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On the Existence of Solutions for Abstract Nonlinear Operator Equations.

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Sunto. – *Forniamo una teoria duale e risultati di esistenza per un'equazione operatore*

$$\nabla T(x) = \nabla N(x),$$

dove T non è necessariamente un operatore monotono. Usiamo la versione astratta del cosiddetto metodo variazionale duale. La soluzione è ottenuta come un limite di una sequenza minimizzante la cui esistenza e convergenza è provata.

Summary. – *We provide a duality theory and existence results for a operator equation*

$$\nabla T(x) = \nabla N(x),$$

where T is not necessarily a monotone operator. We use the abstract version of the so called dual variational method. The solution is obtained as a limit of a minimizng sequence whose existence and convergence is proved.

1. – Problem formulation and assumptions.

We consider the problem of existence of solutions for the following operator equation

$$\nabla T(x) = \nabla N(x), \tag{1.1}$$

where T is not necessarily a monotone mapping. In such a case neither classical variational methods nor topological ones work. Thus we have to come up with a new duality and a new variational method in order to tackle Eq. (1.1) under the following assumptions:

A1 V is reflexive, separable Banach spaces compactly and densely embedded into another Banach space Z. Operator $\nabla T : V \to V^*$ is radially continuous, potential and coercive.

A2 There exists a is radially continuous potential mapping $\nabla S : V \to V^*$, bounded on bounded sets, having a convex lower semicontinuous potential and such that $T + S : V \to \mathcal{R}$ is convex and lower semicontinuous and $\nabla(T + S)$ is radially continuous, strictly monotone and coercive.

A3 $\nabla N : Z \to Z^*$ is radially continuous potential operator whose potential $N : Z \to \mathcal{R}$ is convex lower semicontinuous and bounded on bounded sets.

Let $i : V \to Z$ denotes the embedding of V in Z. We say following [1] that $x \in V$ satisfies equation (1.1) if

$$\nabla T(x) = i^* \nabla N(ix), \tag{1.2}$$

Observe that $\nabla(T + S) : V \to V^*$, $\nabla N : Z \to Z^*$ are demicontinuous [4]. We shall assume that operator i has the following property (analogous to the Poincaré inequality):

A4 $\|ix\|_Z \leq \|x\|_V$ for all $x \in V$

It now follows that i and i^* [6] are continuous. As an embedding operator i, and in a consequence i^*, is invertible.

The study of problem (1.2) is motivated by its applications to a wide class of nonlinear Dirichlet problems governed by partial differential equation in which the differential operator may not be monotone. We provide a suitable example in Section 6. Still the theory we develop applies for problems with various growth conditions. Therefore different types of nonlinearities and nonlinear partial differential operators may be taken into account provided one can prove that certain set is invariant with respect to $(\nabla(T + S))^{-1}$. In case of not necessarily monotone neither the direct nor the dual variational methods can be applied. Therefore the known approaches, e.g. described in [1], [7], [8] may not be applicable.

As far as abstract equations are concerned a similar problem as ours have been considered in [1], where it was assumed that the operator T is monotone and ∇T is a duality mapping. Now we use the method that has been derived in [5] for abstract semilinear Dirichlet problems and which enables us to get rid of monotonicity of T. Instead a certain monotone operator S is assumed to exist. It must be stressed that J is not convex-concave as it is common for a dual variational approach that was derived in [9], [10] and which we modify so that it can be applied for abstract problems. Thus a new duality theory have been constructed and a new dual functional depending on two variables must have been introduced. In our considerations we use the idea of Leray-Schauder linearization trick. The solution is obtained as a critical point of a certain type of the action functional. What is important here, it is the fact that the solution is approximated by a minimizing sequence. It may provide the basis for constructing a numerical approximation in future.

The study of problem (6.1) which we show in Section 6 as an example of our abstract results is interesting in itself. This problem corresponds to such a partial differential equation in which there appear two operators of elliptic type.

Such a problem may not be classical since it may not be tackled by some known methods. Therefore our approach allows us to consider (6.1) and similar problems perhaps with varied growth conditions. Moreover our approach allows for considering problems with non-monotone operator which become monotone if one ads a certain term. Therefore we believe that our approach may contribute somehow to the applications of partial differential equations.

2. – An Equivalent Problem.

We observe that (1.2) corresponds the Euler-Lagrange equation for the functional $J : V \to \mathcal{R}$ given by the formula

$$J(x) = T(x) - N(ix) \tag{2.1}$$

which due to the assumption **A2** may equivalently be written in the following form

$$J(x) = (T(x) + S(x)) - (S(x) + N(ix))$$

We shall seek the solution to (1.2) as a triple $(x, p, q) \in V \times Z^* \times Z^*$ such that

$$\begin{aligned} &\nabla T(x) + \nabla S(x) = i^* p, \\ &\nabla S(x) = i^* q, \\ &p - q = \nabla N(ix). \end{aligned} \tag{2.2}$$

The above system we will obtain by duality relations, i.e. relating critical values and critical points to primal action functional $J : V \to \mathcal{R}$ and a dual action functional $J_D : Z^* \times Z^* \to \mathcal{R}$ given by

$$J_D(p, q) = N^*(p - q) + S^*(i^* q) - (T + S)^*(i^* p). \tag{2.3}$$

Here N^*, T^* denote Fenchel-Young transformations of convex functional N and T, while i^* denotes the adjoint operator.

In order to describe duality theory we shall construct certain subsets of spaces V and $Z^* \times Z^*$ on which we shall investigate the primal and the dual functional. In some cases the geometry of this set will be known, i.e. it is the convex set in our case, see the last section where we provide an example. But with different growth conditions this set will be defined in some other way and may not posses such nice properties. We observe that by **A2** it follows that $\nabla T + \nabla S$ is invertible and its inverse denoted by $(\nabla T + \nabla S)^{-1}: V^* \to V$ is bounded on bounded sets, demicontinuous and strictly monotone. Hence for any $f \in V^*$ equation $\nabla T(x) + \nabla S = f$ has exactly one solution in V, [4]. Thus the following assumption makes sense

A5 There exists a nonempty, weakly compact subset $X \subset V$ such that

$$X \subset (\nabla T + \nabla S)^{-1}(\nabla S(X) + i^*\nabla N(iX))$$

It is obvious that for all $x \in X$ the relation

$$(\nabla T + \nabla S)(\widetilde{x}) = \nabla S(x) + \nabla N(ix). \tag{2.4}$$

implies $\widetilde{x} \in X$. The existence of a nonempty set X must be checked in any case the theory is applied and is crucial in what follows.

The dual functional J_D will be considered on a set X^d which comprises all $(p, q) \in Z^* \times Z^*$ for which there exist $x, \widetilde{x} \in X$ satisfying relation (2.4) and such that

$$\nabla T(\widetilde{x}) + \nabla S(\widetilde{x}) = i^*p$$
$$\nabla S(x) = i^*q.$$

Since the set X is assumed to be nonempty it follows that X^d is also nonempty. It is easily seen that for any $(p, q) \in X^d$ there exists exactly one $x \in X$ and for any $x \in X$ there exists exactly one $(p, q) \in X^d$. This follows since for any $x \in X$ there exists exactly one $\widetilde{x} \in X$ such that $x, \widetilde{x} \in X$ satisfy relation (2.4).

From now on the functional J will be considered on a set X and its dual J_D on a set X^d. It should be noticed that these sets are not subspaces of the respective spaces V and $Z^* \times Z^*$ which makes some standard calculations rather complicated.

3. – Duality results.

Now we construct the duality theory which allows us to obtain relations between critical points and critical values of both action functional. In order to avoid calculation of a Fenchel-Young transform on a nonlinear subset X we will define a kind of perturbation of a functional J. Let $x \in X$. We define a perturbation $J_x : Z \times V \to \mathcal{R}$ of functional J by the formula

$$J_x(v, w) = N(ix + v) + S(x + w) - (T + S)(x)$$

Now J_x is convex and defined on the whole space. Hence a kind of Fenchel-Young transformation of J_x, namely $J_x^{\#} : X^d \to \mathcal{R}$, can be defined with respect to the duality pairing between $Z \times V$ and $Z^* \times V^*$. We put for $(p, q) \in X^d$

$$J_x^{\#}(p, q) = \sup_{v \in Z} \Big\{ \langle p - q, v \rangle_{Z^*, Z} - N(ix + v) \Big\} + (T + S)(x)$$
$$+ \sup_{w \in V} \Big\{ \langle i^*q, w \rangle_{V^*, V} - S(x + w) \Big\}.$$

The above formula is actually a Fenchel-Young transform but with domain restricted to the set X^d. Thus a different symbol is used. Hence we obtain [2]

$$\begin{aligned} J_x^{\#}(p,q) &= N^*(p-q) + (T+S)(x) - \langle p-q, ix\rangle_{Z^*,Z} \\ &+ S^*(i^*q) - \langle i^*q, x\rangle_{V^*,V} \end{aligned}$$

or by a direct calculation

$$J_x^{\#}(p,q) = N^*(p-q) + S^*(i^*q) - \langle p, ix\rangle_{Z^*,Z} + (T+S)(x).$$

In the proof of the duality principle we will make use of the following lemmas.

LEMMA 3.1. – *For any* $(p,q) \in X^d$

$$\inf_{x\in X} J_x^{\#}(p,q) = J_D(p,q).$$

PROOF. – Fix $(p,q) \in X^d$. By Fenchel-Young inequality we obtain

$$\sup_{x\in X}\left\{\langle i^*p, x\rangle_{V^*,V} - (T+S)(x)\right\} \le (T+S)^*(i^*p). \tag{3.1}$$

By definition of X^d we conclude that for a given $(p,q) \in X^d$ there exists $x_p, \widetilde{x}_p \in X$, related by (2.4) and satisfying relations$(\nabla T + \nabla S)\big(\widetilde{x}_p\big) = i^*p$ and $\nabla S\big(x_p\big) = i^*q$. The former relation by convexity of $T+S$ means that [2]

$$(T+S)\big(\widetilde{x}_p\big) + (T+S)^*(i^*p) = \big\langle i^*p, \widetilde{x}_p\big\rangle_{V^*,V}.$$

Hence there is actually equality in (3.1) for $\widetilde{x}_p$ and thus the assertion follows. □

LEMMA 3.2. – *For any* $x \in X$

$$\inf_{(p,q)\in X^d} J_x^{\#}(p,q) = J(x).$$

PROOF. – Fix $x \in X$. By Fenchel-Young inequality we obtain

$$\sup_{(p,q)\in X^d}\Big\{\langle p-q, ix\rangle_{Z^*,Z} - N^*(p-q) + \langle i^*q, x\rangle_{V^*,V} - S^*(i^*q)\Big\} \le N(ix) + S(x) \tag{3.2}$$

For an x considered there exists $(p_x, q_x) \in X^d$ such that

$$\begin{aligned} i^*q_x &= \nabla S(x) \\ p_x - q_x &= \nabla N(ix). \end{aligned} \tag{3.3}$$

Indeed, it suffices to put $i^*p_x = \nabla T(\widetilde{x})$, $i^*q_x = \nabla S(x)$ where $\widetilde{x}, x \in X$ satisfy (2.4) and later use (2.4). By (3.3) and convexity we get

$$N(ix) + N^*(p_x - q_x) + S(x) + S^*(i^*q_x) = \langle i^*q_x, x\rangle_{V^*,V} + \langle p_x - q_x, ix\rangle_{Z^*,Z}.$$

Hence we have actually equality in (3.2) for (p_x, q_x). □

We may now prove the duality principle

THEOREM 3.3. –

$$\inf_{x\in X} J(x) = \inf_{(p,q)\in X^d} J_D(p,q).$$

PROOF. – By lemmas 3.1 and 3.2 we obtain

$$\begin{aligned}\inf_{x\in X} J(x) &= \inf_{x\in X} \inf_{(p,q)\in X^d} J_x^{\#}(p,q) = \inf_{(p,q)\in X^d} \inf_{x\in X} J_x^{\#}(p,q)\\ &= \inf_{(p,q)\in X^d} J_D(p,q) = \inf_{(p,q)\in X^d} J_D(p,q).\end{aligned}$$

□

4. – Necessary conditions.

We shall use the duality results to derive necessary conditions for the existence of solutions to equation 1.2.

THEOREM 4.1. – *Let there exists* $\overline{x}\in X$ *such that* $-\infty < J(\overline{x}) = \inf_{x\in X} J(x) < \infty$. *Then there exists* $(\overline{p},\overline{q})\in X^d$ *such that*

$$\inf_{(p,q)\in X^d} J_D(p,q) = J_D(\overline{p},\overline{q}) = J(\overline{x}) = \inf_{x\in X} J(x).$$

Moreover

$$i^*\overline{p} = (\nabla T + \nabla S)(\overline{x}), \tag{4.1}$$

$$i^*\overline{q} = \nabla S(\overline{x}), \tag{4.2}$$

$$\overline{p} - \overline{q} = \nabla N(i\overline{x}). \tag{4.3}$$

PROOF. – Relations (4.2) and (4.3) are obtained in a similar manner as relations (3.3) in the proof of Lemma 3.2.

By a direct calculation we obtain

$$\begin{aligned}&-J(\overline{x}) = -(T+S)(\overline{x}) - S(\overline{x}) + N(i\overline{x})\\ &= \langle \overline{p}-\overline{q}, i\overline{x}\rangle_{Z^*,Z} - N^*(\overline{p}-\overline{q}) + \langle i^*\overline{q},\overline{x}\rangle_{V^*,V} - S^*(i^*\overline{q}) - (T+S)(\overline{x})\\ &= \langle i^*\overline{p},\overline{x}\rangle_{V^*,V} - (T+S)(\overline{x}) - N^*(\overline{p}-\overline{q}) - S^*(i^*\overline{q})\\ &\le (S+T)^*(i^*\overline{p}) - N^*(\overline{p}-\overline{q}) - S^*(i^*\overline{q}) = -J_D(\overline{p}).\end{aligned}$$

Hence $J(\overline{x}) \ge J_D(\overline{p})$. By Theorem 3.3 it follows that $J(\overline{x}) \le J_D(\overline{p})$. In a consequence

$$J(\overline{x}) = J_D(\overline{p}).$$

It now follows by (4.2), (4.3) that

$$(T+S)(\overline{x}) + (T+S)^*(i^*\overline{p}) = \langle i^*\overline{p}, \overline{x}\rangle_{V^*,V}.$$

By the above relation and Gâteaux differentiability of T relation (4.1) follows. □

The similar result may be derived for minimizing sequences. The below theorem which may be viewed as an $\varepsilon-$variational principle will be used in the proof of the existence theorem. It differs from the above result in the second Hamilton's equation which is now presented in a $\varepsilon-$subdifferential form.

THEOREM 4.2. – *Let $\{x_j\}$, $x_j \in X, j \in N$ be a minimizing sequence for J. Then $\{p_j, q_j\}$ such that $(p_j, q_j) \in X^d$ and*

$$i^*q_j = \nabla S(x_j) \tag{4.4}$$

$$p_j - q_j = \nabla N(ix_j) \tag{4.5}$$

for $j \in N$ is a minimizing sequence for J_D and

$$\inf_{(p,q)\in X^d} J_D(p,q) = \inf_{j\in\mathcal{N}} J_D(p_j, q_j) = \inf_{x\in X} J(x) = \inf_{j\in\mathcal{N}} J(x_j). \tag{4.6}$$

Moreover for any $\varepsilon > 0$ there exists j_0 such that for $j \geq j_0$

$$0 \leq (T+S)(x_j) - \langle i^*p_j, x_j\rangle_{Z^*,Z} + (T+S)^*(i^*p_j) < \varepsilon. \tag{4.7}$$

PROOF. – Relations (4.4) and (4.5) are obtained in a similar manner as relations (3.3) in the proof of Lemma 3.2. We shall show that the sequence $\{(p_j, q_j)\}$ is minimizing for J_D. Reasoning as in the proof of Theorem 4.1 we obtain that for any $j \in \mathcal{N}$

$$J(x_j) \geq J_D(p_j). \tag{4.8}$$

Let us take arbitrary $\varepsilon > 0$. Since

$$-\infty < \inf_{j\in\mathcal{N}} J(x_j) = a < \infty,$$

it follows that there exists j_0 such that for $j \geq j_0$ we have $J(x_j) < a + \varepsilon$. From (4.8) it now follows that for $j \geq j_0$ we have $J_D(p_j, q_j) < a + \varepsilon$. By the latter fact and Theorem 3.3 it follows that $\inf_{j\in\mathcal{N}} J_D(p_j) = a$. Hence $\{p_j\}$ is a minimizing sequence for J_D and relation (4.6) follows.

We will show that (4.7) holds. For any $\varepsilon > 0$ there exists j_0 such that for $j \geq j_0$ we have

$$a \leq J_D(p_j) \leq J(x_j) < a + \varepsilon.$$

From this we obtain

$$0 \leq J(x_j) - J_D(p_j) \leq \varepsilon.$$

By definitions of J and J_D it follows that for j sufficiently large relation (4.7) holds. $\square$

5. – Existence of solutions.

We shall show that there exists an element $\overline{x} \in V$ such that together with a corresponding $(\overline{p}, \overline{q}) \in Z^* \times Z^*$ a triple $(\overline{x}, \overline{p}, \overline{q})$ satisfies system (2.2). We will make use of the ε–variational principle for minimizing sequences and the construction of sets X and X^d. It is not the existence of the minimizing sequences that is a really difficult task to be done in the below consideration but their convergence to the pair satisfying system (2.2). Here the duality theory plays again an important part. We assume that S and N have property (S), see [4].

THEOREM 5.1. – *There exists a triple* $(\overline{x}, \overline{p}, \overline{q}) \in V \times Z^* \times Z^*$ *satisfying the system*

$$i^* \overline{p} = \nabla (T + S)(\overline{x}), \tag{5.1}$$

$$i^* \overline{q} = \nabla S(\overline{x}), \tag{5.2}$$

$$\overline{p} - \overline{q} = \nabla N(i\overline{x}) \tag{5.3}$$

$$\inf_{(p,q)\in X^d} J_D(p,q) = J_D(\overline{p}, \overline{q}) = J(\overline{x}) = \inf_{x \in X} J(x). \tag{5.4}$$

PROOF. – We shall show that J is bounded from below on X. Since $T + S$ is a potential of a monotone and coercive operator and X is relatively weakly compact, there exists a constant c_3 [4], independent of x, such that for any $x \in X$

$$S(x) + T(x) \geq c_3.$$

Since ∇S is bounded on bounded sets it follows by convexity that $S(x) \leq c_2$ on X. By the same argument $N(ix) \leq c_1$ on X. So $J(x) = (T + S)(x) - S(x) - N(ix) \geq c_3 - c_2 - c_1$. Hence we may choose in X a minimizing sequence $\{x_j\}$. It may be assumed that this sequence is weakly convergent in V. By assumption **A1** the sequence $\{ix_j\}$ is strongly convergent in Z. We denote its limit by $\overline{x}$. We now choose the sequence $\{(p_j, q_j)\}$ in such a way that $(p_j, q_j) \in X^d$ for $j \in \mathcal{N}$ satisfies the relations

$$S(x) + i^* q_j = \nabla S(x_j), \tag{5.5}$$

$$p_j - q_j = \nabla N(ix_j). \tag{5.6}$$

By Theorem 4.2 it follows that $\{(p_j, q_j)\}$ is a minimizing sequence for J_D. Since ∇S is bounded on bounded sets it follows that q_j is weakly convergent in V^* (up to a subsequence) and its weak limit we denote by $\overline{q}$. By the above and continuity of

i^* and properties of ∇S we have using (5.5) that relation (5.2) holds. Similar reasoning using (5.1), (5.2) and properties of ∇N leads to (5.3).

By Theorem 4.2 and by relation (5.5) it follows that there exists a numerical sequence $\{\varepsilon_k\}$, $\varepsilon_k > 0$, $\varepsilon_k \to 0$ such that: for every ε_k there exists j_k such that for all $j \geq j_k$

$$(T+S)(x_j) - \langle p_j, ix_j\rangle_{Z^*,Z} + (T+S)^*(i^*p_j) \leq \varepsilon_k.$$

Letting $k \to \infty$ we may choose a subsequence $j_k \to \infty$. Since $\{ix_j\}$ is strongly convergent in Z, $\{p_j\}$ is weakly convergent in Z^*, $T+S$ and $(T+S)^*$ are weakly lower semicontinuous we have

$$\begin{aligned}
0 &\geq \liminf_{k\to\infty}\Big((T+S)(x_{j_k}) - \langle p_{j_k}, ix_{j_k}\rangle_{Z^*,Z} + (T+S)^*(i^*p_{j_k})\Big)\\
&\geq \liminf_{k\to\infty}(T+S)(x_{j_k}) + \liminf_{k\to\infty}(T+S)^*(i^*p_{j_k}) - \lim_{k\to\infty}\langle p_{j_k}, ix_{j_k}\rangle_{Z^*,Z}\\
&\geq (T+S)(\overline{x}) - \langle \overline{p}, i\overline{x}\rangle_{Z^*,Z} + (T+S)^*(i^*\overline{p}) \geq 0.
\end{aligned}$$

The last relation follows by the Fenchel-Young inequality. Hence

$$(T+S)(\overline{x}) - \langle i^*\overline{p}, \overline{x}\rangle_{Z^*,Z} + T^*(i^*\overline{p}) = 0$$

and now relation (5.1) follows by convexity.

To demonstrate (5.4) we need to prove that J is weakly lower semicontinuous. on X. Indeed,

$$V \ni x \mapsto (T+S)(x) \to R$$

is convex and lower semicontinuous. Hence it is weakly lower semicontinuous. Functional N is continuous on Z because it is finite and lower semicontinuous [2]. Since $\{ix_j\}$ is strongly convergent in Z, it follows that $\lim_{n\to\infty} N(ix_n) = N(i\overline{x})$. We need to show that

$$\limsup_{n\to\infty} S(x_n) \leq S(\overline{x})$$

Indeed, by definition of sequence $\{q_n\}$ and by duality we have

$$S(x_n) = -S^*(i^*q_n) + \langle q_n, ix_n\rangle_{Z^*Z}.$$

Now

$$\liminf_{n\to\infty} S^*(i^*q_n) \geq S^*(i^*\overline{q})$$

and

$$\lim_{n\to\infty}\langle q_n, ix_n\rangle_{Z^*Z} = \langle \overline{q}, i\overline{x}\rangle_{Z^*Z}.$$

In a consequence and by (5.2)

$$\begin{aligned}
&\limsup_{n\to\infty} S(x_n) \leq \limsup_{n\to\infty} -S^*(i^*q_n) + \lim_{n\to\infty}\langle q_n, ix_n\rangle_{Z^*Z}\\
&\leq -S^*(i^*\overline{q}) + \langle \overline{q}, i\overline{x}\rangle_{Z^*Z} = S(\overline{x})
\end{aligned}$$

Thus
$$\liminf_{n\to\infty} J(x_n) \geq J(\overline{x}).$$
Hence $J(\overline{x}) = \inf_{x\in X} J(x)$ and relation (5.4) follows by Theorem 3.3. □

6. – Applications.

Now we shall give an example of the problem which may be considered by our methods.

THEOREM 6.1. – *Let us consider the following equation*

$$\begin{aligned} &-div\big(\varphi(y,|\nabla x(y)|)|\nabla x(y)|^{n-2}\nabla x(y)\big)\\ &+div\big(\varphi(y,|\nabla x(y)|)|\nabla x(y)|^{m-2}\nabla x(y)\big) = F_x\big(y,x(y)\big) \end{aligned} \tag{6.1}$$
$$x(y)|_{\partial\Omega} = 0$$

where $n > m > 2$ are fixed, φ_1, $\varphi_2 : \Omega \times R \to R$ are Caratheodory function, i.e. continuous with respect to x for a.e. y and measurable in y for every x; there exist constants $M_{1i}, M_{2i} > 0$ such that for a.e. $y \in \Omega$ and for all $a \in R_+$, $i = 1, 2$
$$M_{1i} \leq \varphi(y,a) \leq M_{2i}.$$
$\Omega \subset R^r$ is a region with a regular boundary. Moreover there exists a constant $m_i > 0$ such that for all $a \geq b$, $a, b \in R$ and a.e. $y \in \Omega$, $i = 1, 2$
$$\varphi_i(y,a)a - \varphi_i(y,b)b \geq m_i(a-b);$$
$F : \Omega \times R \to R$ and $F_x : \Omega \times R \to R$ are Caratheodory functions, there exist constants $q \geq q_1 > 2$, $q \leq n$, k_1, $l_1 > 0$, where a constant k_1 satisfies $\big((vol(\Omega))^{1/q'}\big)/\big((vol(\Omega))^{1/n}\big)k_1 < m$ in case $q = n$, functions k_2, $l_2 \in L^\infty(\Omega, R)$ such that for all $x \in R$ and a.e. $y \in \Omega$

$$|F_x(y,x)| \leq k_1|x|^{q-1} + k_2(y) \tag{6.2}$$

$$F(y,x) \geq l_1|x|^{q_1} + l_2(y). \tag{6.3}$$

Then problem (6.1) *has a solution.*

Here $V = W^{1,n}(\Omega)$, $Z = L^n(\Omega)$. It suffice now to construct a suitable set X. Basing on the lemma

LEMMA 6.2 [3]. – *If a Caratheodory function $f : \Omega \times R \to R$ satisfies for a.e. $y \in \Omega$ and all $x \in R$*
$$|f(y,x)| \leq k_1|x|^{q-1} + k_2(y),$$

where $q \geq 2, k_1 > 0$, $k_2 \in L^{q'}(\Omega, R)$, *then the Niemytskij operator* N_f *defined by* f, *namely*

$$(N_f x)(y) = f(y, x(y)) \text{ for } a.e.\ y \in \Omega$$

is continuous and bounded from $L^q(\Omega, R)$ *to* $L^{q'}(\Omega, R)$. *Moreover it holds*

$$\|N_f x\|_{L^{q'}} \leq k_1 \|x\|_{L^q}^{q-1} + \|k_2\|_{L^{q'}}$$

for all $x \in L^q(\Omega, R)$, *where* $\frac{1}{q} + \frac{1}{q'} = 1$.

We get that there exists constants $a, \beta > 0$ such that for all $x \in V$

$$\|\nabla F(x)\|_{V^*} \leq a \|x\|_V^{q-1} + \beta, \tag{6.4}$$

where $a = k_1$ and $\beta = \|k_2\|_{L^{n'}}$. Here we denote by ∇F, the Niemytskij operator defined by F_x. Now let $x \in V$ be fixed. And denote by u the solution to the following Dirichlet problem

$$\begin{aligned} & -div\big(\varphi(y, |\nabla u(y)|)|\nabla u(y)|^{n-2} \nabla u(y)\big) \\ & = -div\big(\varphi(y, |\nabla x(y)|)|\nabla x(y)|^{m-2} \nabla x(y)\big) + F_x\big(y, x(y)\big) \end{aligned} \tag{6.5}$$

$$u(y)|_{\partial\Omega} = 0$$

which exists by classical arguments, [4] since it is an equation with a fixed right hand side. Using the properties of functions φ_i, $i = 1, 2$ and integrating by parts we get the following estimation by (6.4)

$$M_{11} \|\nabla u\|_{L^n}^n \leq M_{22} \|\nabla x\|_{L^{m'}} \|\nabla u\|_{L^m} + \Big(a \|\nabla x\|_{L^q}^{q-1} + \beta\Big) \|\nabla u\|_{L^{q'}}$$

$$M_{11} \|\nabla u\|_{L^n}^n \leq M_{22} d_1 \|\nabla x\|_{L^n} \|\nabla u\|_{L^n} + d_2 \Big(a \|\nabla x\|_{L^q}^{q-1} + \beta\Big) \|\nabla u\|_{L^n},$$

where $d_1 = \big((vol(\Omega))^{1/m'}\big) / \big((vol(\Omega))^{1/n}\big)$, $d_2 = \big((vol(\Omega))^{1/q'}\big) / \big((vol(\Omega))^{1/n}\big)$ are certain constants. Thus

$$M_{11} \|\nabla u\|_{L^n}^{n-1} \leq M_{22} d_1 \|\nabla x\|_{L^n} + d_2 \big(a \|\nabla x\|_{L^q}^{q-1} + \beta\big) \tag{6.6}$$

and since $n > q$ we have $\lim_{t \to \infty} \big(M_{11} t^{n-1} - M_{22} d_1 t - d_2 \big(a t^{q-1} + \beta\big)\big) = +\infty$. In case $n = q$ the same result holds due to the assumptions on the constants. Hence there exists k such that

$$M_{11} k^{n-1} \geq M_{22} d_1 k + d_2 \big(a k^{q-1} + \beta\big) \tag{6.7}$$

Thus we may take

$$\overline{X} = \{u \in V : \|\nabla u\|_{L^n} \leq k\}.$$

Taking any $x \in \overline{X}$ we observe that the solution u to the equation (6.5) satisfies due to (6.6) and definition of $\overline{X}$

$$M_{11}\|\nabla u\|_{L^n}^{n-1} \leq M_{22} d_1 k + d_2\big(a k^{q-1} + \beta\big).$$

Now using (6.7) we get that $\|\nabla u\|_{L^n} \leq k$. So the set $X = \overline{X}$. Since all other assumptions are satisfied, we have proved the theorem.

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