PO}(X, \tau)$ (*resp.* $A \in \alpha\text{-}\mathrm{O}(X, \tau)$; $A \in \mathrm{SPO}(X, \tau)$; $A \in \mathrm{SO}(X, \tau)$) *is $\mathcal{S}$-closed relative to* (X, τ), *then the set* $\mathrm{int}_X(\mathrm{scl}_X(A))$ (*respectively* $\mathrm{int}_X(\mathrm{spcl}_X(A))$; $\mathrm{int}_X(\alpha\text{-}\mathrm{cl}_X(A))$; $\mathrm{int}_X(\mathrm{pcl}_X(A))$) *is an $\mathcal{S}$-closed subspace of* (X, τ).

PROOF. – Follows from Theorem 2.4(b), Lemma 2.3, and [27, Theorem 3.1]. □

REMARK 2.6. – Without difficulties one checks that [27, Theorem 3.5] is also true if we replace "A is ... open ... " by "A is an $\mathcal{S}$-closed α-open ... ". We obtain below that similar results hold also for weaker kinds of closure of α-open sets.

THEOREM 2.7. – *Let A be an $\mathcal{S}$-closed α-open subspace of* (X, τ). *Then,* $\mathrm{scl}_X(A)$, $\mathrm{spcl}_X(A)$, $\alpha\text{-}\mathrm{cl}_X(A)$, *and* $\mathrm{pcl}_X(A)$ *are $\mathcal{S}$-closed subspaces of* (X, τ).

PROOF. – We apply respective parts of Lemma 2.3 for each considered case of weak closures, which are of the form $\mathrm{int}\,(\mathrm{cl}\,(.))$ or $\mathrm{cl}\,(\mathrm{int}\,(.))$. So, $\mathrm{scl}_X(A)$, $\mathrm{spcl}_X(A)$, $\alpha\text{-}\mathrm{cl}_X(A)$, and $\mathrm{pcl}_X(A)$ are semi-open in (X, τ). The proof for each case is quite similar to that of [27, Theorem 3.5] and hence we can leave details to the reader. □

THEOREM 2.8. – *Let* (X, τ) *be an $\mathcal{S}$-closed space.*

(A) *Let A be a semi-closed* (*resp. a semi-preclosed*) *subset of* (X, τ). *If* $A \in \mathrm{PO}(X, \tau)$ (*resp.* $A \in \alpha\text{-}\mathrm{O}(X, \tau)$) *then A is an $\mathcal{S}$-closed subspace of* (X, τ).

(B) *Let A be an α-closed* (*resp. a preclosed*) *subset of* (X, τ). *If* $A \in \mathrm{SPO}(X, \tau)$ (*resp.* $A \in \mathrm{SO}(X, \tau)$) *and* $\mathrm{Fr}(A)$ *is $\mathcal{S}$-closed relative to* (X, τ), *then A is $\mathcal{S}$-closed relative to* (X, τ).

PROOF. – (A). This follows from Lemma 2.3 and [27, Corollary 3.2].

(B) follows from Lemma 2.3, [27, Theorem 3.3], and [27, Theorem 3.6] (see the proof of [27, Theorem 3.7]). □

Recall that a space (X, τ) is called *locally $\mathcal{S}$-closed* [27, Definition 4.1], if each point of X has an open neighbourhood which is an $\mathcal{S}$-closed subspace of (X, τ).

THEOREM 2.9. – (*see* [27, Theorem 4.1]). *For a space* (X, τ) *the following are equivalent:*

(1) (X, τ) *is locally $\mathcal{S}$-closed.*

(2) *Each point of X has an open neighbourhood which is $\mathcal{S}$-closed relative to (X, τ).*

(3) *Each point of X has an open neighbourhood V such that* a-$\mathrm{cl}_X(V)$ (*resp.* $\mathrm{scl}_X(V)$; $\mathrm{pcl}_X(V)$; $\mathrm{spcl}_X(V)$) *is $\mathcal{S}$-closed relative to (X, τ).*

(4) *Each point of X has an open neighbourhood V such that* $\mathrm{int}_X(a\text{-}\mathrm{cl}_X(V))$ (*resp.* $\mathrm{int}_X(\mathrm{scl}_X(V))$; $\mathrm{int}_X(\mathrm{pcl}_X(V))$; $\mathrm{int}_X(\mathrm{spcl}_X(V))$) *is $\mathcal{S}$-closed relative to (X, τ).*

(5) *Each point of X has an open neighbourhood V such that* $\mathrm{int}_X(a\text{-}\mathrm{cl}_X(V))$ (*resp.* $\mathrm{int}_X(\mathrm{scl}_X(V))$; $\mathrm{int}_X(\mathrm{pcl}_X(V))$; $\mathrm{int}_X(\mathrm{spcl}_X(V))$) *is an $\mathcal{S}$-closed subspace of (X, τ).*

PROOF. – (1)⇒(2) and (4)⇒(5) follow from [27, Theorem 3.1]. (2)⇒(3): the case (a) of Theorem 2.4. (3)⇒(4): Theorem 2.4. (5)⇒(1) is obvious. □

LEMMA 2.10. – [13] (*see also* Acta Math. Hungar., **105** (3) (2004), *p*. 235). *In every space (X, τ)*

$$V \cap \mathrm{scl}(S) \subset \mathrm{cl}\,(\mathrm{scl}(V \cap S))$$

for each $S \subset X$ and $V \in \mathrm{SO}(X, \tau)$.

REMARK 2.11. – Recall that

$$\mathrm{RO}(X, \tau) \cup \mathrm{RC}(X, \tau) \subset \mathrm{SR}(X, \tau),$$

[39, Lemma 2.3]. This inclusion is proper, in general.

The following theorem is a slight improvement of [27, Theorem 3.3] for the case of spaces that are not e.d.

THEOREM 2.12. – *Assume that a space (X, τ) is not e.d. Let an $A \subset X$ be $\mathcal{S}$-closed relative to (X, τ) and a set $B \in \mathrm{RO}(X, \tau)$ or $B \in \mathrm{SR}(X, \tau) \setminus \mathrm{RO}(X, \tau)$ with* $\mathrm{cl}\,(B) = \mathrm{scl}(B)$. *Then $A \cap B$ is $\mathcal{S}$-closed relative to (X, τ).*

PROOF. – Suppose $A \cap B \subset \bigcup_{a \in \mathcal{A}} V_a$, where $V_a \in \mathrm{SO}(X, \tau)$ for each $a \in \mathcal{A}$. Since $B \in \mathrm{SR}(X, \tau)$, thus $X \setminus B \in \mathrm{SO}(X, \tau)$ [39, Lemma 2.2 (ii)] and

$$A \subset (X \setminus B) \cup \bigcup_{a \in \mathcal{A}} V_a.$$

But A is an $\mathcal{S}$-closed relative to (X, τ), thus there exists a finite subfamily $\mathcal{A}_1 \subset \mathcal{A}$ such that

$$A \subset \mathrm{cl}\,(X \setminus B) \cup \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}\,(V_a).$$

Utilizing Lemma 2.10 we obtain

$$A \cap B \subset \big(B \cap \mathrm{scl}(X \setminus B)\big) \cup \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}\,(V_a) = \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}\,(V_a).$$

This shows that $A \cap B$ is $\mathcal{S}$-closed relative to (X, τ). □

REMARK 2.13. – (a). The author proved in [14] that a space (X, τ) is e.d. if and only if for each $S \in \mathrm{SO}(X, \tau)$, $\mathrm{scl}(S) = \mathrm{int}\,(\mathrm{cl}\,(S)) = \mathrm{cl}\,(\mathrm{int}\,(S))$. Thus, by [24, Lemma 2] we obtain that in e.d. spaces $\mathrm{cl}\,(S) = \mathrm{scl}(S)$ for each $S \in \mathrm{SO}(X, \tau)$. The reversed implication is also true. This equivalence was proved in [9, Proposition 2.4].

(b). The author proved in [14] that a space (X, τ) is e.d. if and only if $\mathrm{RO}(X, \tau) = \mathrm{RC}(X, \tau)$. On the other hand, by [10, Proposition 2(i)] ($\mathrm{SR}(X, \tau) = \mathrm{RO}(X, \tau) \cap \mathrm{RC}(X, \tau)$) and [39, Lemma 2.3] (see Remark 2.11), we have $\mathrm{RO}(X, \tau) \cup \mathrm{RC}(X, \tau) = \mathrm{SR}(X, \tau)$ in any e.d. space. Consequently, in these spaces we have $\mathrm{RO}(X, \tau) = \mathrm{SR}(X, \tau)$. To give an example of a set $B \in \mathrm{SR}(X, \tau) \setminus \mathrm{RO}(X, \tau)$ for which $\mathrm{cl}\,(B) = \mathrm{scl}(B)$, it is enough to consider the space of reals with usual topology and $B = [0, 1]$.

COROLLARY 2.14. – (see [27, Corollary 3.2]). *If (X, τ) is an $\mathcal{S}$-closed not e.d. space and an $A \in \mathrm{RO}(X, \tau)$ or $A \in \mathrm{SR}(X, \tau) \setminus \mathrm{RO}(X, \tau)$ with $\mathrm{cl}\,(A) = \mathrm{scl}(A)$, then A is $\mathcal{S}$-closed relative to (X, τ).*

PROOF. – To see $\mathcal{S}$-closedness of A relative to (X, τ) we apply [27, Theorem 3.1] and Theorem 2.12. Notice that if $A \in \mathrm{SR}(X, \tau)$ and $\mathrm{cl}\,(A) = \mathrm{scl}(A)$, then $A \in \mathrm{RC}(X, \tau)$. □

COROLLARY 2.15. – (*see* [27, Corollary 3.3]). *Let (X, τ) be not an e.d. space. If an A is $\mathcal{S}$-closed relative to (X, τ) and a set $B \in \mathrm{RO}(X, \tau)$ or $B \in \mathrm{SR}(X, \tau) \setminus \mathrm{RO}(X, \tau)$ with $\mathrm{cl}\,(B) = \mathrm{scl}(B)$, then*

(1) *$A \cap B$ is $\mathcal{S}$-closed relative to B.*
(2) *B is $\mathcal{S}$-closed relative to (X, τ), if $B \subset A$.*

PROOF. – (1) follows from Theorem 2.12 and [27, Theorem 3.2] (strong sufficiency). (2): Theorem 2.12. □

The following corollary is an improvement of [27, Corollary 3.1].

COROLLARY 2.16. – *Let A and X_0 be a-open subsets of a space (X, τ) such that $A \subset X_0$. Then, A is an $\mathcal{S}$-closed subspace of (X_0, τ_{X_0}) if and only if A is an $\mathcal{S}$-closed subspace of (X, τ).*

PROOF. – We use [33, Lemma 2], [27, Theorem 3.1], and [27, Theorem 3.2]. □

The next corollary is an immediate consequence of [27, Theorem 3.2].

COROLLARY 2.17. – *Let* $A \subset X_0 \subset X_1 \subset X$ *and* X_0, X_1 *be* α*-open subsets of* (X, τ)*. Then,* A *is an* $\mathcal{S}$*-closed relative to* (X_0, τ_{X_0}) *if and only if* A *is an* $\mathcal{S}$*-closed relative to* (X_1, τ_{X_1})*.*

Using [33, Lemma 2] and Corollary 2.16 we infer what follows.

COROLLARY 2.18. – *Let* $A \subset X_0 \subset X_1 \subset X$ *and* A, X_0, X_1 *be* α*-open subsets of* (X, τ)*. Then,* A *is* $\mathcal{S}$*-closed subspace of* (X_0, τ_{X_0}) *if and only if* A *is* $\mathcal{S}$*-closed subspace of* (X_1, τ_{X_1})*.*

THEOREM 2.19. – *Let* A *be an* $\mathcal{S}$*-closed* α*-open subspace of* (X, τ)*. Then,* $\mathrm{scl}_X(A)$ *is* $\mathcal{S}$*-closed relative to* $\big(\mathrm{cl}_X(A), \tau_{\mathrm{cl}_X(A)}\big)$*.*

PROOF. – Let $\{V_a : a \in \mathcal{A}\} \subset \mathrm{SO}\big(\mathrm{cl}_X(A), \tau_{\mathrm{cl}_X(A)}\big)$ be a cover of $\mathrm{scl}_X(A)$. Obviously, $\{V_a : a \in \mathcal{A}\}$ is a cover of A. Since $A \in \tau^\alpha$, $\mathrm{cl}_X(A) \in \mathrm{SO}(X, \tau)$. Hence $V_a \in \mathrm{SO}(X, \tau)$ for each $a \in \mathcal{A}$ [24, Theorem 1]. By [27, Theorem 3.1] the set A is $\mathcal{S}$-closed relative to (X, τ). Thus, there exists a finite subset $\mathcal{A}_1 \subset \mathcal{A}$ with $A \subset \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}_X(V_a)$. This inclusion implies that $\mathrm{scl}_X(A) \subset \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}_X(V_a)$. So, we obtain $\mathrm{scl}_X(A) \subset \bigcup_{a \in \mathcal{A}_1} \big(\mathrm{cl}_X(V_a) \cap \mathrm{cl}_X(A)\big) = \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}_{\mathrm{cl}_X(A)}(V_a)$ and the proof is complete. □

REMARK 2.20. – Let an $A \in \mathrm{RO}(X, \tau)$ be such that $\mathrm{cl}_X(A) \in \tau^\alpha$. The set $\mathrm{scl}_X(A)$ is $\mathcal{S}$-closed relative to $\big(\mathrm{cl}_X(A), \tau_{\mathrm{cl}_X(A)}\big)$ if and only if A is an $\mathcal{S}$-closed subspace of (X, τ).

PROOF. – It is enough to show that A is closed in (X, τ). Namely, we have

$$\mathrm{cl}_X(A) \subset \mathrm{int}_X\big(\mathrm{cl}_X\big(\mathrm{int}_X\big(\mathrm{cl}_X(A)\big)\big)\big) \subset \mathrm{cl}_X(A).$$

So, $\mathrm{cl}_X(A) = \mathrm{int}_X\big(\mathrm{cl}_X(A)\big)$ and by hypothesis $\mathrm{cl}_X(A) = A$. □

COROLLARY 2.21. – *Let an* $A \in \mathrm{RO}(X, \tau)$ *be such that* $\mathrm{cl}_X(A) \in \tau^\alpha$*. Then,* A *is an* $\mathcal{S}$*-closed subspace of* (X, τ) *if and only if* $\mathrm{scl}_X(A)$ *is* $\mathcal{S}$*-closed relative to* $\big(\mathrm{cl}_X(A), \tau_{\mathrm{cl}_X(A)}\big)$*.*

Recall that a topological space (X, τ) is said to be *semi-connected* [32], if X cannot be written as a union of two nonempty disjoint semi-open sets in (X, τ). In the opposite case a space is called *semi-disconnected*.

THEOREM 2.22. – *Let $A \neq \emptyset$ be $\mathcal{S}$-closed relative to (X, τ) and $\mathrm{cl}\,(A) \subsetneq X_0 \subset X$. If there exists a subfamily $\{V_a : a \in \mathcal{A}\} \subset \mathrm{SO}(X, \tau)$ such that $(\mathbf{a}_1)$ $A \supset \bigcup_{a \in \mathcal{A}} V_a$ and $(\mathbf{a}_2)$ $A \subset \bigcup_{a \in \mathcal{A}} \mathrm{cl}\,(V_a)$, then (X_0, τ_{X_0}) is semi-disconnected.*

PROOF. – We have $\mathrm{cl}\,(V_a) \in \mathrm{SO}(X, \tau)$ for each $a \in \mathcal{A}$. Since A is an $\mathcal{S}$-closed relative to (X, τ), thus by $(\mathbf{a}_2)$ there exists a finite subset $\mathcal{A}_1 \subset \mathcal{A}$ such that $A \subset \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}\,(V_a)$. Hence $\mathrm{cl}\,(A) \subset \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}\,(V_a)$. On the other hand, by $(\mathbf{a}_1)$ we have $\mathrm{cl}\,(A) \supset \bigcup_{a \in \mathcal{A}} \mathrm{cl}\,(V_a) \supset \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}\,(V_a)$. Thus, $\mathrm{cl}\,(A) = \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}\,(V_a)$ and hence

$$\mathrm{cl}_{X_0}(A) = \bigcup_{a \in \mathcal{A}_1} \mathrm{cl}_{X_0}(V_a). \tag{1}$$

By $(\mathbf{a}_1)$, $V_a \subset X_0$ for each $a \in \mathcal{A}$, thus by [19, Theorem 6] every set $V_a \in \mathrm{SO}(X_0, \tau_{X_0})$. So, (1) implies that $\mathrm{cl}_{X_0}(A) \in \mathrm{SO}(X_0, \tau_{X_0})$ [19, Theorem 2]. To finish the proof it is enough to observe that $\emptyset \neq X_0 \setminus \mathrm{cl}_{X_0}(A) \in \tau_{X_0}$. □

It is known that the family τ^a induced by τ forms a topology on X [23], which is different than τ, in general. Recall that for any $S \in \mathrm{SO}(X, \tau)$ we have a-$\mathrm{cl}(S) = \mathrm{cl}\,(S)$ [18, Proposition 2.2].

THEOREM 2.23. – *Let (X, τ) be a space, $A \in \mathrm{SO}(X, \tau)$, $B \in \tau^a$, and $A \cap B = \emptyset$. If the set $A \cup B$ is $\mathcal{S}$-closed relative to (X, τ), then the set B is $\mathcal{S}$-closed relative to (X, τ).*

PROOF. – Let $\mathcal{F} = \{U_a : a \in \nabla\} \subset \mathrm{SO}(X, \tau)$ be a cover of B. Then, the family $\mathcal{F} \cup \{A\}$ covers $A \cup B$. By hypothesis there exists a finite subfamily $\mathcal{F}' = \{U_{a_i} : i = 1, \ldots, n\} \subset \mathcal{F}$ such that

$$A \cup B \subset \bigcup_{i=1}^{n} \mathrm{cl}\,(U_{a_i}) \cup \mathrm{cl}\,(A).$$

So, we obtain

$$B \subset \bigcup_{i=1}^{n} \mathrm{cl}\,(U_{a_i}) \cup a\text{-}\mathrm{cl}(A \cap B) = \bigcup_{i=1}^{n} \mathrm{cl}\,(U_{a_i}).$$

This shows that B is S-closed relative to (X, τ). □

In [15] the author has proved that a space (X, τ) is semi-disconnected if and only if there exist nonempty sets $U_1 \in \mathrm{SO}(X, \tau)$, $U_2 \in \tau^a$ such that $X = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Directly from this result and Theorem 2.23 one obtains the following corollary.

COROLLARY 2.24. – *Let* (X, τ) *be a semi-disconnected and* S*-closed space. Then there exists a nonempty* a*-open proper subset of* X*,* S*-closed relative to* (X, τ)*.*

REMARK 2.25. – Let (X, τ) be $\mathcal{S}$-closed and A be clopen. Then $X \setminus A$ is S-closed relative to (X, τ) (hence S-closed subspace of (X, τ) [27, Theorem 3.1]).

PROOF. – Obvious since $X \setminus A$ is clopen in (X, τ). □

THEOREM 2.26. – *Let a subset* A *of a space* (X, τ) *be clopen and be an* S*-closed subspace of* (X, τ)*. Then* (X, τ) *is* S*-closed if and only if* $X \setminus A$ *is an* S*-closed subspace of* (X, τ)*.*

PROOF. – It follows from Remark 2.25, [27, Theorem 3.1], and [27, Theorem 3.6]. □

In [5] Cameron introduced the concept of I-compactness of a space. It was established [5, Corollary 3] that I-compact spaces are precisely the S-closed spaces which are e.d. Recall that a subset S of a space (X, τ) is *I-compact relative to* (X, τ) if every cover of S with semi-open sets has a finite subfamily interiors of closures of whose members cover S [37]. A subset A of a space (X, τ) is *N-closed* if every cover with regular open sets has a finite subcover [6]. A space is said to be *weakly Hausdorff* if for each point $x \in X$, $\{x\}$ is the intersection of all regular closed sets containing x [38].

THEOREM 2.27. – *Let* (X, τ) *be weakly Hausdorff. If* $A \in \mathrm{PO}(X, \tau)$ *is* I*-closed relative to* (X, τ) *and* $X \setminus A$ *is* N*-closed, then there exists a finite partition of* X *by regular open subsets of* (X, τ)*.*

PROOF. – This is an immediate consequence of [37, Lemma 4.13] and [37, Theorem 4.16] (we use [11, Lemma 2.2(4)] and the well known fact that the intersection of two regular open sets is regular open too [12, p. 92, 22g]). □

3. – Mappings and $\mathcal{S}$-closedness.

DEFINITION 3.1. – [17]. *A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is said to be* ***almost continuous*** *(in the sense of Husain), if for each* $x \in X$ *and each neighbourhood* V *of* $f(x)$*,* $\mathrm{cl}\left(f^{-1}(V)\right)$ *is a neighbourhood of* x*.*

Mashhour et al. observed [20] that almost continuity in the sense of Husain coincides with *precontinuity* (i.e., $f^{-1}(V) \subset \mathrm{int}\left(\mathrm{cl}\left(f^{-1}(V)\right)\right)$ for each $V \in \sigma$).

DEFINITION 3.2. – [19]. *A mapping* $f:(X,\tau)\to(Y,\sigma)$ *is* ***semi-continuous*** *if* $f^{-1}(V)\in \mathrm{SO}(X,\tau)$ *for every set* $V\in\sigma$.

A. Neubrunnová showed [22] that precontinuity and semi-continuity are independent of each other.

DEFINITION 3.3. – [29]. *A mapping* $f:(X,\tau)\to(Y,\sigma)$ *is* ***strongly semi-continuous*** (*Mashhour et al.* [21] *call these mappings* α-***continuous***) *if* $f^{-1}(V)\in\tau^{\alpha}$ *for each* $V\in\sigma$.

DEFINITION 3.4. – [8]. *A mapping* $f:(X,\tau)\to(Y,\sigma)$ *is* ***irresolute*** *if* $f^{-1}(V)\in \mathrm{SO}(X,\tau)$ *for each* $V\in \mathrm{SO}(Y,\sigma)$.

Each irresolute mapping is semi-continuous ($\tau\subset \mathrm{SO}(X,\tau)$). Each α-continuous mapping is semi-continuous and precontinuous ($\tau\subset \mathrm{SO}(X,\tau)\cap \mathrm{PO}(X,\tau)=\tau^{\alpha}$ [30, Lemma 3.1]).

Janković showed the following.

THEOREM 3.5. – [18, Corollary 4.14]. *Let* $f:(X,\tau)\to(Y,\sigma)$ *be a precontinuous and irresolute mapping. If G is a subset S-closed relative to* (X,τ), *then* $f(G)$ *is S-closed relative to* (Y,σ).

Without difficulties it may be observed that this theorem can be obtained with the use of [18, Proposition 3.1(c)].

Notions of precontinuity and irresoluteness are independent of each other as the following examples show.

EXAMPLE 3.6. – We apply [30, Example 3.11]. Let $X=Y=\{a,b,c\}$, $\tau=\big\{\emptyset,X,\{a\},\{b\},\{a,b\}\big\}$ and $\sigma=\big\{\emptyset,X,\{a\},\{b,c\}\big\}$, and let $f:(X,\tau)\to(Y,\sigma)$ be the identity mapping. Then f is irresolute and it is not precontinuous because $f^{-1}(\{b,c\})\notin \mathrm{PO}(X,\tau)$.

EXAMPLE 3.7. – (**a**). [30, Theorem 3.12] shows that there exists an α-continuous mapping which is not irresolute. We shall give an example of such a mapping. (**b**). Let $X=Y=\{a,b,c\}$, $\tau=\big\{\emptyset,X,\{a\},\{a,b\}\big\}$ and $\sigma=\big\{\emptyset,X,\{a\},\{b\},\{a,b\}\big\}$, and let $f:(X,\tau)\to(Y,\sigma)$ be defined as follows: $f(a)=f(b)=a$, $f(c)=c$. Then f is continuous but it is not irresolute since $f^{-1}(\{b,c\})=\{c\}\notin \mathrm{SO}(X,\tau)$.

The above examples show that α-continuity and irresoluteness are independent of each other, as it was observed in [30]. We recall now definitions of some weak forms of openness of mappings.

DEFINITION 3.8. – [36]. *A mapping* $f:(X,\tau)\to(Y,\sigma)$ *is said to be* ***almost open in the sense of Singal*** (*briefly* **a.o.S.**), *if* $f(U)\in\sigma$ *for each* $U\in \mathrm{RO}(X,\tau)$.

DEFINITION 3.9. – [41]. *A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is said to be* ***almost open in the sense of Wilansky*** *(briefly* **a.o.W.***), if* $f^{-1}(\mathrm{cl}\,(V)) \subset \mathrm{cl}\,\big(f^{-1}(V)\big)$ *for each* $V \in \sigma$.

Rose has proved [35, Theorem 11], that a mapping $f : (X, \tau) \to (Y, \sigma)$ is a.o.W. if and only if $f(U) \in \mathrm{PO}(Y, \sigma)$ for each subset $U \in \tau$.

DEFINITION 3.10. – [3]. *A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is* ***semi-open*** *if* $f(U) \in \mathrm{SO}(Y, \sigma)$ *for each* $U \in \tau$.

Notions of a.o.S., a.o.W., and of semi-openness (as given above), are independent of each other (see respective examples in [28]).

DEFINITION 3.11. – [34]. *A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is* ***weakly open*** *if* $f(U) \subset \mathrm{int}\,(f(\mathrm{cl}\,(U))$ *for each set* $U \in \tau$.

Each a.o.S. mapping is weakly open [28, Lemma 1.4], but the converse is not true, in general [28, Example 1.5]. Notions of weak openness and a.o.W. are independent of each other (respective examples in [28]).

LEMMA 3.12. – [25, Theorem 1]. *Every a.o.W. and semi-continuous mapping is irresolute.*

Combining Theorem 3.5 and Lemma 3.12 we obtain the following generalization of [26, Theorem 2.1].

THEOREM 3.13. – *If a mapping* $f : (X, \tau) \to (Y, \sigma)$ *is a-continuous and a.o.W., and if a G is S-closed relative to* (X, τ)*, then* $f(G)$ *is S-closed relative to* (Y, σ).

REMARK 3.14. – Notions of a.o.W. and a-continuity are independent of each other. The mapping f from [28, Example 1.6] is a.o.W., while it is not a-continuous. The mapping f from Example 3.7(**b**) is a-continuous and it is not a.o.W., since $f^{-1}\big(\mathrm{cl}\,(\{b\})\big) \not\subset \mathrm{cl}\,\big(f^{-1}(\{b\})\big) = \emptyset$.

LEMMA 3.15. – [28, Theorem 1.12]. *Every a.o.S. and semi-continuous mapping is irresolute.*

Combining Theorem 3.5 and Lemma 3.15 we get the following.

THEOREM 3.16. – *If a mapping* $f : (X, \tau) \to (Y, \sigma)$ *is a-continuous and a.o.S., and if a G is S-closed relative to* (X, τ)*, then* $f(G)$ *is S-closed relative to* (Y, σ).

REMARK 3.17. – Notions of a.o.S. and α-continuity are independent of each other. [28, Example 1.7] shows that there exists an a.o.S. mapping which is not α-continuous. In Example 3.7(**b**) the mapping f is not a.o.S. because $f(X)$ is not open in the range.

LEMMA 3.18. – [28, Theorem 1.14]. *If a space (Y,σ) is e.d. and a mapping $f:(X,\tau)\to(Y,\sigma)$ is semi-open and semi-continuous, then f is irresolute.*

Recall that a semi-open semi-continuous (hence α-continuous) mapping, must not be irresolute if the range is not e.d. [31, Example 19].

Applying Theorem 3.5 and Lemma 3.18 we obtain what follows.

THEOREM 3.19. – *Let a mapping $f:(X,\tau)\to(Y,\sigma)$ be α-continuous and semi-open. If (Y,σ) is e.d. and $G\subset X$ is S-closed relative to (X,τ), then $f(G)$ is S-closed relative to (Y,σ).*

REMARK 3.20. – Semi-openness and α-continuity of an f are independent notions, even if the range of f is e.d. (**a**). [28, Example 1.8] shows that the f (from this example) is α-continuous and not semi-open. (**b**). [28, Example 1.9] shows that a mapping may be semi-open and not α-continuous, but the range in this example is not e.d. (**c**). Let $X=\{a,b,c\}$, $\tau=\big\{\emptyset,X,\{a\},\{b\},\{a,b\}\big\}$, $Y=\{a,b\}$, and $\sigma=\big\{\emptyset,Y,\{a\},\{b\}\big\}$. The mapping $f:(X,\tau)\to(Y,\sigma)$ defined as follows: $f(a)=f(b)=a, f(c)=b$, is semi-open and not α-continuous.

DEFINITION 3.21. – [16]. *A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be **somewhat continuous** if for each set $V\in\sigma$ with $f^{-1}(V)\neq\emptyset$, there exists a set $U\in\tau$ such that $\emptyset\neq U\subset f^{-1}(V)$.*

Each semi-continuous mapping is somewhat continuous [16] (semi-continuity and quasi-continuity are equivalent [22]), but the converse is not true in general [16, Example 1].

LEMMA 3.22. – [28, Theorem 1.11]. *If $f:(X,\tau)\to(Y,\sigma)$ is a weakly open somewhat continuous injection, then it is irresolute.*

Using once again Theorem 3.5 and Lemma 3.22 we obtain the following.

THEOREM 3.23. – *Let $f:(X,\tau)\to(Y,\sigma)$ be an α-continuous weakly open injection. If G is S-closed relative to (X,τ), then $f(G)$ is S-closed relative to (Y,σ).*

REMARK 3.24. – Weak openness and α-continuity are independent notions. (**a**). The mapping f from [28, Example 1.5] is weakly open, but it is not α-continuous (in fact, it is not semi-continuous). (**b**). Let $X = \{a, b\} = Y$, $\tau = \{\emptyset, X, \{a\}, \{b\}\}$, and $\sigma = \{\emptyset, Y, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity mapping. Then f is continuous and it is not weakly open, because $f(\{b\}) \not\subset \mathrm{int}\,(f(\mathrm{cl}\,(\{b\}))) = \emptyset$.

The next theorem is an immediate consequence of Theorems 3.13, 3.16, 3.19, 3.23 (for the respective parts).

THEOREM 3.25. – *Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping.*

(1) *If f is a-continuous and a.o.W., and if (X, τ) is S-closed, then $f(X)$ is S-closed relative to (Y, σ).*

(2) *If f is a-continuous and a.o.S., and if (X, τ) is S-closed, then $f(X)$ is S-closed relative to (Y, σ).*

(3) *If f is a-continuous and semi-open, (Y, σ) is e.d., and if (X, τ) is S-closed, then $f(X)$ is S-closed relative to (Y, σ).*

(4) *If f is an a-continuous weakly open injection, and if (X, τ) is S-closed, then $f(X)$ is S-closed relative to (Y, σ).*

Using Theorem 3.25 (3) one trivially obtains the following corollary.

COROLLARY 3.26. – *Let $f : (X, \tau) \to (Y, \sigma)$ be a surjection and (Y, σ) be e.d. If f is a-continuous, semi-open and if (X, τ) is S-closed, then (Y, σ) is I-compact.*

It is interesting to compare this corollary with [37, Theorem 5.5].

DEFINITION 3.27. – *A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be* ***contra-semi-open*** *if $f(U) \in \mathrm{SC}(Y, \sigma)$ for every $U \in \tau$.*

LEMMA 3.28. – *Let $f : (X, \tau) \to (Y, \sigma)$ be a.o.W. and contra-semiopen. Then $f(U) \in \mathrm{RO}(Y, \sigma)$ for each $U \in \tau$.*

PROOF. – By [35, Theorem 11] and by Definition 3.27 we have

$$f(U) \subset \mathrm{int}\,(\mathrm{cl}\,(f(U))) \subset f(U).$$ □

THEOREM 3.29. – *Let $f : (X, \tau) \to (Y, \sigma)$ be a-continuous, a.o.W., and contra-semiopen. If (X, τ) is an S-closed space and $G \in \mathrm{RO}(X, \tau)$, then $f(G)$ is an S-closed subspace of (Y, σ).*

PROOF. – This follows from Lemma 3.28, Theorem 3.13, [27, Corollary 3.2 and Theorem 3.1]. □

Recall that a mapping $f : (X, \tau) \to (Y, \sigma)$ is semi-continuous if and only if $f(\mathrm{scl}(A)) \subset \mathrm{cl}\,(f(A))$ for every subset $A \subset X$ [7, Theorem 1.6].

LEMMA 3.30. – *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a.o.S., contra-semiopen, and semi-continuous. Then* $f(U) \in \mathrm{RO}(Y, \sigma)$ *for each* $U \in \tau$.

PROOF. – By [7, Theorem 1.16] and [18, Proposition 2.7(a)] we have $f(\mathrm{int}\,(\mathrm{cl}\,(U)) \subset \mathrm{cl}\,(f(U))$. But f is contra-semiopen, therefore applying [34, Theorem 4] we obtain what follows

$$f(U) \subset \mathrm{int}\,(f(\mathrm{int}\,(\mathrm{cl}\,(U)))) \subset \mathrm{int}\,(\mathrm{cl}\,(f(U))) \subset f(U).$$

This shows that $f(U) \in \mathrm{RO}(Y, \sigma)$ for any $U \in \tau$. □

THEOREM 3.31. – *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a-continuous, a.o.S., and contra-semiopen. If* (X, τ) *is an S-closed space and* $G \in \mathrm{RO}(X, \tau)$, *then* $f(G)$ *is an S-closed subspace of* (Y, σ).

PROOF. – We apply Lemma 3.30, Theorem 3.16, [27, Corollary 3.2 and Theorem 3.1]. □

LEMMA 3.32. – *Let a space* (Y, σ) *be e.d. and a mapping* $f : (X, \tau) \to (Y, \sigma)$ *be semi-open and contra-semiopen. Then* $f(U) \in \mathrm{RO}(Y, \sigma)$ *for each* $U \in \tau$.

PROOF. – For any $U \in \tau$ we have what follows:

$$\begin{aligned}\mathrm{int}\left(\mathrm{cl}\left(f(U)\right)\right) &\subset f(U) \subset \mathrm{cl}\left(\mathrm{int}\left(f(U)\right)\right)\\ &= \mathrm{int}\left(\mathrm{cl}\left(\mathrm{int}\left(f(U)\right)\right)\right) \subset \mathrm{int}\left(\mathrm{cl}\left(f(U)\right)\right).\end{aligned}$$ □

THEOREM 3.33. – *Let a space* (Y, σ) *be e.d. and a mapping* $f : (X, \tau) \to (Y, \sigma)$ *be a-continuous, semi-open, and contra-semiopen. If* (X, τ) *is an S-closed space and* $G \in \mathrm{RO}(X, \tau)$, *then* $f(G)$ *is an S-closed subspace of* (Y, σ).

PROOF. – This follows from Lemma 3.32, Theorem 3.19, [27, Corollary 3.2 and Theorem 3.1]. □

LEMMA 3.34. – *Let* $f : (X, \tau) \to (Y, \sigma)$ *be weakly open, contra-semiopen, and precontinuous. Then* $f(U) \in \mathrm{RO}(Y, \sigma)$ *for each* $U \in \tau$.

PROOF. – By hypothesis and by [18, Proposition 3.1 (c)] we have

$$f(U) \subset \mathrm{int}\,(f(\mathrm{cl}\,(U))) \subset \mathrm{int}\,(\mathrm{cl}\,(f(U))) \subset f(U).$$ □

THEOREM 3.35. – *Let* $f : (X, \tau) \to (Y, \sigma)$ *be an a-continuous, weakly open and contra-semiopen injection. If* (X, τ) *is an S-closed space and* $G \in \mathrm{RO}(X, \tau)$, *then* $f(G)$ *is an S-closed subspace of* (Y, σ). □

REMARK 3.36. – It is easy to see that the statement "(X, τ) ***is an S-closed space and*** $G \in \mathrm{RO}(X, \tau)$" in conclusions of Theorems 3.29, 3.31, 3.33, and 3.35 may be replaced by "G ***is regular open*** S***-closed subspace of*** (X,τ)". Thus, these theorems give conditions under which a mapping preserves regular open S-closed subspaces.

By Remark 3.36, respective theorems mentioned in this remark, and by [27, Theorem 4.1(5)], we obtain the following result.

THEOREM 3.37. – *Let* (X,τ) *be a locally* S*-closed space and* $f:(X,\tau)\to(Y,\sigma)$ *be a mapping.*

(1) *If* f *is an a-continuous, a.o.W. (or a.o.S.), and contra-semiopen surjection, then* (Y,σ) *is locally* S*-closed.*

(2) *If* (Y,σ) *is e.d. and* f *is an a-continuous, semi-open, and contra-semiopen surjection, then* (Y,σ) *is locally* S*-closed.*

(3) *If* f *is an a-continuous, weakly open, and contra-semiopen bijection, then* (Y,σ) *is locally* S*-closed.*

The reader is advised to compare Theorem 3.37 with [27, Theorem 4.4].

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