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Sunto. – *Scriviamo* $[x, y] = [x,_1 y]$ *e* $[x,_{k+1} y] = [[x,_k y], y]$. *Nel presente mostriamo certe proprietà ed esempi dei gruppi con i generatori* x, y *tali che* $x = [x,_n y], y = [y,_n x]$ *o* $[x, y] = [x,_n y], [y, x] = [y,_n x]$.

Summary. – *We use notations:* $[x,_1 y] = [x, y]$ *and* $[x,_{k+1} y] = [[x,_k y], y]$ *for* $k \geq 1$. *We consider groups generated by* x, y *satisfying relations* $x = [x,_n y], y = [y,_n x]$ *or* $[x, y] = [x,_n y], [y, x] = [y,_n x]$. *We call groups of the first type mep-groups and of the second type nmep-groups. We show many properties and examples of mep- and nmep-groups. We prove that if p is a prime then the group* $\mathrm{Sl}_2(p)$ *is a nmep-group. We give the necessary and sufficient conditions for metacyclic group to be a nmep-group and we show that nmep-groups with presentation* $\langle x, y | [x, y] = [x,_2 y], [y, x] = [y,_2 x], x^n, y^m \rangle$ *are finite.*

1. – Introduction.

We use standard notations: $[x, y] = [x,_1 y] = x^{-1}y^{-1}xy$ and for $k > 1$, $[x,_k y] = [[x,_{k-1} y], y]$. Two elements x, y will be called a mutually Engel periodic pair (in short a mep-pair) if there exist positive integers m, n, such that $x = [x,_m y]$, $y = [y,_n x]$ and x, y will be called a near mep-pair (a nmep-pair) if there are $m, n > 1$ such that $[x, y] = [x,_m y], [y, x] = [y,_n x]$. We say that a group G is a mep-group (a nmep-group resp.) if it is generated by a mep-pair (a nmep-pair resp.).

If $a = [a,_n b]$ then $[a, b] = [a,_{n+1} b]$, so every mep-group is also nmep-group. The converse statement is not true because every non-trivial two-generator abelian group is a nmep-group but is not a mep-group.

Groups generated by mep-pairs were considered by H. Heineken in [2]. He showed that if a group G is generated by a mep-pair x, y then G is perfect (that is $G = [G, G]$) and elements x, y^{-1}, xy^{-1} are conjugated in G. He also find mep-pairs in groups $\mathrm{Sl}_2(q)$ in some cases: for every prime q of the form $8t + 5$, for some prime q of the form $8t + 1$, for every prime q such that $q^3 - q$ is divisible by 7 and for $q = p^3$, where p is a prime such that 7 does not divide $p^3 - p$.

In this paper we show some properties and examples of mep- and nmep-groups. We prove that for every prime p, $\mathrm{Sl}_2(p)$ is the nmep-group. We prove also that if x, y is a nmep-pair and $x^2 = y^2 = 1$ then the group generated by x, y is a dihedral group of order $2^{m+1} \pm 4$ and we prove that groups with presentation $\langle x, y | [x, y] = [x,_2 y], [y, x] = [y,_2 x], x^n, x^m \rangle$ are finite metabelian (or abelian in the case when m, n are coprime). We give necessary and sufficient conditions for finite metacyclic group to be a nmep-group. Finally we construct some matrix nmep-groups.

As usual if G is a group then $\gamma_n(G)$ is a member of the lower central series of G, that is $\gamma_1(G) = G$ and for $n > 1$ we have $\gamma_n(G) = [\gamma_{n-1}(G), G]$.

Let us start with some examples.

2. – Examples and properties of mep and nmep-groups.

EXAMPLE 1. – Let $G = \langle x, y | yx = xy^2 \rangle$. We show that G is a nmep-group with a trivial centre. If $a = xy, b = x$ then $[a,_2 b] = [a, b]$ and $[b,_2 a] = [b, a]$, so a, b is a nmep-pair, generating G. We show that the centre $Z(G)$ of G is trivial. The relation $yx = xy^2$ implies $y^m x^n = x^n y^{m2^n}$ and $x^{-n} y^m = y^{m2^n} x^{-n}$ for every integer m and every positive integer n. Hence every element $g \in G$ can be written in the form $g = x^k y^l x^{-n}$, where l is an integer and k, n are non-negative. If $1 \neq g = x^k y^l x^{-n} \in Z(G)$ then g cannot be a power of x, so $l \neq 0$. A relation $gx = xg$ implies $x^{-1} g x = g$, and we get $x^k y^{2l} x^{-n} = x^k y^l x^{-n}$, so $y^l = 1$, but it is possible in G only if $l = 0$, which contradicts $l \neq 0$.

EXAMPLE 2. – If C is an arbitrary cyclic group then $A_5 \times C$ is a nmep-group with non-trivial centre (see Corollary 2).

EXAMPLE 3. – The alternating group A_5 is a mep-group since every pair x, y of the form $x = (a_1, a_2, a_3, a_4, a_5), y = (a_1, a_3, a_2, a_5, a_4)$ generates A_5 and satisfies $x = [x,_5 y]$ and $y = [y,_5 x]$.

EXAMPLE 4. – The alternating group A_6 is not a mep-group because every mep-pair in A_6 generates a subgroup isomorphic to A_5.

PROOF. – We can deduce from [2] that if x, y is a mep-pair then x, y^{-1} and xy^{-1} are conjugated. So if x, y is a mep-pair generating S_6 then x, y, xy^{-1} have the same cycle structure and they have the same order. Now we see that x, y cannot be of order $2, 3, 4$, since in the first case the group would be dihedral, in the second case 3-cycles never generate A_6 and also two permutations which have the structure $(i, j, k)(l, m, n)$ do not generate A_6 (since A_6 has an outer automorphism mapping $(1, 2, 3)$ onto $(1, 2, 3)(4, 5, 6)$), in case of order 4 if $x = (1, 2, 3, 4)(5, 6)$ then there is no

y such that y^{-1} and xy^{-1} are both of order 4. In case of order 5, it is enough to assume that $x = (1, 2, 3, 4, 5)$ and that y is a 5-cycle fixing the object 1. Such y can be chosen on 24 ways, but only in 9 cases xy^{-1} have order 5. If y belongs to the set $\{(2, 3, 5, 4, 6), (2, 3, 6, 5, 4), (2, 4, 5, 6, 3), (2, 6, 4, 5, 3)\}$ then x, y satisfy the relation $[x, y] = [x,_{11} y]$ and since $x \neq [x,_k y]$ for $k < 11$, x, y is not a mep-pair. If y belongs to the set $\{(2, 3, 5, 6, 4), (2, 4, 5, 6, 3)\}$ then x, y satisfy the relation $[x, y] = [x,_{13} y]$ and since $x \neq [x,_k y]$ for $k < 13$, x, y is not a mep-pair. If y belongs to the set $\{(2, 5, 3, 4, 6), (2, 5, 6, 3, 4)\}$ then x, y satisfy the relation: $[y,_3 x] = [y,_{13} x]$ and $y \neq [y,_k x]$ for $k < 13$, so x, y also is not a mep-pair (it can be interesting that in this case x and y satisfy the relation $x = [x,_{10} y]$ but they do not satisfy $y = [y,_{10} x]$). Finally if $x = (1, 2, 3, 4, 5)$ and $y = (2, 6, 5, 3, 4)$ then x, y is a mep-pair satisfying relations: $x = [x,_5 y]$, $y = [y,_5 x]$, $xyx = yxy$, $x^3 = yx^2y$ and they generate the group isomporphic to A_5 (see the Theorem 4 and remarks below that theorem).

EXAMPLE 5. – There are three forms of mep-pairs that generate A_7:

(1) $x = (a_1, a_7, a_2, a_4, a_5, a_3, a_6),\ y = (a_1, a_4, a_3, a_7, a_6, a_2, a_5)$
(2) $x = (a_1, a_2)(a_3, a_4, a_5, a_6),\ y = (a_2, a_3)(a_4, a_7, a_5, a_6)$
(3) $x = (a_1, a_2, a_3)(a_4, a_5)(a_6, a_7),\ y = (a_1, a_6, a_5)(a_2, a_7)(a_3, a_4)$

Moreover, the pair $x = (a_1, a_2)(a_3, a_4, a_5, a_6),\ y = (a_5, a_6)(a_2, a_3, a_7, a_4)$ is also a mep-pair in A_7 and it generates a simple group of order 168 (all calculations were done by M. Żabka).

EXAMPLE 6. – For $n \in \{3, 4, 5, 6, 7\}$, S_n is a nmep-group.

PROOF. – (1) If $n = 3$ then for $x = (1\ \ 2)$ and $y = (1\ \ 3)$ we have $[x, y] = [x,_3 y]$ and $[y, x] = [y,_3 x]$.

(2) If $n = 4$ then the pair $x = (1\,2\,3\,4), y = (1\,3\,2\,4)$ satisfies $[x, y] = [y,_5 x]$, $[y, x] = [y,_5 x]$ and since $x^2y = (1\,2)$, S_4 is a nmep-group.

(3) If $n = 5$ then the pair $x = (1\,2)(3\,4\,5)$ and $y = (1\,3)(2\,5\,4)$ generates S_5, because $x^3 = (1\,2)$ and $xy^3 = (1\,2\,3\,4\,5)$. We can check that $[x, y] = [x,_{13} y]$ and $[y, x] = [y,_{13} x]$, so S_5 is a nmep-group.

(4) If $n = 6$ then for the pair: $x = (1\,4)(3\,2\,6)$ and $y = (1\,5)(2\,3\,4)$, we get $[x, y] = [x,_{73} y]$ and $[y, x] = [y,_{73} x]$. We can observe that x, y generate S_6 because $x^3y^3x^3 = (4\,5)$ and $x^{-1}yxy^2 = (1\,6\,4\,5\,3\,2)$. It follows that S_6 is a nmep-group.

(5) If $n = 7$ then the pair $x = (1\,2)(3\,4\,5\,6\,7)$, $y = (1\,3)(2\,4\,7\,6\,5)$ satisfies $[x, y] = [x,_{31} y]$, $[y, x] = [y,_{31} x]$. Because $x^5 = (1\,2)$ and $xy^5 = (1\,2\,3\,4\,5\,6\,7)$, S_7 is also a nmep-group. □

PROPOSITION 1. – *Every quotient group of a mep-group (nmep-group) is also a mep-group (nmep-group).*

PROPOSITION 2. – *If G is a mep-group, then $G = G'$. If G is a nmep-group, then $G' = \gamma_3(G) = \gamma_i(G)\ \ \forall i > 1$.*

PROOF. – For the first statement it is enough to show that $G \subseteq G'$ and for the second that $G' \subseteq \gamma_3(G)$. If G is a mep-group then G is generated by elements which are contained in G' so $G \subseteq G'$. If G is generated by a nmep-pair x, y that satisfies $[x, y] = [x, {}_n\, y]$, $[y, x] = [y, {}_n\, x]$ then $[x, y] \subseteq \gamma_{n+1}(G) \subseteq \gamma_3(G)$. Since $\gamma_3(G)$ is normal in G and G' is normally generated by $[x, y]$, then $G' \subseteq \gamma_3(G)$. □

PROPOSITION 3. – *If G is a solvable (nilpotent) mep-group, then G is trivial.*

PROOF. – By the above Proposition a non-trivial mep-group cannot be solvable. □

PROPOSITION 4. – *If G is a nilpotent nmep-group, then G is abelian.*

PROPOSITION 5. – *If G is generated by x and y which satisfy relations $[x, y] = x$ and $[y, x] = y$, then G is trivial.*

PROOF. – We have $x = [x, y] = [y, x]^{-1} = y^{-1}$ and $y^{-1} = x = [x, y] = [y^{-1}, y] = 1$, so $G = \mathrm{gp}(x, y)$ is trivial. □

PROPOSITION 6. – *Relations $x = [x, {}_n\, y], y = [y, {}_n\, x]$ and $[x, y] = [x, {}_m\, y], [y, x] = [y, {}_m\, x]$ are equivalent to $x = [x, {}_d\, y], y = [y, {}_d\, x]$, where $d = \gcd(n, m-1)$. Moreover if $\gcd(n, m-1) = 1$ then the group generated by x and y is trivial.*

PROOF. – Let $d = \gcd(n, m-1)$ and let elements x, y satisfy relations $x = [x, {}_n\, y], y = [y, {}_n\, x]$ and $[x, y] = [x, {}_m\, y], [y, x] = [y, {}_m\, x]$. Then there exist positive integers s, t such that $d = sn - t(m-1)$. We use assumed relations and we get

$$[x, y] = [x, y, {}_{m-1}\, y] = [x, y, {}_{2(m-1)}\, y] = \ldots = [x, y, {}_{t(m-1)}\, y], \tag{1}$$

$$x = [x, {}_n\, y] = [x, {}_{2n}\, y] = \ldots = [x, {}_{sn}\, y]. \tag{2}$$

By (1) and (2) we get

$$x = [x, {}_{sn}\, y] = [x, {}_{d+t(m-1}\,)y] = [x, y, {}_{t(m-1)}\, y, {}_{d-1}\, y] = [x, y, {}_{d-1}\, y] = [x, {}_d\, y],$$

and similarly $[y, {}_d\, x] = y$. If $d = 1$, then by Proposition 5 the group generated by x and y is trivial. □

PROPOSITION 7. – *Let x, y be nontrivial elements of a group G. If k, l are the minimal numbers such that $x = [x, {}_k\, y]$ and $[x, y] = [x, {}_l\, y]$ then $l = k + 1$.*

PROOF. – By Proposition 6 if $x = [x,_k y]$ and $[x, y] = [x,_l y]$ then $x = [x,_d y]$ for $d = \gcd(k, l-1)$. Hence $d \leq k, d \leq l-1$ and by the minimality of k and l $d = k = l-1$, so $l = k+1$. □

THEOREM 1. – *Let x, y be elements of a group G. If N is a subgroup of G, such that $N \subseteq Z(G)$ and $xN, yN \in G/N$ is a mep pair then x, y is a nmep-pair.*

PROOF. – Since xN, yN is a mep pair there exists k such that $xN = [xN,_k yN] = [x,_k y]N$. So, $[x,_k y]x^{-1} \in N \subseteq Z(G)$ and then

$[x_{k+1}y] = [x,_k y]^{-1}y^{-1}[x,_k y]y = x^{-1}(x[x,_k y]^{-1})y^{-1}([x,_k y]x^{-1})xy = x^{-1}y^{-1}xy = [x, y]$

and similarly $[y,_{k+1} x] = [y, x]$. So x, y is a nmep pair. □

COROLLARY 1. – *Let $x, y \in G$ be a mep-pair and let a, b belong to the centre of G. Then $H = \mathrm{gp}(x, y, a, b)$ is a nmep-group generated by a nmep-pair xa, yb.*

PROOF. – By assumptions $N = \mathrm{gp}(a, b)$ is contained in $Z(H)$, and xaN, ybN is a mep-pair. So by the previous Theorem xa, xb is a nmep-pair. Now we show that the subgroup $H = \mathrm{gp}(x, y, a, b)$ is generated by xa, yb. Clearly, $\mathrm{gp}(xa, yb) \subseteq \mathrm{gp}(x, y, a, b)$, so it is enough to prove that $x, y \in \mathrm{gp}(xa, yb)$. From $x = [x,_n y]$ and $y = [y,_m x]$ we get $x = [x,_n y] = [xa,_n yb] \in \mathrm{gp}(xa, yb)$ and $y = [yb,_m xa] \in$ $\in \mathrm{gp}(xa, yb)$. □

COROLLARY 2. – *If G is a mep-group and A is either cyclic or two-generated abelian group, then $G \times A$ is a nmep-group.*

PROOF. – Since G is generated by a mep-pair x, y, $A \subseteq Z(G \times A)$ and A is generated by a, b, we have by Corollary 1 that G is generated by a nmep-pair xa, yb. □

THEOREM 2. – *Pairs u, v of a group G_1 and a, b of a group G_2 are mep-pairs if and only if the pair ua, vb is the mep-pair in $G_1 \times G_2$. Moreover, if G_1 is generated by u, v and G_2 is generated by a, b and G_1, G_2 do not posses the normal subgroups M_1, M_2 such that $G_1/M_1 \cong G_2/M_2 \neq 1$ then ua, vb generates $G_1 \times G_2$ (so $G_1 \times G_2$ is a mep-group).*

PROOF. – First part of Theorem is clear. We can consider G_1 and G_2 as quotient groups of the two-generated free group F, so $G_1 \cong F/N_1$ and $G_2 \cong F/N_2$. If G_1 and G_2 satisfy assumptions of the theorem then $N_1N_2 = F$ (because $F/N_1N_2 \cong (F/N_1)/(N_1N_2/N_1) \cong (F/N_2)/(N_1N_2/N_2)$). So there exists a word $w(x, y)$, such that $u = w(u, v)$ and $w(a, b) = 1$. If we consider the subgroup gp(ua, vb) of the group $G_1 \times G_2$ then $w(ua, vb) = w(u, v)w(a, b) = w(u, v) = u$, so $u \in$ gp(ua, vb), and similarly $v \in$ gp(ua, vb) and hence gp$(ua, vb) = G_1 \times G_2$. □

LEMMA 1. – *If $x, y \in G$ have order 2, then $[x,_m y] = (yx)^{(-2)^m}$.*

PROOF. – If x, y have order 2 then $xy = (yx)^{-1}$. To prove the equality we use an induction on m. For $m = 1$ we get $[x, y] = x^{-1}y^{-1}xy = (xy)^2 = (yx)^{-2}$. Let us assume that the formula is true for all integers less or equal m then

$$[x,_{m+1} y] = [x,_m y]^{-1}y^{-1}[x,_m y]y = (yx)^{-(-2)^m}y^{-1}(yx)^{(-2)^m}y = (yx)^{-(-2)^m}(xy)^{(-2)^m} = (yx)^{(-2)^{m+1}}.$$

□

Let us remind that the dihedral group D_n has the presentation $\langle x, y | x^2, y^2, (xy)^n\rangle$ (see [1] 1.5).

THEOREM 3. – *Let $G = \langle x, y | [x, y] = [x,_m y], [y, x] = [y,_m x], x^2, y^2\rangle$. Then for even m, G is isomorphic to a dihedral group D_{2^m+2} and for odd m, G is isomorphic to a dihedral group D_{2^m-2}.*

PROOF. – By Lemma 1, from the relation $[x, y] = [x,_m y]$, we get $(yx)^{-2} = (yx)^{(-2)^m}$. For even m we get $(yx)^{2^m+2} = 1$, so G is isomorphic to D_{2^m+2}. For odd m we get $(yx)^{2^m-2} = 1$, so G is isomorphic to D_{2^m-2}. □

3. – Mep-pairs and nmep-pairs x, y satisfying the relation $xyx = yxy$.

The group A_5 is generated by a mep-pair $a = (1\,2\,3\,4\,5), b = (1\,3\,2\,5\,4)$ and it is easy to check that a and b satisfy relations $aba = bab$, $a^5 = 1$ and $a = [a,_5 b], b = [b,_5 a]$. The relation $xyx = yxy$ simplifies commutators $[x,_n y]$, so one can ask if there exist many mep-groups in which map-pair x, y satisfies relation $xyx = yxy$? The answer is: there are only two such non-trivial groups: A_5 and $\mathrm{Sl}_2(5)$.

First we observe that the relation $xyx = yxy$ implies

$$xyx^{-1} = y^{-1}xy \tag{3}$$

LEMMA 2. – *If elements x, y satisfy the relation $xyx = yxy$, then for every positive integer n, $[x,_n y] = y^{-(2n-3)}x^{-1}y^{2n-2}$.*

PROOF. – We use an induction on n. It follows from $xyx = yxy$ that $[x, y] = yx^{-1}$, so the statement is true for $n = 1$. Let us assume that it is true for n, then using the relation $xyx = yxy$ we get:

$$[x,_{n+1} y] = [x,_n y]^{-1}y^{-1}[x,_n y]y = y^{2-2n}xy^{-1}x^{-1}y^{2n-1} =$$
$$y^{2-2n}xy^{-1}x^{-1}y^{-1}y^{2n} = y^{2-2n}xx^{-1}y^{-1}x^{-1}y^{2n} =$$
$$y^{-(2n-1)}x^{-1}y^{2n} = y^{-(2(n+1)-3)}x^{-1}y^{2(n+1)-2}$$

□

LEMMA 3. – *If elements x and y satisfy $xyx = yxy$, then relations $x = [x,_n y], y = [y,_n x]$ are equivalent to $x^3 = yx^2y$ and $x^{2n-5} = y^{2n-5}$.*

PROOF. – If $xyx = yxy$, then by Lemma 2, $x = [x,_n y] = y^{-(2n-3)}x^{-1}y^{2n-2}$ and by symmetry, $y = x^{-(2n-3)}y^{-1}x^{2n-2}$. Hence, we get relations $y^{2n-2} = xy^{2n-3}x$ and $x^{2n-2} = yx^{2n-3}y$. From the relation $y^{2n-2} = xy^{2n-3}x$ follows $xy^{2n-2}x^{-1} = x^2y^{2n-3}$ and $(xyx^{-1})^{2n-2} = x^2y^{2n-3}$. Now, by using the relation (3), we get $y^{-1}x^{2n-2}y = x^2y^{2n-3}$ and $x^{2n-2} = yx^2y^{2n-4}$. We can combine this last relation with $x^{2n-2} = yx^{2n-3}y$ obtaining $yx^2y^{2n-4} = yx^{2n-3}y$, so $x^{2n-5} = y^{2n-5}$. It means that x^{2n-5} lies in the centre, and from the relation $x^{2n-2} = yx^{2n-3}y$ we get $x^3 = x^{2n-2}x^{-(2n-5)} = yx^{2n-3}yx^{-(2n-5)} = yx^{2n-3}x^{-(2n-5)}y = yx^2y$. □

LEMMA 4. – *Relations $xyx = yxy$ and $x^3 = yx^2y$ imply $x^5 = y^5$.*

PROOF. – We show first that relations $xyx = yxy$ and $x^3 = yx^2y$ imply $y^3 = yx^2y$. The relation $x^3 = yx^2y$ can be written in the form $y^{-1}x^3y = x^2y^2$. Hence we get $(y^{-1}xy)^3 = y^2x^2$, then by (3), we get $xy^3x^{-1} = x^2y^2$ and finally $y^3 = yx^2y$. Now to show that the relation $x^5 = y^5$ holds we use relations $xyx = yxy$, $x^3 = yx^2y$ and $y^3 = xy^2x$ and we get: $(xyx)^2 = x(yx^2y)x = xx^3x = x^5$ and $(xyx)^2 = (yxy)^2 = y(xy^2x)y = y^5$ so $x^5 = y^5$. □

THEOREM 4. – *Let G_n be the group with presentation:*

$$G_n = \langle x, y | xyx = yxy, x = [x,_n y], y = [y,_n x]\rangle.$$

Then

(i) *if* $5|n$ *then* G_n *is isomorphic to* $\mathrm{Sl}_2(5)$,
(ii) *if* $5\nmid n$ *then* G_n *is trivial.*

PROOF. – By Lemmas 3 and 4 the presentation of G_n can be written in the form:

$$G_n = \langle x, y | xyx = yxy, x^3 = yx^2y, x^{2n-5} = y^{2n-5}\rangle$$

It can be checked, using the Coxeter-Todd algorithm (see [1]), that G_5 has 120 elements. Moreover, since matrices $A, B \in \mathrm{Sl}_2(5)$:

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

generate $\mathrm{Sl}_2(5)$ (see [5], Theorem 8.8) and satisfy relations $ABA = BAB$, $A^3 = BA^2B$ and $\mathrm{Sl}_2(5)$ has 120 elements, G_5 is isomorphic to $\mathrm{Sl}_2(5)$.

We have shown in Lemma 4 that in G_n the relation $x^5 = y^5$ holds. Hence if $5|n$ then the relation $x^{2n-5} = y^{2n-5}$ is a consequence of $x^5 = y^5$, so in this case G_n is isomorphic to G_5.

If $5 \nmid n$ then numbers $2n - 5$ and 5 are coprime. Since in G_n we have $x^{2n-5} = y^{2n-5}$ and $x^5 = y^5$, then x^{2n-5} and x^5 belong to the centre of G_n, so ($2n - 5$ and 5 are coprime) x is in the centre and y is also in the centre. It means that G_n is abelian and a mep-group, so it is trivial. □

Let us notice that A_5 is isomorphic to $\mathrm{PSl}_2(5) \cong \mathrm{Sl}_2(5)/Z(\mathrm{Sl}_2(5))$ (see [5] p. 226), and since $Z(\mathrm{Sl}_2(5))$ has order two and is generated by x^5, then A_5 has the presentation:

$$\langle x, y | xyx = yxy, x = [x,_5 y], y = [y,_5 x], x^5 = 1\rangle$$

Next we want to investigate nmep-groups generated by nmep-pairs x, y satisfying the relation $xyx = yxy$.

Lemma 5. – *If $xyx = yxy$, then relations $[x, y] = [x,_n y], [y, x] = [y,_n x]$ are equivalent to the relation $x^{2n-2} = y^{2n-2}$.*

Proof. – By Lemma 2, $[x,_n y] = y^{-(2n-3)}x^{-1}y^{2n-2}$ and $[x, y] = yx^{-1}$, so the relation $[x, y] = [x,_n y]$ implies $yx^{-1} = y^{-(2n-3)}x^{-1}y^{2n-2}$. Hence, we get $xy^{2n-2} = y^{2n-2}x$, which can be written in the form $xy^{2n-2}x^{-1} = y^{2n-2}$ and

$$(xyx^{-1})^{2n-2} = y^{2n-2}. \tag{4}$$

From relations (3) and (4) we get $(y^{-1}xy)^{2n-2} = y^{2n-2}$ and finally $x^{2n-2} = y^{2n-2}$. □

Corollary 3. – *If G is a group generated by elements x, y that satisfy a relation $xyx = yxy$ and x is of finite order, then y has the same order as x and G is a nmep-group. Particularly if G is finite, then G is a nmep-group.*

Proof. – By (3) x and y are conjugated, so they have the same order. If elements x and y satisfy $xyx = yxy$ and have finite order k, then for $n = k + 1$ we have $x^{2n-2} = y^{2n-2}$. So by Lemma 5 G is a nmep-group. □

Theorem 5. – *For every prime number p, $\mathrm{Sl}_2(p)$ and $\mathrm{PSl}_2(p)$ are nmep-groups.*

Proof. – It can be deduced from [5] (Theorem 8.8) that $\mathrm{Sl}_2(p)$ is generated by two matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

and matrices A, B satisfy the relation $ABA = BAB$, so by Corollary 3 the pair A, B is a nmep-pair.

Let us observe that $\mathrm{Sl}_2(p)$ is also generated by the nmep-pair:

$$X = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, Y = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

also satisfying $XYX = YXY$.

4. – Groups generated by nmep-pairs that satisfy $[y, x] = [y, x, x], [x, y] = [x, y, y]$.

We will use the following commutator identity:

$$[a, bc] = [a, b][a, b]^c \tag{5}$$

LEMMA 6. – *If n, m are positive integers and $d = \gcd(n, m)$, then we have $2^d - 1 = \gcd(2^n - 1, 2^m - 1)$.*

THEOREM 6. – *Let $G_{n,m}$ be the group which has a presentation*

$$G_{n,m} = \langle x, y \mid [y, x] = [y, x, x], [x, y] = [x, y, y], x^n, y^m \rangle,$$

where n, m are positive integers. Then $G_{n,m}$ is a finite, metabelian or cyclic group of order $nm(2^d - 1)$, where $d = \gcd(m, n)$. Elements x^d, y^d belong to the centre of G. Moreover, $G_{n,m}$ is cyclic if and only if n, m are coprime.

PROOF. – Since $[y, x, x] = [x, y]^2[y, x^2]$, relations $[y, x] = [y, x, x]$ and $[x, y] = [x, y, y]$ are equivalent to $[y, x]^3 = [y, x^2]$ and $[x, y]^3 = [x, y^2]$. Hence the presentation of $G_{n,m}$ can be written in the form:

$$\langle x, y \mid [y, x]^3 = [y, x^2], [x, y]^3 = [x, y^2], x^n, y^m \rangle.$$

We show now that the commutator subgroup $G'_{n,m}$ of $G_{n,m}$ is cyclic and has order $2^d - 1$, where $d = \gcd(m, n)$. By ([4], 4.3) if $G_{n,m}$ is generated by x and y, then the commutator subgroup $G'_{n,m}$ of $G_{n,m}$ is generated by commutators $[x^k, y^l]$, where $k, l \in \mathbb{Z}$. Since in $G_{n,m}$ generators have finite orders, its commutator subgroup is generated by all commutators $[x^k, y^l]$, where k, l are nonzero, positive integers. We use relations:

$$\begin{aligned} [y, x]^3 &= [y, x^2], \\ [x, y]^3 &= [x, y^2]. \end{aligned} \tag{6}$$

to obtain some new relations:

(*a*) $[x, y]^y = [x, y]^2$.

Indeed using the second relation from (6) we have $[x,y]^y = y^{-1}x^{-1}y^{-1}xy^2 = y^{-1}x^{-1}yx[x,y^2] = [y,x][x,y^2] = [y,x][x,y]^3 = [x,y]^2$.

(*b*) $[x,y^l] = [x,y]^{2^l-1}, \quad l \geq 1$.

We show this by induction on l. It is true for $l = 1$. Let us assume that it is true for all numbers less or equal to l. Then, using commutator identity (5), the relation (a) and the inductional assumption we get:

$$[x,y^{l+1}] = [x,y][x,y^l]^y = [x,y]([x,y]^{2^l-1})^y = [x,y]([x,y]^y)^{2^l-1}$$
$$[x,y][x,y]^{2(2^l-1)} = [x,y]^{2^{l+1}-1}.$$

Since relations (6) are symmetric we have also for $k \geq 1$ relations $[x^k,y] = [x,y]^{2^k-1}$ and

(*c*) $[x^k,y^l] = [x,y]^{(2^k-1)(2^l-1)}, \quad k \geq 1, l \geq 1$.

So we have shown that every commutator $[x^k,y^l]$, for $k,l \geq 1$ is a power of $[x,y]$. It means that $[x,y]$ generates $G'_{n,m}$ and $G'_{n,m}$ is cyclic. Hence $G_{n,m}$ is cyclic-by-abelian. Since $x^n = 1$ and $y^m = 1$ we get:

$$1 = [x^n,y] = [x,y]^{2^n-1},$$
$$1 = [x,y^m] = [x,y]^{2^m-1}.$$

By Lemma 6, if $d = \gcd(m,n)$ then $2^d-1 = \gcd(2^n-1, 2^m-1)$, so $[x,y]^{2^d-1} = 1$. It means that $G'_{n,m}$ is cyclic and has order 2^d-1, moreover $G_{n,m}/G'_{n,m}$ is abelian and of order mn, so $G_{n,m}$ has order $mn(2^d-1)$.

Since $[x^d,y] = [x,y]^{2^d-1} = 1$, the element x^d is in the centre of $G_{n,m}$ and similarly y^d is in the centre.

If m,n are coprime, then the order of $G'_{n,m}$ is 1, so $G_{n,m}$ is cyclic of order mn.

In Example 7 we show the matrix representation of the group $G_{n,n}$

EXAMPLE 7. – Let A, B be following 2×2 matrices over the ring $\mathbb{Z}_{2^n-1}$:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2^{-1} \end{bmatrix}$$

Then A and B satisfy relations $[A,B] = [A,{}_2B], [B,A] = [B,{}_2A]$. Moreover $A^k = \begin{bmatrix} 2^k & 0 \\ 2^k-1 & 1 \end{bmatrix}$ and $B^k = \begin{bmatrix} 1 & 0 \\ 0 & 2^{-k} \end{bmatrix}$ and since $2^n \equiv 1 \bmod (2^n-1)$, we have $A^n = 1$ and $B^n = 1$. So A and B have order n. The commutator $[A,B]$ is equal to

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

so $[A,B]$ has order 2^n-1. To show that $\mathrm{gp}(A,B) \simeq G_{n,n}$ it is enough to prove, by the above Theorem, that $\mathrm{gp}(A,B)$ has exactly $n^2(2^n-1)$ elements. Since $\mathrm{gp}(A,B)$

has a cyclic commutator subgroup, its every element has a form $A^rB^s[A,B]^t$, and it is easy to see that A^rB^s belongs to the commutator subgroup of $\mathrm{gp}(A,B)$ if and only if $r=s=n$. Hence $\mathrm{gp}(A,B)$ has $n^2(2^n-1)$ elements and it is isomorphic to $G_{n,n}$.

It can be deduced from the above Theorem that if $d=\gcd(n,m)$ then $H=\mathrm{gp}(x^d,y^d)$ is contained in the centre of $G_{n,m}$, so it is normal in $G_{n,m}$ and $G_{n,m}/H\simeq G_{d,d}$.

5. – When is a finite metacyclic group a nmep-group?

We say that a group G is metacyclic if it is cyclic-by-cyclic (that is if there exists a normal cyclic subgroup H in G, such that G/H is cyclic).

By Hölder's Theorem (see [6] on page 129) finite metacyclic group has a presentation:

$$G=\langle x,y \mid x^n=y^t, y^m=1, yx=xy^k\rangle, \tag{7}$$

where $k^n\equiv 1 \bmod m$ and $t(k-1)\equiv 0 \bmod m$.

LEMMA 7. – *Let G be a finite metacyclic group with presentation* (7). *Then* $G'=\mathrm{gp}(y^{k-1})$, *and* $\gamma_{n+1}(G)=\mathrm{gp}(y^{(k-1)^n})$.

PROOF. – If G has a presentation (7) then $[y,x]=y^{-1}x^{-1}yx=y^{k-1}$, so $\mathrm{gp}(y^{k-1})\subseteq G'$. It is clear that $\mathrm{gp}(y^{k-1})$ is normal in G. Then in $G/\mathrm{gp}(y^{k-1})$ we have: $\bar{y}\bar{x}=\bar{x}\bar{y}^k=\bar{x}\bar{y}\bar{y}^{k-1}=\bar{x}\bar{y}$, so $G/\mathrm{gp}(y^{k-1})$ is abelian, which means that $G'\subseteq \mathrm{gp}(y^{k-1})$.

To prove that $\gamma_{n+1}(G)=\mathrm{gp}(y^{(k-1)^n})$, we use an induction on n. We proved this statement for $n=1$. Let us assume that $\gamma_n(G)=\mathrm{gp}(y^{(k-1)^{n-1}})$, then $\gamma_{n+1}(G)=\mathrm{gp}([y^{s(k-1)^{n-1}},x^r])$. Since

$$\begin{aligned}[][y^{s(k-1)^{n-1}},x^r]&=y^{-s(k-1)^{n-1}}x^{-r}y^{s(k-1)^{n-1}}x^r\\&=y^{-s(k-1)^{n-1}}y^{sk^r(k-1)^{n-1}}=y^{(k^r-1)s(k-1)^{n-1}}=y^{(k-1)^n u}\end{aligned}$$

we have $\gamma_{n+1}(G)\subseteq \mathrm{gp}(y^{(k-1)^n})$. It is easy to calculate that $[y,_n x]=y^{(k-1)^n}$, so $\mathrm{gp}(y^{(k-1)^n})\subseteq\gamma_{n+1}(G)$, which finishes the proof. □

COROLLARY 4. – *The group G with presentation* (7) *is nilpotent if and only if $k-1$ is nilpotent in the ring Z_m.*

LEMMA 8. – *Let t be an element of the ring Z_m. Then there exists an integer $s>1$ such that $t^s=t$ in Z_m if and only if* $\gcd(t,m)=\gcd(t^2,m)$.

PROOF. – Since Z_m is finite, there exists a pair (i,j) of different integers, such that $t^i \equiv t^j \bmod m$. Let $j > i$ and j minimal. If $i > 1$, then m divides $t^i(1-t^{j-i})$ and by relatively primes factors $m = \gcd(t^i, m)\gcd(1-t^{j-i}, m)$ with $\gcd(t^i, m) = \gcd(t, m)$, so m divides $t(1-t^{j-i})$ contrary to minimality of j. On the other hand, if $t - t^k$ is divisible by m then $\gcd(t, m) \le \gcd(t^2, m) \le \ldots \le \gcd(t^k, m) = \gcd(t, m)$ gives the result. □

THEOREM 7. – *Let G be a finite metacyclic group with presentation*

$$G = \langle x, y \mid x^n = y^t, y^m, yx = xy^k, k^n \equiv 1 \quad \bmod m,\ t(k-1) \equiv 0 \bmod m\rangle.$$

Then following statements are equivalent:

(i) *G is nmep-group,*
(ii) *$G' = \gamma_3(G)$.*
(iii) *$\gcd(k-1, m) = \gcd((k-1)^2, m)$,*

PROOF. – (i) $\Rightarrow$ (ii) This implication holds in general not only for metacyclic groups (we prove it in Proposition 2).

(ii) $\Rightarrow$ (iii) If $G' = \gamma_3(G)$, then by Lemma 7 $\langle y^{(k-1)^2}\rangle = \langle y^{k-1}\rangle$, so there exists integer x, such that $(k-1)^2 x \equiv k-1 \bmod m$, which means that $\gcd((k-1)^2, m) | k-1$. Hence, $\gcd((k-1)^2, m) = \gcd(k-1, m)$, as required.

(iii) $\Rightarrow$ (i) Let $a = x$ and $b = xy$. It is easy to calculate that $[a,_n b] = y^{-(k-1)^n}$ and $[b,_n a] = y^{(k-1)^n}$. If $\gcd(k-1, m) = \gcd((k-1)^2, m)$ then by Lemma 8 there exists $s > 1$ such that $(k-1)^s \equiv k-1 \bmod m$, so $[a,_s b] = y^{-(k-1)^s} = y^{-(k-1)} = [a, b]$ and $[b,_s a] = y^{(k-1)^s} = y^{k-1} = [b, a]$. Hence G is generated by a mep-pair a, b. □

If in the presentation (7), $m = p$ is a prime number, then $\gcd(k-1, p) = \gcd((k-1)^2, p)$, so by using criteria (iii) in Theorem 7 we get two Corollaries:

COROLLARY 5. – *If p, q are different primes, then every group of order pq is a nmep-group.*

PROOF. – If G is abelian, then G is a nmep-group. So let us assume that G is not abelian. If $q > p$, then it is easy to see that G has the presentation:

$$\langle x, y \mid x^p, y^q, yx = xy^k, k^p \equiv 1 \bmod q\rangle$$

Since q is a prime, we have $\gcd(k-1, q) = \gcd((k-1)^2, q)$, so by Theorem 7, G is a nmep-group. □

COROLLARY 6. – *Let C_p be a cyclic group of prime order p. Then every subgroup of its holomorph is metacyclic (or abelian) nmep-group.*

PROOF. – The automorphism group of C_p is cyclic group of order $p-1$, so the holomorph of C_p has a presentation:

$$\mathrm{Hol}\,(C_p) = \langle x, y \mid x^{p-1} = y^p = 1, yx = xy^k \rangle$$

and it satisfies the condition (iii) from Theorem 7, so it is a nmep-group. Let H be a subgroup of Hol(C_p). It is enough to assume that H is not abelian. The subgroup H is metacyclic and it is generated by some powers of x and y. Indeed the subgroup $H \cap C_p$ is normal and cyclic and the quotient group $H/H \cap C_p \cong HC_p/C_p$ is also cyclic and H is generated by the power of y (which generates $H \cap C_p$) and the power of x (whose image generates $H/H \cap C_p$). Since every power of y has order p, so H also satisfies the condition (iii) of Theorem 7 and H is a nmep-group. □

COROLLARY 7. – *The dihedral group D_n is a nmep-group if and only if $4 \nmid n$.*

PROOF. – A dihedral group D_n has the presentation (see [1] 1.5):

$$\langle x, y \mid x^2, y^n, yx = xy^{n-1} \rangle$$

and to show that D_n is a nmep-group it is enough to show that $\gcd(4, n) = \gcd(2, n)$, which holds if and only if $4 \nmid n$. □

COROLLARY 8. – *If G is a finite group whose all Sylow subgroups are cyclic then G is metacyclic, nmep-group.*

PROOF. – By Theorem 11 from [6] (on page 175) a group whose all Sylow subgroup are cyclic is metacyclic and has a presentation:

$$\langle x, y \mid x^n, y^m, xy = y^r x \rangle$$

with conditions: $r^n \equiv 1 \bmod m, \gcd((r-1)n, m) = 1$. If $\gcd((r-1)n, m) = 1$ then $\gcd(r-1, m) = 1$ and also $\gcd((r-1)^2, m) = 1$, so this group satisfies condition (iii) of Theorem 7, so it is a nmep-group. □

6. – Matrix groups generated by mep- or nmep-pairs.

Now we want to consider groups generated by two matrices of the form $\begin{bmatrix} a & 0 \\ x & b \end{bmatrix}$ over commutative ring R. Such groups are metabelian and since matrices of the form $\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ are in their centres it is enough to consider matrices of the form $\begin{bmatrix} a & 0 \\ x & 1 \end{bmatrix}$.

THEOREM 8. – *Let R be a commutative ring with unity and let a, b be units in R and $x, y \in R$. If $(a-1)^{m-1} = 1$ and $(b-1)^{n-1} = 1$ then matrices*

$$A = \begin{bmatrix} a & 0 \\ x & 1 \end{bmatrix}, B = \begin{bmatrix} b & 0 \\ y & 1 \end{bmatrix}$$

form a non-commutative mep-pair that satisfies $[A, B] = [A,_n B], [B, A] = [B,_m A]$.

PROOF. – For given A, B we have:

$$[A, B] = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix},$$

where $s = x(b-1) - y(a-1)$ and for $n \geq 1$:

$$[A,_n B] = \begin{bmatrix} 1 & 0 \\ s(b-1)^{n-1} & 1 \end{bmatrix}, \ [B,_m A] = \begin{bmatrix} 1 & 0 \\ -s(a-1)^{m-1} & 1 \end{bmatrix}$$

and $[A,_n B] = [A, B]$ if and only if $s = s(b-1)^{n-1}$. So, if $(b-1)^{n-1} = 1$ then the first relation relation holds, and it can be shown similarly the second one. □

Let us notice that if R has no zero divisors then the converse is also true (that is if matrices A, B are as in the above Theorem and satisfy $[A, B] = [A,_n B]$ and $[B,_m A] = [B, A]$ then $(b-1)^{n-1} = 1$ and $(a-1)^{m-1} = 1$ or A and B commute).

It is a well-known question: for which complex numbers x, y, matrices $X = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}$ generate a free group? By a classical result this group is free for $x = y$ and $|x| \geq 2$ and $|y| \geq 2$ (see for example [3] in the case if x is an integer). We can ask the question: is it possible that such matrices generate a nmep-group? And the answer is: no. For $C = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$, we have $CXC^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $CYC^{-1} = \begin{bmatrix} 1 & 0 \\ xy & 1 \end{bmatrix}$, hence it is enough to consider matrices of the form $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$.

THEOREM 9. – *Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$. Then for any complex number b, A, B is not a nmep-pair.*

PROOF. – We have $[B,_k A] = \begin{bmatrix} * & * \\ b^{2^k} & kb^{2^k} + \sum_{s=0}^{k} b^{2^k - 2^s} \end{bmatrix}$. So the equation $[B, A] = [B,_k A]$ holds if and only if

$$b^2 = b^{2^k} \text{ and } b^2 + b + 1 = kb^{2^k} + \sum_{s=0}^{s=k} b^{2^k - 2^s}. \tag{8}$$

Using the first equation we get

$$b^2 + b + 1 = kb^{2^k} + b^{2^k-1} + b^{2^k-2} + b^{2^k-2^2} + \dots + b^{2^k-2^{k-1}} + 1,$$
$$b^2 + b + 1 = kb^2 + b + 1 + b^{2^k-2^2} + \dots + b^{2^k-2^{k-1}} + 1,$$
$$-(k-1)b^2 = b^{2^k-2^2} + \dots + b^{2^k-2^{k-1}} + 1.$$

From $b^2 = b^{2^k}$ we have $|b| = 1$ and so $|-(k-1)b| = k-1$, on the right side we have $k-1$ summands b^z with $|b^z| = 1$, one even equal to 1. So all summands on the right side satisfy $b^z = 1$, and $b^2 = -1$. Now $b^2 = b^{2^k}$ is only possible for $k = 1$, and hence A, B is not a mep-pair ($b = 1$ is not a solution). $\square$

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REFERENCES

[1] H.S.M. Coxeter, W.O.J. Moser, *Generators and Relations for Discrete Groups*, Berlin-Heildeberg-New York 1980.
[2] H. Heineken, *Groups generated by two mutually Engel Periodic elements*, Bolletino U.M.I., (8) 3-B (2000), 461-470.
[3] M.I. Kargapolov, Ju. I. Merzljakov, *Fundamentals of the Theory of Groups*, Springer-Verlag, New York, 1979.
[4] W. Magnus, A. Karrass, D. Solitar, *Combinatorial Group Theory*, Dover Publications, Inc. New York 1976.
[5] J.J. Rotman, *An Introduction to the Theory of Groups*, Springer-Verlag, Inc. New York 1995.
[6] H.J. Zassenhaus, *The Theory of Groups*, Dover Publications, Inc., Mineola, New York 1999.

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