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Remarks on the Existence of Many Solutions of Certain Nonlinear Elliptic Equations.

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Sunto. – *In questo lavoro, si mostra come i cambi di variabile unitamente ai metodi utilizzati per trovare soluzioni ad uno o più picchi, possono essere usati per provare che varie equazioni alle derivate parziali non lineari hanno molte soluzioni.*

Summary. – *We show how a change of variable and peak solution methods can be used to prove that a number of nonlinear elliptic partial differential equations have many solutions.*

1. – Introduction.

Consider the problem

$$
\begin{aligned}
-\Delta u &= au - |u|^p \ \text{ in } \ \Omega \\
u &= 0 \ \text{ on } \ \partial\Omega
\end{aligned}
\tag{1}
$$

where Ω is a smooth bounded domain in R^n.

There are many works concerning the multiplicity of solutions for (1). A popular method to study (1) has been to use order theoretic and Morse index techniques as in [1]. See the discussion in Remark 1. But it seems very difficult to obtain more than three non-trivial solutions by these techniques. So, to find more solutions for (1), a new idea is needed.

In this paper, we continue a recent theme in work of ours ([7], [8]) by showing that (1) has many solutions by using a change of variable and then using peak solution ideas. The many peak solutions come about for rather different reasons to those in [7] and [8] and the locations of the peaks are very different. Here the locations are determined directly by the shape of the domain and not by the first eigenfunction of the Laplace operator on the domain Ω with Dirichlet boundary condition. Once again, we obtain many more solutions than can be obtained by the usual variational and topological techniques, such as the order theoretic and Morse index techniques discussed above.

We mainly consider the case $p = 2$ and $n \leq 5$. Many generalizations will also be discussed in Section 3.

The main result of this paper is

THEOREM 1. – *If $p = 2, n \leq 5$, then, for any integer $k > 0$,* (1) *has at least k solutions when a is large.*

2. – Proof of the Main Result.

PROOF OF THEOREM 1. STEP 1. – We use the change of variable $u = \varepsilon^{-1} v$, where $\varepsilon^2 = a^{-1}$. Then our problem becomes

$$-\varepsilon^2 \Delta v = v - v^2, \text{ in } \Omega; \quad v = 0, \text{ on } \partial\Omega. \tag{2}$$

This is the type of equation where we can apply peak solution methods.

STEP 2. – Let $f(t) = t - t^2$. Then, $f(t) = 0$ has a solution $t = 1$ with $f'(1) < 0$. By standard techniques, (2) has a unique positive solution ϕ_ε satisfying

$$0 < \phi_\varepsilon < 1, \quad \text{in } \Omega,$$

and

$$\phi_\varepsilon = 1 + O\big(e^{-\delta/\varepsilon}\big) \quad \text{uniformly on } K \subset\subset \Omega, \tag{3}$$

where $\delta > 0$ is a constant depending on K.

STEP 3. – We then obtain many solutions for (2) for small ε by looking for solutions with k negative peaks superimposed on ϕ_ε in the interior of Ω (for arbitrary k). That is, we look for solution of the form

$$\phi_\varepsilon - \sum_{i=1}^{k} \widetilde{w}\Big(\frac{x - x_i}{\varepsilon}\Big) + \text{higher order terms}, \tag{4}$$

where $\widetilde{w}$ and $x_i \in \Omega$ are to be determined.

By (3), we see that the limit equation for a peak is

$$\Delta w = (1 - w) - (1 - w)^2 \text{ on } R^n, \quad w \in H^1(R^n).$$

That is,

$$-\Delta w = w^2 - w \text{ in } R^n, \quad w \in H^1(R^n). \tag{5}$$

Since $n \leq 5$, we see $2 < \dfrac{n+2}{n-2}$. Thus equation (5) has a unique (up to a translation) positive solution $\widetilde{w}$. Note that $\widetilde{w}$ decays exponentially and it is non-degenerate in the sense that the linearized operator:

$$-\Delta v + v - 2\widetilde{w}v \tag{6}$$

has kernel (in the space of decaying functions) spanned by $\left\{\dfrac{\partial \widetilde{w}}{\partial x_i} : i = 1, \cdots n\right\}$. See [13], [2] and [4]. Note that we assume that $n \leq 5$ here because if $n \geq 6$, (3) has no positive decaying solution by a Pohozaev identity.

Now, it is clear that the function $\widetilde{w}$ in (4) is the radial positive solution of (5). So, we are in the situation of § 5 of [9] except that we are putting negative peaks on a positive solution rather than vice versa as in [9].

Since $\widetilde{w}$ is non-degenerate, we can easily carry out a Liapounov-Schmidt reduction argument. So the problem of finding a solution with the form (4) is reduced to a kn dimensional problem (which is variational). To solve the corresponding finite dimensional problem, we can use a maximization process as in [7], [8] and [9] to prove that the finite dimensional problem has a solution $x = (x_1, \cdots, x_k) \in R^{kn}$, such that x is close to a point which maximizes $d(z, \partial\Omega)$ if $k = 1$, $z \in \Omega$; and maximizes $\tilde{p}(z)$ if $k > 1$, where

$$\tilde{p}(z) = \min\{d(z_j, \partial\Omega),\ |z_i - z_j|,\ i, j = 1, \cdots, k,\ i \neq j\}.$$

Note that the maximum is easily seen to exist and occurs at $(\tilde{z}_1, \cdots, \tilde{z}_k)$ with $d(\tilde{z}_i, \partial\Omega) \geq \delta > 0$, $\tilde{z}_i \neq \tilde{z}_j$, $i \neq j$. Since these solutions are clearly distinct for different k, we obtain the required result. We will give more details in Appendix B.

To close this section, let us make a few remarks.

REMARK 1. – A popular method to study (1) has been to use order theoretic and Morse index techniques as in [1]. The general procedure can be described as follows:

(i) For $a > \lambda_1$, where λ_i is the i-th eigenvalue of the $-\Delta$ in Ω with Dirichlet boundary condition, one can prove that (1) has a positive solution ψ by using the sub-solution and super-solution method;

(ii) If $a > \lambda_2$, then one can apply the mountain pass theorem in the order interval

$$(-\infty, \psi] = \left\{u \in H_0^1(\Omega), u \leq \psi\right\},$$

to obtain a second solution, which cannot be the zero solution because zero has Morse index strictly larger than 1 if $a > \lambda_2$.

(iii) If $a > \lambda_2$ and a is not an eigenvalue, one can find a third non-trivial solution by a simple degree calculation. In fact, by homotopy invariance (by varying a), the sum of the indices relative to the order interval $(-\infty, \psi]$ is 0 for all $a > \lambda_1$. But ψ has index 1 for $a > \lambda_1$ since it is stable, and a mountain pass solution has index -1. So the existence of the third non-trivial solution u_3 follows.

One can obtain a little more information about the critical groups of u_3 (and sometimes allow a to be an eigenvalue) by Conley index techniques, but it seems very difficult to obtain more than three non-trivial solutions by these methods. Critical groups are defined in [1]. Here, we show that if a is large, we can obtain many more solutions by using the peak solution methods and a good deal more information on what the solutions look like.

REMARK 2. – It follows from the definition of $\tilde{p}(z)$ that the asymptotic locations of the peaks are determined by a modified sphere packing problem, while in [7] and [8], the locations are determined by the locations of the global maxima of the first eigenfunction of $-\Delta$, and at least for generic Ω, the peaks are close to each other which contrasts with the case here.

REMARK 3. – None of the above solutions is the mountain pass solution (which must exist) for small ε. It is easy to see that solutions with more than one peak are not mountain pass solutions. On the other hand, the method in [9] can be easily modified to prove that the mountain pass solution has a peak within order ε of the boundary.

REMARK 4. – If $n = 2$, the methods in [3] and [9] can be used to prove that the only solutions of (2) of bounded Morse index for small positive ε are solutions with a finite number of negative peaks on the positive solution, where some of these peaks could be within order ε of the boundary (as in the mountain pass solution).

3. – Some Generalizations.

Our method in Section 1 can be generalized to the subcritical case $1 < p < \dfrac{n+2}{n-2}$. To carry out the reduction argument, the only property we need is that the following problem

$$-\Delta w = |w-1|^p + (w-1) \ \text{ in } R^n, \quad w \in H^1(R^n), \tag{7}$$

where $p < (n+2)/(n-2)$, has a positive decaying solution (necessarily radial up to translation), which is also non-degenerate. The existence of a solution is standard. The question is the non-degeneracy. By the results in [13], [2] and [4], which also give the uniqueness, a sufficient condition is that the nonlinearity $f(t) = |t-1|^p + (t-1)$ satisfies the following conditions:

(f): $\Theta(t) \le p$ on $(0,1)$, $\Theta(t) \ge \Theta(a)$ on $(1,a)$, and Θ is non-increasing on $[a,\infty)$. Here $\Theta(t) = tf'(t)/f(t)$, and $a > 1$ is the constant with $\int_0^a f = 0$.

From the above discussion, we have the following generalization of Theorem 1.

THEOREM 2. – *Suppose that* $1 < p < \frac{n+2}{n-2}$, *and* $f(t)$ *satisfies* (f). *Then for any integer* k, (7) *has at least* k *solutions when* a *is large positive.*

In Appendix A, we will check that condition (f) holds if $p \in [2,3]$. The non-degeneracy and uniqueness continue to hold if p is subcritical and close to $[2,3]$ (by a limit argument), and we suspect it holds for all subcritical p. For our purposes, it would suffice to obtain one non-degenerate solution in cases where uniqueness fails.

REMARK 5. – By Theorem 2, (7) has more and more solutions as a become larger. Moreover, if Ω is a ball, then solutions with more than one peak are not radial. Thus, if Ω is a ball and a is suitably large, (7) has a solution with two peaks, and thus (7) has infinitely many solutions by rotation.

On the other hand, one can easily obtain bounds for all the solutions of (7) if p is subcritical for fixed a. We clearly have an upper bound because the positive solution is the maximal solution. Thus, the difference of this maximal solution and any other solution is positive. By using a standard blowing up argument, the proof of the bound reduces to the non-existence of bounded positive solutions of $-\Delta u = u^p$ on R^n or on a half space with Dirichlet boundary conditions. See [10] and [11]. We can then deduce for fixed a and "generic" Ω (in the sense of [14]) that the number of solutions for (7) is finite (by using the results of [14]).

Our method applies to more general nonlinearities. For example, there are similar results for

$$-\Delta u = au - u^2 - |u|^3, \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega. \tag{8}$$

We proceed as before.

STEP 1. – Using the change of variable $v = \varepsilon^{-1} v$ and $\varepsilon = a^{-1/2}$, (8) becomes

$$-\varepsilon^2 \Delta v = v - \varepsilon v^2 - |v|^3, \text{ in } \Omega, \quad v = 0, \text{ on } \partial\Omega. \tag{9}$$

STEP 2. – We let $\widetilde{v}_\varepsilon$ be the positive solution of (9). Then $\widetilde{v}_\varepsilon$ is exponentially close to b_ε in the interior of Ω, where b_ε satisfies

$$b_\varepsilon - \varepsilon b_\varepsilon^2 - b_\varepsilon^3 = 0.$$

Note that b_ε is close to 1.

STEP 3. – We now put k negative peaks on $\widetilde{v}_\varepsilon$. To determine the limit problem for a peak, as in the proof of Theorem 1, one may first use the following problem as the limit problem:

$$\Delta w = (1-w) - |1-w|^3, \ \text{ in } R^n, \quad w \in H^1(R^n). \tag{10}$$

As we have seen in the proof of Theorem 1, the terms in the energy expansions are exponentially small. But both εv in (9) and $b_\varepsilon - 1$ are algebraically small. We modify the limit problem to the following one to avoid the awkward algebraic terms of ε:

$$\Delta w = (b_\varepsilon - w) - \varepsilon(w - b_\varepsilon)^2 - |w - b_\varepsilon|^3, \ \text{ in } R^n, \quad w \in H^1(R^n). \tag{11}$$

By using the implicit function theorem in the space of radial functions, we can prove that (11) has a solution w_ε in the neighborhood of the positive solution of (10). This solution is also non-degenerate. We then look for peak solutions of the form

$$\tilde{v}_\varepsilon(y) - \sum_{i=1}^{k} w_\varepsilon\Big(\frac{y - x_i}{\varepsilon}\Big) + \text{higher order terms},$$

and we can proceed as before. We obtain the following theorem.

THEOREM 3. – *Suppose that $n \leq 3$. Then for any integer k,* (8) *has at least k solutions when a is large positive.*

Note that we can replace $-u^2$ by u^2 because our arguments do not depend upon the convexity. It is also clear that our methods can be applied for many other nonlinearities, such as the simple cubic nonlinearities which appear in many applications. We stress that, when we study $-\varepsilon^2 \Delta w = f(w)$ in Ω, it is critical for our arguments that, if the positive solution ϕ_ε is close to 1 on compact sets, then $f'(1) < 0$.

Appendix A.

Verification of the Condition for Non-degeneracy

Let $f(t) = |t-1|^p + (t-1)$ and let $\Theta(t) = tf'(t)/f(t)$. In this section, we will check that if $p \in [2,3]$, then condition (f) holds. Thus, if $p \in [2,3]$ and $p < \frac{n+2}{n-2}$, the positive solution of (8) is non-degenerate and unique.

Here, we will prove that if $p \in [2,3]$, then

(i) $\Theta(t) < p$, in $(0,1)$;
(ii) Θ is non-increasing in $(1,+\infty)$.

We only prove (ii). It is similar but easier to prove (i).

Verification of (ii). We have

$$\Theta(t) = \frac{t\big(p(t-1)^{p-1}+1\big)}{(t-1)^p + t - 1}, \quad t > 1.$$

Let $z = \frac{1}{t-1}$. Then

$$\Theta(z) = \frac{z(z+1)\big(p + z^{p-1}\big)}{z^p + z}, \quad z > 0.$$

We have

$$\Theta'(z) = \frac{z^2 h(z)}{(z^p+z)^2},$$

where

$$h(z) = p - (p-1)^2 z^{p-2} + p(3-p)z^{p-1} + z^{2p-2}.$$

Note that $\frac{d\Theta(t)}{dt} \le 0$ is equivalent to $h'(z) \ge 0$ for $z > 0$. Direct calculation shows

$$h'(z) = -(p-1)^2(p-2)z^{p-3} + p(p-1)(3-p)z^{p-2} + 2(p-1)z^{2p-3}.$$

Hence the critical points of $h(z)$ are given by the non-negative zeros of

$$\eta(z) = -(p-2)(p-1) + p(3-p)z + 2z^p.$$

Now η is strictly increasing for $z > 0$ (for $p \le 3$), $\eta(0) \le 0$ (for $p \ge 2$) and $w(1) = 2p(3-p) \ge 0$. Hence the only nonnegative critical point $z^\star$ of h lies in $[0,1]$. Now

$$\begin{aligned} h(z^\star) &= h(z^\star) - (p-1)^{-1} z^\star h'(z^\star) \\ &= p - (z^\star)^{2p-2} - (p-1)(z^\star)^{p-2} \\ &> 0, \end{aligned}$$

since $0 < z^\star < 1$. Hence our claim follows.

Appendix B.

Proof of the Existence of Peak Solutions

In this section, we give a little more details than in [9] on how the computations are conducted. We do not give complete details because it is a very straightforward modification of the ideas in [9, 12, 17]. Here $f(t) = t - t^2$.

Since $w_{\varepsilon,x_i}(x) = \widetilde{w}(\varepsilon^{-1}(x - x_i))$ is not zero on the boundary of Ω, where $\widetilde{w}$ is the positive decreasing solution of (5), we define $P_{\varepsilon,\Omega} w_{\varepsilon,y}$ as the solution of

$$-\varepsilon^2 \Delta v - f'(1)v = -f(1 - w_{\varepsilon,y}) - f'(1) w_{\varepsilon,y} \ \text{ in } \Omega, \quad v = 0, \text{ on } \partial\Omega. \tag{12}$$

Let ϕ_ε be the positive solution of (2). We look for k-peak solutions of (2) of the form

$$\phi_\varepsilon - \sum_{i=1}^{k} P_{\varepsilon,\Omega} w_{\varepsilon,x_i} + v. \tag{13}$$

The function v in (13) is a higher order term, and

$$v \in E_{\varepsilon,x} = \left\{ \left\langle \frac{\partial P_{\varepsilon,\Omega} w_{\varepsilon,x_i}}{\partial x_{i,j}}, v \right\rangle_\varepsilon = 0, \ 1 \le i \le k, \ 1 \le j \le n \right\},$$

where

$$\langle u, v \rangle_\varepsilon = \int_\Omega (\varepsilon^2 \nabla u \nabla v + uv), \quad \|u\|_\varepsilon = \langle u, u \rangle_\varepsilon^{1/2}.$$

Note that $P_{\varepsilon,\Omega} w_{\varepsilon,y}$ is a modification of $w_{\varepsilon,y}$ to satisfy the boundary conditions and, by the maximum principle, it is easily seen

$$P_{\varepsilon,\Omega} w_{\varepsilon,y} = w_{\varepsilon,y} + O\big(e^{-d(y,\partial\Omega)/\varepsilon}\big).$$

We are interested in solution where $\{x_i\}$ almost maximizes

$$\widetilde{p}(z) = \min\big(d(z_i, \partial\Omega), \ |z_i - z_j|, \ i,j = 1, \cdots, k, \ i \neq j\big\}.$$

This ensures that x_i are bounded away from each other and the boundary. In fact, we will look for those $(x_1, \cdots, x_k)$ in the δ neighborhood of the $(z_1, \cdots, z_k)$, which maximize $\widetilde{p}$. (These may not be unique). Denote this δ neighborhood by S. Now, by a standard Liapounov-Schmidt reduction, we can solve for $v_{\varepsilon,x}$ uniquely, and obtain the following estimate:

$$\|v_{\varepsilon,x}\|_\varepsilon \le C e^{-\frac{1}{2}(1+\sigma)\gamma\varepsilon^{-1}\tilde{p}(x)}, \tag{14}$$

where $\gamma^2 = -f'(1)$, $\gamma > 0$ and $\sigma > 0$ is positive and fixed. This is very similar to the arguments in [9], [12] [17]. By estimate (14), we can easily see, as in [12] or [17], that $v_{\varepsilon,x}$ makes a higher order contribution to the energy. In the estimates below, we will omit the contribution of $v_{\varepsilon,x}$.

Now, firstly note that

$$\phi_\varepsilon(z) = 1 - e^{-\gamma\varepsilon^{-1}(d(z,\partial\Omega)+o(1))}$$

if z is not close to $\partial\Omega$. This follows from the proof of Theorem 2.1 in [5].

We will work with the energy

$$I(v) = \int_\Omega \frac{1}{2}\varepsilon^2|\nabla v|^2 - \widetilde{G}_\varepsilon(y,v)$$

where $\widetilde{G}_\varepsilon(y,t) = F(\phi_\varepsilon(y)-t) - F(\phi_\varepsilon(y)) + f'(\phi_\varepsilon(y))t^2, F' = f, F(0) = 0$. Then, if x_j is not close to $\partial\Omega$, we have

$$I(P_{\varepsilon,\Omega}w_{\varepsilon,x_j}) = \varepsilon^n A - \tau_{\varepsilon,x_j} + O\big(e^{-\varepsilon^{-1}\gamma(1+\sigma)d(x_j,\partial\Omega)}\big), \tag{15}$$

where

$$\tau_{\varepsilon,x_j} = \int_\Omega (f'(1-w_{\varepsilon,x_j}) + f'(1)w_{\varepsilon,x_j})(a-\phi_\varepsilon),$$

$\sigma > 0$, and

$$A = \int_{R^n} \frac{1}{2}|\nabla\widetilde{w}|^2 - F_1(\widetilde{w}), \text{ where } F_1(0) = 0, F_1'(t) = f(1-t)\ .$$

Moreover,

$$C_0 e^{-(1+\theta)\gamma\varepsilon^{-1}d(x_j,\partial\Omega)} \le \tau_{\varepsilon,x_j} \le C_1 e^{-(1-\theta)\gamma\varepsilon^{-1}d(x_j,\partial\Omega)} \tag{16}$$

for any $\theta > 0$, where C_0 and C_1 are positive constants.

To prove (15), we multiply equation (12) by $P_{\varepsilon,\Omega}w_{\varepsilon,y}$ to remove gradient terms in $I(P_{\varepsilon,\Omega}w_{\varepsilon_1 x_i})$ above. Then estimate (15) can be proved by repeating Step 1 of the proof of Lemma 3.1 in [9]. To prove (16), we first replace Ω by R^n (which only affects the remainder term). It is best to use Theorem 1.2 and the argument on p. 100-101 of [6]. (This avoids the convexity which is used at the end of Step 1 of Lemma 3.1 in [9]). (As in [6], we can actually prove a more precise estimate than Proposition 5.2 in [9]).

Next we consider, the energy of a sum for x_i not close to each other or to the boundary

$$\begin{aligned} &I\left(\sum_{j=1}^k P_{\varepsilon,\Omega}w_{\varepsilon,x_y}\right)\\ &= \sum_{j=1}^k I(P_{\varepsilon,\Omega}w_{\varepsilon,x_j}) - \int_\Omega \left[F_1\left(\sum_{j=1}^k w_{\varepsilon,x_j}\right) - \sum_{j=1}^k F_1(w_{\varepsilon,j}) - \sum_{i<j} f_1(w_{\varepsilon,x_j})x_j\right]\\ &\quad + \text{ higher order terms,} \end{aligned} \tag{17}$$

where $f_1(t) = f(1-t)$ and $F_1(t) = \int_0^t f_1(s)ds$. This follows as in Proposition 5.3 of [9] (where there is a more detailed discussion of the higher order terms) or in [12]. The key point is to multiply (12) by $P_{\varepsilon,\Omega}w_{\varepsilon,x_j}$, integrate by parts and use this to remove the gradient terms.

We now follow the argument on p10-14 of [12] to prove that the integral in (17) is

$$-\frac{1}{2}\widetilde{\gamma}\varepsilon \sum_{1\leq i<j\leq k}^{n} \widetilde{w}(\varepsilon^{-1}|x_i - x_j|) + \text{ higher order terms}$$

where $\widetilde{\gamma} > 0$. (To prove that $\widetilde{\gamma} > 0$ in this formula, we need to assume that $f(1-y) < f'(1)y$ for $y < 1$ though in fact this could be removed by modifying the idea on p. 100-101 of [6]). Hence we see that

$$I\left(\sum_{j=1}^{k} P_{\varepsilon,\Omega}w_{\varepsilon,x_j}\right) = k\varepsilon^n A - \sum_{i=1}^{k} \tau_{\varepsilon,x_i} - \frac{1}{2}\widetilde{\gamma}\varepsilon^n \sum_{1\leq i<j\leq k} \widetilde{w}(\varepsilon^{-1}|x_i - x_j|)$$
$$+\text{higher order terms.}$$

Note that the energy decreases if some x_j gets close to the boundary or some x_i, x_j gets close. Hence, exactly as in [12] §4, we can maximize

$$I\left(\sum_{j=1}^{k} P_{\varepsilon,\Omega}w_{\varepsilon,x_j} + v_{\varepsilon,x}\right)$$

over S and show that, for small ε, the maximum occurs in the interior of S and hence is a critical point of I. Hence we have a solution of our original equation.

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