Giuliano Parigi

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Some Remarks on Prym-Tyurin Varieties.

GIULIANO PARIGI

to Fabio

Sunto. – *Gli scopi del presente lavoro sono i seguenti:*

a) In [2] *Beauville ha dimostrato che se un certo endomorfismo u di una Jacobiana $J(C)$ ha nucleo connesso, la polarizzazione principale su $J(C)$ induce un multiplo di una polarizzazione principale sull'immagine di u. Si riformula e si completa questo teorema provando "costruttivamente" il seguente:*

Teorema. *Sia $Z \subset J(C)$ una sottovarietà abeliana e Y la sua varietà complementare. Z è una varietà di Prym-Tyurin rispetto a $J(C)$ se e solo se la sequenza:*

$$0 \to Y \hookrightarrow J(C) \to Z \to 0$$

è esatta.

b) In [5] *Izadi pose la questione se ogni p.p.a.v. fosse una varietà di Prym-Tyurin rispetto ad una corrispondenza simmetrica senza punti fissi. In questo lavoro si fornisce un contributo ad una possibile risposta negativa a questa domanda costruendo una classica varietà di Prym-Tyurin esplicitamente tale che una tale varietà non possa mai essere definita da una corrispondenza senza punti fissi.*

Summary. – *The aims of the present paper can be described as follows:*

a) In [2] *Beauville showed that if some endomorphism u a Jacobian $J(C)$ has connected kernel, the principal polarization on $J(C)$ induces a multiple of the principal polarization on the image of u. We reformulate and complete this theorem proving "constructively" the following:*

Theorem. *Let $Z \subset J(C)$ be an abelian subvariety and Y its complementary variety. Z is a Prym-Tyurin variety with respect to $J(C)$ if and only if the following sequence*

$$0 \to Y \hookrightarrow J(C) \to Z \to 0$$

is exact.

b) In [5] *Izadi set the question whether every p.p.a.v. is a Prym-Tyurin variety for a symmetric fixed point free correspondence. In this work a contribution to a possible negative answer to this question is provided by building a classical Prym-Tyurin variety explicitly, but this variety can never be defined through a fixed point free correspondence.*

Introduction.

Prym-Tyurin abelian were initially introduced by Tyurin in [10] as a natural generalization of Prym varieties.

Given a smooth and projective curve C of genus g and its Jacobian variety $(J(C), \Theta)$, we shall refer to those abelian subvarieties Z in $J(C)$ for whom the induced polarization is multiple of a principal polarization Ξ on Z.

In his basic work, Welters ([11]) proves that every principally polarized abelian variety (p.p.a.v.) is isomorphic to some Prym-Tyurin variety and suggests that a rigorous study of correspondences on curves is needed for a deeper knowledge of geometry of the abelian varieties.

Now, it is well-known that for a generic algebraic curve C the Jacobian $J(C)$ does not contain proper abelian subvarieties. Hence, we have been brought to consider curves which could have special symmetric correspondences, that is to say to consider special curves such that the period ω of their Jacobian satisfies the Hurwitz relation:

$$a\omega = \omega A, \qquad \text{where} \det(A) = 0.$$

Those correspondences were introduced and studied by Scorza and Rosati in the first decades of 1900 but were made "handier" by Albert ([1], 1935) who constructively showed that the period matrix ω of a curve having a special correspondence is actually an impure matrix, i.e. such that

$$\omega = \begin{pmatrix} \omega_1 & 0 \\ \omega_3 & \omega_2 \end{pmatrix}.$$

On the other hand, in terms of $\mathrm{End}(J(C))$, that corresponds to the existence of a non-surjective endomorphism of $J(C)$ which is fixed by Rosati's involution, whose image defines an abelian subvariety Z in $J(C)$. Moreover, it is easy to realize that given a principally polarized abelian variety Z, there exists a smooth and projective curve C whose Jacobian $J(C)$ has an endomorphism σ such that Z can be constructed as $\mathrm{Im}(\sigma) \hookrightarrow J(C)$ (e.g. [8], page 374).

The aims of the present work can be summarized as follows:

a) In [2] Beauville showed that if some endomorphism u of a Jacobian $J(C)$ has connected kernel, the principal polarization on $J(C)$ induces a principal polarization on the image of u.

Consequently, given an abelian subvariety Z of $J(C)$, considering the norm-endomorphism N_Z of Z, and the abelian subvariety Y of $J(C)$ complementary to Z (see [8], pp. 125-127), our methods allow us to reformulate and to extend the theorem stated by Beauville, proving "constructively" (i.e. using the matrix of the canonical polarization of $J(C)$ the following:

THEOREM. – *Let $Z \subset J(C)$ be an abelian subvariety, Y its complementary variety. Z is a Prym-Tyurin variety with respect to $J(C)$ if and only if the fol-*

lowing sequence:

$$0 \to Y \hookrightarrow J(C) \xrightarrow{N_Z} Z \to 0$$

is exact.

b) Anyway we remark that it does not seem straightforward to characterize the principally polarized abelian subvarieties of $J(C)$ by means of particular special correspondences on C. The only known result in this direction is the one achieved by Kanev ([6], 1987). In that work Kanev proves that, given a principally polarized abelian subvariety Z of a Jacobian $J(C)$, an effective fixed point free correspondence on C defines a multiple of the given principal polarization on Z.

In 2001 Izadi in [5] set the question whether every p.p.a.v. is a Prym-Tyurin variety for a symmetric fixed point free correspondence. In this work we provide a contribution for a possible negative answer to this question by building a Prym variety explicitly, but this variety can never be defined through a fixed point free correspondence, consequently confirming that Kanev's condition is only sufficient.

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1. – Subvarieties of abelian varieties.

1.1 – *Basic concepts.*

Let C be a smooth and projective curve of genus g and let $J(C)$ be its Jacobian. Let us suppose that there exists an endomorphism $\sigma \in \mathrm{End}(J(C))$, non-surjective and symmetric with respect to Rosati's involution. If we consider its rational representation $\rho_r(\sigma)$, the rank of the matrix A_σ, matricial representation of $\rho_r(\sigma)$, cannot be maximum, so that $\det(A_\sigma) = 0$. Let us now recall the well-known theorem by Scorza and Albert:

THEOREM 1.1 (Scorza [9], page 278 and Albert [1], theorem 3, page 154). – *Let ω be a Riemann matrix; ω is impure (e.g. Introduction) if and only if there exist $a \in M_{g,g}(\mathbb{C})$ and $A \in M_{2g,2g}(\mathbb{Z})$ such that the Hurwitz condition, i.e. $a\,\omega = \omega A$, where* $\det(A) = 0$, *is satisfied (e.g.* [4]*).* □

It follows that the Jacobian $J(C)$ can be represented by an impure period matrix ω, that is of the following kind:

$$\omega = \begin{pmatrix} \omega_1 & 0 \\ \omega_3 & \omega_2 \end{pmatrix}.$$

Furthermore $Z := \mathrm{Im}\,(\sigma)$ turns out to be a proper abelian subvariety of $J(C)$.

REMARK 1.2. – Supposing $\omega_1 \in M_{r,2r}(\mathbb{C})$ and $\omega_2 \in M_{g-r,2(g-r)}(\mathbb{C})$ for $r \in \mathbb{Z}$ such that $0 < r < g$. Since ω is the period matrix of a complex torus, we have that the matrix $P := \begin{pmatrix} \omega \\ \overline{\omega} \end{pmatrix}$ is nonsingular ($\overline{\omega}$ is the complex conjugate matrix of ω). On the other hand, we have

$$0 \neq \det\begin{pmatrix} \omega \\ \overline{\omega} \end{pmatrix} = \det\begin{pmatrix} \omega_1 & 0 \\ \omega_3 & \omega_2 \\ \overline{\omega}_1 & 0 \\ \overline{\omega}_3 & \overline{\omega}_2 \end{pmatrix} = \pm\det\begin{pmatrix} \omega_1 & 0 \\ \overline{\omega}_1 & 0 \\ \omega_2 & \omega_2 \\ \overline{\omega}_3 & \overline{\omega}_2 \end{pmatrix}$$

$$= \pm\det\begin{pmatrix} \omega_1 \\ \overline{\omega}_1 \end{pmatrix} \cdot \det\begin{pmatrix} \omega_2 \\ \overline{\omega}_2 \end{pmatrix}.$$

That implies that ([8], proposition 1.1.2) both ω_1 and ω_2 are the period matrices of two complex tori.

REMARK 1.3. – If we consider the vector space V_1 spanned by the first r vectors of the basis $\{e_1, \ldots, e_g\}$ of $V = \mathbb{C}^g$, that is to say:

$$V_1 = \mathbb{L}\{e_1, \ldots, e_r\} \cong \mathbb{C}^r,$$

and if $P_{V_1} : V \to V_1$ is the natural prjection, it is obvious that asserting that ω_1 is the period matrix of a complex torus is equivalent to asserting that, if $\eta_i := P_{V_1}(\lambda_i)$, $i = 1, \ldots, 2r$, the elements $\{\eta_1, \ldots, \eta_{2r}\}$ form a complete basis for the lattice $\Lambda_1 \subseteq \Lambda$ defined by

$$\Lambda_1 := \langle \eta_1, \ldots, \eta_{2r} \rangle_{\mathbb{Z}}\,.$$

Hence the complex torus we have defined is

$$Y := V_1/\Lambda_1.$$

At the same time, if

$$V_2 := \mathbb{L}\{e_{r+1}, \ldots, e_g\} \cong \mathbb{C}^{g-r},$$

since the matrix ω has the form (1), the elements $\lambda_{2r+1}, \ldots, \lambda_{2g}$ have non-zero components only with respect to $\{e_{r+1}, \ldots, e_g\}$, so they are elements in V_2.

Anyway we shall call $\tilde{\lambda}_{2r+1}, \ldots, \tilde{\lambda}_{2g}$ those elements in V_2, but we shall keep calling $\lambda_{2r+1}, \ldots, \lambda_{2g}$ the same elements in $\mathbb{C}^g$, that is to say with the first r components all equal to 0. Now, asserting that ω_2 is the period matrix of a complex torus means that.

$$\Lambda_2 := \langle \tilde{\lambda}_{2r+1}, \ldots, \tilde{\lambda}_{2g} \rangle_{\mathbb{Z}} .$$

is a lattice in V_2 and that

$$Z := V_2 / \Lambda_2$$

is a $(g-r)$-dimensional complex torus.

1.2 – *A natural embeddimg.*

Let $J(C)$ be a Jacobian for a curve C and X be an abelian variety with $\dim(X) = \rho$, $0 < \rho < g$, such that an embedding $F : X \hookrightarrow J(C)$ exists. We want to see if it is possible to describe the map F "in coordinates" in order to point out its analytic and rational representation.

More precisely it is easy to prove the following:

PROPOSITION 2.1. – *Let $F : X \hookrightarrow J(C)$ be an embedding with $X \cong \mathbb{C}^p / \Gamma$. Then, unless we use a holomorphic coordinate change with non-zero determinat in $\mathbb{C}^g$, F must necessarily be describable in one of the following ways:*

$$F([z_1, \ldots, z_p]_\Gamma) := [z_1, \ldots, z_p, \underbrace{0, \ldots, 0}_{g-p}]_\Lambda \tag{1}$$

or

$$F([z_1, \ldots, z_p]_\Gamma) := [\underbrace{0, \ldots, 0}_{g-p}, z_1, \ldots, z_p]_\Lambda . \tag{2}$$

□

REMARK 2.2. – Really, we should have described F this as follows:

$$F([z_1, \ldots, z_p]) := ([e_1, \ldots, e_g)]_\Lambda ,$$

where $g - p$ coefficients are zero and the other p coefficients coincide with $z_1, \ldots, z_p$, in this order. Anyway, the two forms (1) and (2) are sufficient for our goal.

Now we define an application $i_Z : \mathbb{C}^{g-r} \to \mathbb{C}^g$ such that:

$$i_Z(z_1, \ldots, z_{g-r}) := (0, \ldots, 0, z_1, \ldots, z_{g-r}),$$

and immediately we notice that $i_Z : \mathbb{C}^{g-r} \to \mathbb{C}^g$ defines a complex matrix

$\tilde{a} \in M_{g,g-r}(\mathbb{C})$:

$$\tilde{a} = \begin{pmatrix} 0 \\ \mathrm{Id}_{g-r} \end{pmatrix}$$

in relation to the standard bases.

$i_Z(\Lambda_2) \subset \Lambda$, so this matrix is the analytic representation $\rho_a(i_Z)$ of the morphism $i_Z : Z \hookrightarrow J(C)$.

Besides, if we call:

$$\tilde{A} = \begin{pmatrix} 0 \\ \mathrm{Id}_{2(g-r)} \end{pmatrix} \in M_{2g,2(g-r)}(\mathbb{Z}),$$

it is straighforward to understand that the equality

$$\tilde{a} \cdot \omega_2 = \omega \cdot \tilde{A} \tag{3}$$

holds. Consequently, the two matrices $\tilde{a}$ and $\tilde{A}$ define a morphism $Z \to J(C)$, which by construction will be the above-defined morphism i_Z and the matrix $\tilde{A}$ will be the matrix of the rational representation $\rho_r(i_Z)$ of that morphism. On the other hand, it is easy to see that i_Z is injective.

1.3 – *Polarizations in* $J(C)$.

Since $J(C)$ is principally polarized, there exists a non-singular, unimodular, skew-symmetric matrix $H \in M_{2g,2g}(\mathbb{Z})$ such that for $C := H^{-1} \in M_{2g,2g}(\mathbb{Z})$ the Riemann conditions:

$$\omega \cdot C \cdot \omega^T = 0$$
$$\omega \cdot C \cdot \overline{\omega}^T > 0$$

both hold. The skew-symmetric matrix C is called a principal matrix for ω

Hence, let $H = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}$, whith $H_4 \in M_{2(g-r),2(g-r)}(\mathbb{Z})$ be the non-singular, unimodular, skew,symmetric matrix associated to the principal polarization Θ in $J(C)$ and let $C(C := H^{-1})$ be a principal matrix for ω.

$C \in M_{2g,2g}(\mathbb{Z})$, and it can always be written in the form:

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \text{ with } C_1 \in M_{2r,2r}(\mathbb{Z})$$

and $C_3 \in M_{2(g-r),2r}(\mathbb{Z})$.

In [1, theorem 1], Albert shows that C_1 is a principal matrix for the Riemann matrix ω_1, consequently, $\det(C_1) \neq 0$ and $C_1^{-1} \in M_{2r,2r}(\mathbb{Q})$. Then we can assert the following:

PROPOSITION 3.1. – *Let $J(C)$ be a g-dimesional Jacobian, whose period matrix ω is an impure Riemann matrix having the form:* $\omega = \begin{pmatrix} \omega_1 & 0 \\ \omega_3 & \omega_2 \end{pmatrix}$, *where $\omega_1 \in M_{r,2r}(\mathbb{C})$. Let H be a polarization on $J(C)$, where $H_4 \in M_{2(g-r),2(g-r)}(\mathbb{Z})$, and let $C := H^{-1}$ be a principal matrix for $\omega \in M_{2g,2g}$. Then if $i : Z \to J(C)$ is the immersion defined in* (3), we have that:

$$i_Z^*(\Theta) = H_4. \tag{4}$$

□

2. – Prym-Tyurin varieties.

2.1 – *Some definitions.*

Before deepending the analysis of principally polarized abelian varieties, it will be useful to describe the set of abelian subvarieties of an abelian variety X in terms of the endomorphism algebra $\mathrm{End}_a(X)$. In order to do that, Lange-Birkenhake (e.g. [8], page 125), given a polarization on X, associate to every abelian subvariety Z of X an endomorphism N_Z, the norm-endomorphism, and a symmetric idempotent ε_Z and then prove that the symmetric idempotents are in one-to-one correspondence to the abelian subvarieties of X. This leads to a criterions for an endomorphism to be a norm-endomorphism.

For a more comfortable reading, here we enunciate the definition and the main proporties of the norm-endomorphisms. However, although such endomorphisms can be defined for abelian subvarieties of any abelian variety endowed with any polarization, also in order to fix our notations, here we prefer to consider only the case in which X is a Jacobian variety $J(C)$ and the polarization on $J(C)$ is the canonical principal polarization Θ. Finally we recall that, given any abelian variety Z, to fix a polarization L on Z induces an isogeny $\phi_L : Z \to \hat{Z}$ depending only on the class of L in $NS(Z)$. The exponent $e(L)$ of the finite group $\mathrm{Ker}(\phi_L)$ is called the exponent of the polarization L. Then, there exists a unique isogeny $\psi_L : \hat{Z} \to Z$ such that $\psi_L \cdot \phi_L = e(L)_Z$ and $\phi_L \cdot \psi_L = e(L)_{\hat{Z}}$, the multiplications by the integer $e(L)$ on Z.

Then, consider $(J(C), \Theta)$, and let Z be an abelian subvariety of $J(C)$ with canonical embedding $i_Z : Z \hookrightarrow J(C)$. Define the exponent of the abelian subvariety Z to be the exponent $e(i^*(\Theta))$ of the induced polarization on Z and write $e(Z) := e(i^*(\Theta))$. We have the isogeny:

$$\psi_{i^*(\Theta)} = e(Y)\phi^{-1}_{i^*(\Theta)} : \hat{Z} \to Z.$$

With this notation, we have:

DEFINITION 1.1. – *We define the* ***norm-endomorphism*** *of $J(C)$ associated to Z (with respect to Θ) by:*

$$N_Z = i \cdot \psi_{i^*(\Theta)} \cdot \hat{i} \cdot \phi_\Theta \in \mathrm{End}(J(C))$$

i.e. as the composition:

$$J(C) \xrightarrow{\Phi_\Theta} \widehat{J(C)} \xrightarrow{\hat{i}_Z} \hat{Z} \xrightarrow{\Psi_{i_Z^*(\Theta)}} Z \overset{i_Z}{\hookrightarrow} J(C)$$

We immediately have:

LEMMA 1.2 (see [8], lemma 5.3.1). – *For any abelian subvariety Z of $J(C)$:*

$$N'_Z = N_Z \qquad \text{and} \qquad N_Z^2 = e(Z)N_Z, \tag{1}$$

where ' denotes the Rosati involution with respect to the polarization Θ. □

Moreover, we can prove (see [8], corollary 5.3.3) that the relations in (1) completely characterize the norm-endomorphisms.

In other words, we obtain the following criterion for an endomorphism to be a norm-endomorphism.

PROPOSITION 1.3. – *For $f \in End(J(C))$ and $Z := f(J(C))$ the following statements are equivalent:*

i) $f = N_Z$;
ii) $f' = f$ *and* $f^2 = e(Z)f$. □

We end this paragraph with an observation pointing out how the norm-endomorphisms depend on the polarization Θ on $J(C)$.

Back to the previous notation, if Z is the abelian subvariety embedded in $J(C)$ associated to the Riemann matrix:

$$\omega = \begin{pmatrix} \omega_1 & 0 \\ \omega_3 & \omega_2 \end{pmatrix},$$

that is $Z = \mathbb{C}^{g-r}/\langle \omega_2 \rangle_{\mathbb{Z}}$ and $\dim(Z) = g - r$, and if $H \in m_{2g,2g}(\mathbb{Z})$ is the nonsingular, unimodular skew-symmetric matrix associated to the polarization Θ on $J(C)$, H can be written in the form

$$H = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix},$$

which $H_1 \in M_{2r,2r}(\mathbb{Z})$ and $H_4 \in M_{2(g-r),2(g-r)}(\mathbb{Z})$.

As usual, we set $C := H^{-1} \in M_{2r,2r}(\mathbb{Z})$ and we write:

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix},$$

with $C_1 \in M_{2r,2r}(\mathbb{Z})$ and $C_4 \in M_{2(g-r),2(g-r)}(\mathbb{Z})$. It is easy to prove that:

PROPOSITION 1.4. – *The rational representation of the norm-endomorphism N_Z relative to the variety Z and to the Jacobian $J(C)$ defined by the impure matrix ω can be written, with respect to suitable bases, in the form:*

$$\rho_r(N_Z) = e(Z)\begin{pmatrix} 0 & 0 \\ C_3C_1^{-1} & I_{2(g-r)} \end{pmatrix} = e(Z)\begin{pmatrix} 0 & 0 \\ H_4^{-1}H_2^T & I_{2(g-r)} \end{pmatrix}.$$

□

REMARK 1.5. – It immediately follows from the previous proposition that the correspondence σ associated to N_Z in the isomorphism between $\mathrm{Corr}(C)$ and $\mathrm{End}(J(C))$ is necessarily a special correspondence.

2.2 – *The Beauville's criterion.*

In this paragraph we analyze the principally polarized abelian subvarieties of a Jacobian $J(C)$. First we remember that in [13], Welters proved that every principally polarized abelian variety Z is a so-called Prym-Tyurin variety, that is to say that given a principally polarized abelian variety (Z, Ξ) there exists a smooth and projective curve C with a Jacobian $(J(C), \Theta)$ such that Z is an abelian subvariety of $J(C)$ with:

$$j_Z^*(\Theta) = e\Xi \tag{2}$$

for some integer e. Necessarily e is the exponent of Z in $J(C)$. We also say that Z is a Prym-Tyurin variety for the curve C and that e is the exponent of the Prym-Tyurin variety Z.

The problem we have to face now is to state a criterion to understand if an abelian variety Z of a Jacobian $J(C)$ can be principally polarized. The decisive step in that direction is given by the following:

THEOREM 2.1 (Beauville [2], page 607). – *Let u be an endomorphism of a Jacobian $J(C)$ and p a positive integer. Assume*

i) *u is symmetric;*
ii) *$u^2 = pu$;*
iii) *the kernel of u is connected.*

Then the principal polarization on $J(C)$ induces p times a principal polarization of the image of u. (In particular, the image of u is principally polarized).

□

Keeping what we stated in lemma 1.2 in mind, this theorem can be reformulated this way:

THEOREM 2.1′. – *Let Z be an abelian subvariety of $J(C)$. If $Ker(N_Z)$ is connected, then Z is a Prym-Tyurin variety.*

PROOF. – We want to give a "more constructive" proof of this theorem, involving the unimodular and skew-symmetric matrix

$$H = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \in M_{2g,2g}(\mathbb{Z}),$$

which is associated to the polarization Θ on $J(C)$ (with

$$C := \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = H^{-1}).$$

To do that, we need the following:

LEMMA 2.2. – *$Ker(N_Z)$ is connected if and only if the rational representation*

$$\rho_r(N_Z) : H_1(C, \mathbb{Z}) \to H_1(\mathbb{Z}, \mathbb{Z})$$

is surjective.

PROOF. – Since $N_Z : J(C) \to Z$ is a surjective homomorphism, we can use the Stein factorization of N_Z:

$$\begin{array}{ccc} J(C) & \xrightarrow{N_Z} & Z \\ g \searrow & & \nearrow h \\ & J(C)/(\mathrm{Ker}\,(N_Z))_\circ & \end{array} \tag{5}$$

where $h : J(C)/(\mathrm{Ker}(N_Z))_\circ \to Z$ is an isogeny and the fibers of g are connected.

So we obtain a commutative diagram:

$$\begin{array}{ccc} \pi_1(C, c_0) & \xrightarrow{\rho_r(N_Z)} & \pi_1(Z, z_0) \\ \searrow & & \nearrow h_* \\ & \pi_1(X, x_0) & \end{array} \tag{6}$$

where we indicated $X := J(C)/(\mathrm{Ker}(N_Z))_\circ$, where $(\mathrm{Ker}(N_Z))_\circ$ is a finite-indexed subtorus of $J(C)$ in $\mathrm{Ker}(N_Z)$.

Now being the diagram (6) commutative and $\rho_r(N_Z)$ surjective, h_* is surjective too.

On the other hand, $h : X \to Z$ is a covering of Z and we can suppose that $h(x_0) = z_0$.

Then it can be claimed that h_* is injective too; that is to say, there exists an isomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Z, z_0)$. Consequently, the fiber of h in the diagram (5) is only constituted by a point, therefore h is an isomorphism too. Since g has connected fibers, $\mathrm{Ker}(N_Z)$ is connected.

Let us now show the opposite implication. Now, supposing that $\mathrm{Ker}(N_Z)$ is connected, we have an exact sequence of complex tori:

$$0 \to \mathrm{Ker}(N_Z) \to J(C) \to Z \to 0.$$

If we call Λ_1 the lattice of the torus $\mathrm{Ker}(N_Z)$, by applying the snake's lemma we find that the induced sequence of lattices is exact too:

$$0 \to \Lambda_1 \to H_1(C, \mathbb{Z}) \to H_1(Z, \mathbb{Z}) \to 0.$$

Hence $\phi_r(n_Z)$ is injective. $\square$

Let us go back to the proof of the theorem. Considering N_Z as an application $N_Z : J(C) \to Z$, by proposition 1.4 the matrix representing it with respect to suitable bases is:

$$M(\rho_r(N_Z)) = (eC_3C_1^{-1}, e\,\mathrm{Id}_{2(g-r)}) \in M_{2(g-r),2g}(\mathbb{Z}),$$

with $eC_3C_1^{-1} \in M_{2(g-r),2r}(\mathbb{Z})$, $C_1^{-1} \in M_{2r,2r}(\mathbb{Q})$. Since $\rho_r(N_Z)$ is surjective, the application $\mathbb{Z}^{2r} \to \mathbb{Z}^{2(g-r)}$ induced by the matrix $eC_3C_1^{-1}$ has to be surjective too, therefore (e.g. [7], page 419) one can determinate a unimodular minor of order $2(g-r)$ in $eC_3C_1^{-1} \in M_{2(g-r),2r}(\mathbb{Z})$, since by construction obviously the inequality $g - r < r$ must hold. We also recall that C_1 is a skew-symmetric matrix in $M_{2r,2r}(\mathbb{Z})$, whereas $C_3 \in M_{2(g-r),2r}(\mathbb{Z})$.

For our aim, by the skew-symmetry of C_1, to the canonical form:

$$C_1 = \begin{pmatrix} 0 & & & x_1 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & x_r \\ -x_1 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & -x_r & & & 0 \end{pmatrix}. \tag{7}$$

On the other hand, by Schur's formula,

$$\det(C_1) \cdot \det(C_4 - C_3C_1^{-1}C_3^T) = 1$$

holds, and with a straightforward calculation immediately implies

$$H_4 = (C_4 - C_3C_1^{-1}C_3^T)^{-1} \in M_{2(g-r),2(g-r)}(\mathbb{Z}),$$

so we have that: $\det(H_4) = \det(C_1)$. We know that $\det(H_4) = d_1^2 \cdot \ldots \cdot d_{g-r-1}^2 \cdot d_{g-r}^2$, with $d_i | d_{i+1}$, $i = 1, \ldots, g-r-1$, and (see [8], page 368), $d_{g-r} = e$, where $e = e(Z)$ is the exponent of the variety Z in $J(C)$. Consequently, e^2 divides $\det(C_1)$. By (7) we obtain:

$$x_1^2 \cdot \ldots x_r^2 = d_1^2 \cdot d_{g-r-1}^2 \cdot e^2.$$

Since by hypothesis $eC_3C_1^{-1}$ has a unimodular minor of range $2(g-r)$ it is easy to realize that we have:

$$d_1 = \ldots = d_{g-r} = e,$$

that is to say:

$$H_4 = \begin{pmatrix} 0 & & & & e & & & \\ & \ddots & & & & \ddots & & \\ & & \ddots & & & & \ddots & \\ & & & 0 & & & & e \\ -e & & & & 0 & & & \\ & \ddots & & & & \ddots & & \\ & & \ddots & & & & \ddots & \\ & & & -e & & & & 0 \end{pmatrix}. \tag{8}$$

Since we already know from proposition 1.3.1 that $i_Z^*(\Theta) = H_4$, the proof is complete. $\square$

This theorem can easily be inverted. In fact, let Z be a principally polarized abelian variety. Then (Z, Ξ) defines a Prym-Tyurin variety for some curve C. Since Ξ defines a principal polarization we can identify Z with its dual variety via the isomorphism ϕ_Ξ. Then $\psi_Z = \mathrm{Id}_Z$ and if $i_Z : Z \to J(C)$ is the natural embedding, the equation defining the norm-endomorphism N_Z of Z reads:

$$N_Z = i_Z \hat{i}_Z. \tag{9}$$

Now, let $\tilde{Y}$ be the abelian subvariety of $J(C)$ complementary to Z, i.e.:

$$\tilde{Y} = \mathrm{Im}\,(e(Z)\mathrm{Id}_{J(C)} - N_Z). \tag{10}$$

Then, we have $\tilde{Y} = \mathrm{Im}\,(N_{\tilde{Y}}) \subseteq (\ker(N_Z))_\circ$, since $N_Z \cdot N_{\tilde{Y}} = 0$. As $\tilde{Y}$ and $(\ker(N_Z))_\circ$ are abelian subvarieties of the same dimension, $\tilde{Y} = (\ker(N_Z))_\circ$. Moreover, $(\ker(N_Z))_\circ = (\ker(\hat{i}_Z))_\circ$ since $N_Z = i\psi_Z\hat{i}$, i_Z ia a closed immersion and ψ_j is an isogeny. In order to show that $\ker(\hat{i}_Z)$ is connected, consider the exact sequence:

$$0 \to Z \hookrightarrow J(C) \to J(C)/Z \to 0.$$

But the dual sequence is exact too:

$$0 \to (\widehat{J(C)}/Z) \to \widehat{J(C)} \xrightarrow{\hat{i}_Z} \hat{Z} \to 0$$

so $\ker(\hat{i}_Z) \simeq (\widehat{J(C)}/Z)$. In particular, $\ker(\hat{i}_Z)$ is connected. Moreover, by (9), $\ker(N_Z)$ is connected too.

Finally we can conclude:

THEOREM. – *Let Z be an abelian subvariety of a Jacobian $J(C)$. Z is a Prym-Tyurin variety if and only if* $\ker(N_Z)$ *is connected, that is if and only if the following sequence is exact:*

$$0 \to \tilde{Y} \to J(C) \to Z \to 0,$$

where $\tilde{Y}$ is the abelian subvariety of $J(C)$ complementary to Z. □

2.3 – *Kanev's condition.*

First of all, let us recall that our definition of a Prym-Tyurin variety coincides with the one given by Bloch-Murre ([3]): we suppose that $\sigma \in \mathrm{End}(J(C))$ is symmetric with respect to the Rosati involution of $(J(C), \Theta)$ and verifying the equation:

$$\sigma^2 + (m-2)\,\sigma - (m-1) = 0 \tag{11}$$

for $m \in \mathbb{Z}_+$. Then the abelian subvariety $Z = \mathrm{Im}\,(\sigma - 1)$ of $J(C)$ is a generalized Prym variety in the sense of Bloch-Murre if the induced polarization is multiple of some principal polarization of Z.

Obviously the two definitions are equivalent: if Z is a Prym-Tyurin with exponent e for the curve C, then the endomorphism $\sigma = 1 - N_Z$ satisfies $Z = \mathrm{Im}\,(1-\sigma)$ and the equation (11) with $m = e$.

So the problem is to characterize the principally polarized abelian subvarieties of $J(C)$ by means of particular special correspondences on C. The only known result in this direction are those achieved by Kanev in 1987, particularly the following:

THEOREM 3.1 (Kanev [6]). – *Let Z be an abelian subvariety of exponent e of the Jacobian $J(C)$. Suppose there is an effective fixed point free correspondence L on $C \times C$ of bidegree (d, d) with $\gamma_L = 1_{J(C)} - N_Z$. Then Z is a Prym-Tyurin variety for the curve C. Moreover, there are theta divisors on $J(C)$ and Ξ on Z such that $i_Z^*(\Theta) = e\Xi$.* □

It is better to recall the following:

DEFINITION 3.2 (e.g. [8], page 400). – *A correspondence Γ between the points of a curve C is said to be* ***fixed point free*** *if*

$$\Gamma \cdot \Delta_C = 0,$$

where Δ_C is the diagonal of $C \times C$.

The question proposed in Izadi in [5] whether every p.p.a.v. is a Prym-Tyurin variety for a symmetric fixed point free correspondence is very natural. The following example intends to be a contribution for a hypothetic negative answer to this question and confirms that Kanev's conditions is only sufficient.

EXAMPLE 3.3. – Let us suppose that Z is a p-dimensional Prym-Tyurin subvariety of a g-dimensional Jacobian $J(C)$, with $e(Z) = q$. Let us also suppose that there exists a correspondence L, effective and fixed point free on $C \times C$ of bidegree (d, d), with $\gamma_L = 1_{J(C)} - N_Z$. Then $g = pq + d$ (e.g. [8], prop. 11.5.2).

Let $f : C \to C'$ be a duble (and branched in two points) covering of smooth and projective curves with $g(C) = g$ and $g(C') = g'$. Let Z be the abelian subvariety of the Jacobian $J(C)$, complementary to the abelian subvariety $Y = \mathrm{Im}(f^*)$, and let $i : C \to C$ be the involution corresponding to the double convering f.

It is clear that (see [8], theorem 12.3.3) Z is a Prym-(Tyurin) variety of the kind: $i_Z^*(\Theta) = 2\Xi$ and $\dim(Y) = \dim(Z) = g'$

By Hurwitz's formula, we have that $g' = g/2$, so this implies (see also [8], prop. 11.5.2): $g = 2 \cdot (g/2) + d$, and consequently $d = 0$. Hence for such a Z no fixed point free correspondence L on $C \times C$ representing it may exist, i.e. this is the only way of constructing the p.p.a.v. Z as a classical Prym variety.

In conclusion, in order to answer Izadi's question, one should in addition prove that there is no other Prym-Tyurin construction of Z which can be obtained from another correspondence which is fixed point free and defines on Z m

times the theta divisor, $m > 2$. At the present time, such a construction is not known yet.

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Dipartimento di Matematica,
Piazza di Porta S. Donato 5, 40127, Bologna, Italia
e-mail: parigi@dm.unibo.it

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