# Bollettino Unione Matematica Italiana



Andrija Raguž


## A Note on Calculation of Asymptotic Energy for a Functional of Ginzburg-Landau Type with Externally Imposed Lower-Order Oscillatory Term in One Dimension


*Bollettino dell'Unione Matematica Italiana, Serie 8*, Vol. **10-B** (2007), n.3, p. 1125–1142.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_1125_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 1125-1142

# A Note on Calculation of Asymptotic Energy for a Functional of Ginzburg-Landau Type with Externally Imposed Lower-Order Oscillatory Term in One Dimension.

ANDRIJA RAGUŽ

**Sunto.** – *In questa nota consideriamo il funzionale di Ginzburg-Landau*

$$I_a^\varepsilon(v) = \int_0^1 \left(\varepsilon^2 v''^2(s) + W(v'(s)) + a(\varepsilon^{-\beta}s)v^2(s)\right)ds$$

*ove $\beta > 0$ e $a$ è 1-periodica. Mostreremo come la minima energia asintotica (ridimensionata) associata a $I_a^\varepsilon$ dipenda dal parametro $\beta > 0$ per $\varepsilon \longrightarrow 0$. In particolare, la nostra analisi mostra che i minimizzatori di $I_a^\varepsilon$ sono quasi $\varepsilon^{1/3}$-periodici.*

**Summary.** – *In this note we consider the Ginzburg-Landau functional*

$$I_a^\varepsilon(v) = \int_0^1 \left(\varepsilon^2 v''^2(s) + W(v'(s)) + a(\varepsilon^{-\beta}s)v^2(s)\right)ds$$

*where $\beta > 0$ and $a$ is 1-periodic. We determine how (rescaled) minimal asymptotic energy associated to $I_a^\varepsilon$ depends on parameter $\beta > 0$ as $\varepsilon \longrightarrow 0$. In particular, our analysis shows that minimizers of $I_a^\varepsilon$ are nearly $\varepsilon^{1/3}$-periodic.*

## 1. – Introduction.

In this note we deal with the asymptotic behavior of a family of functionals of Ginzburg-Landau type in one dimension. Our consideration relies on techniques and results developed in paper [1] by G. Alberti and S. Müller. In that paper the authors introduced a concept of *Young measure on micropatterns* (or *two-scale Young measure*) to describe properties of minimizers of variational problems which lead to creation of multiple small scales depending on small parameter $\varepsilon$. As an example of the approach, they studied the functional $I_{\varepsilon,a} : \mathrm{H}^2_{per}\langle 0,1\rangle \longrightarrow [0,+\infty]$ defined by

$$I_{\varepsilon,a}(v) := \int_0^1 \left(\varepsilon^2 v''^2(s) + W(v'(s)) + a(s)v^2(s)\right)ds\ , \tag{1}$$

where $v \in \mathrm{H}^2_{per}\langle 0,1\rangle$, $W \in \mathrm{C}(\boldsymbol{R};[0,+\infty\rangle)$, $W(\rho)=0$ if and only if $\rho \in \{-1,1\}$, $W$ has superlinear growth in infinity and $a \in \mathrm{L}^1_{per}\langle 0,1\rangle$ satisfies $a(s) \geq \alpha > 0$ (a.e. $s \in \langle 0,1\rangle$). Functional (1) can be regarded as a simplified version of functional of Cahn-Hillard type (cf. [8], [3]) which appears in modeling of complex physical systems like block copolymer melts. It is a well-known fact that minimizers of such functionals develop fine structure as a result of an attempt to minimize different terms. In particular, micro-phase separation occurs. Due to the competition between formation of microstructure and highest gradient regularization (cf. [1], p. 762.), minimizers of (1) exhibit oscillation on two fast scales (namely on the scale of order $\varepsilon^{1/3}$ and on the scale of order $\varepsilon$). A thorough description of such behavior, as well as calculation of associated (rescaled) asymptotic energy, is obtained in [1] in the following way: basic idea is to rewrite $I_{\varepsilon,a}(v)$ in terms of carefully chosen rescalings $s \mapsto R^\varepsilon_s v$,

$$R^\varepsilon_s v(\tau) := \varepsilon^{-1/3} v(+\varepsilon^{1/3}\tau)\ , \quad \tau \in \boldsymbol{R}\ , \tag{2}$$

as an integral functional in $s$, where for every $s \in \langle 0,1\rangle$ integrated function is a functional itself and it is evaluated in $R^\varepsilon_s v$. Then such a functional can be extended to the space of Young measures and we can pass to the limit as $\varepsilon \longrightarrow 0$ by means of the Modica-Mortola theorem (cf. [6]), which results in a non-trivial $\Gamma$-limit. Finally, the Young measure which minimizes the $\Gamma$-limit is identified.

Our goal is to apply similar reasoning to slightly general situation. More precisely, we study a variant of energy in [1], which is perturbed by the highly oscillatory term $a(\varepsilon^{-\beta}s)$, where $\beta > 0$. The original functional (1) is now replaced by

$$I^\varepsilon_a(v) := \int_0^1 \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(\varepsilon^{-\beta}s)v^2(s)\Big)ds\ . \tag{3}$$

Since the period of map $s \mapsto a(\varepsilon^{-\beta}s)$ vanishes as $\varepsilon \longrightarrow 0$, Alberti and Müller expected that an additional structure of the minimizers emerges (cf. [1], section 6). Indeed, apart from the creation of fast scale of order $\varepsilon^{1/3}$, minimizers of (3) now comply with the constraint coming from fast scale of order $\varepsilon^\beta$. To understand what exactly happens when $\varepsilon \longrightarrow 0$ we formulate two objectives. First, we want to determine the rescaled asymptotic energy associated to (3) as $\varepsilon \longrightarrow 0$. To this end, we consider the following quantities: $\mathcal{E}^\varepsilon_{a,per}(\beta) := \min\limits_{v\in \mathrm{H}^2_{per}\langle 0,1\rangle} \varepsilon^{-2/3} I^\varepsilon_a(v)$, $\mathcal{E}^\varepsilon_a(\beta) := \min\limits_{v\in \mathrm{H}^2\langle 0,1\rangle} \varepsilon^{-2/3} I^\varepsilon_a(v)$, $\mathcal{E}_{a,per}(\beta) := \lim\limits_{\varepsilon\longrightarrow 0} \mathcal{E}^\varepsilon_{a,per}(\beta)$ and $\mathcal{E}_a(\beta) := \lim\limits_{\varepsilon\longrightarrow 0} \mathcal{E}^\varepsilon_a(\beta)$. As now a hierarchy of small scales appears, we distinguish cases $\beta \in \langle 0,1/3\rangle$ (the subcritical case), $\beta = 1/3$ (the critical case) and $\beta > 1/3$ (the supercritical case). The main result of this note, obtained by a mild modification of techniques in [1], states that there holds

$$\text{(4)} \quad \mathcal{E}_a(\beta) = \mathcal{E}_{a,per}(\beta) = E_0\overline{a^{1/3}}\chi_{[0,1/3\rangle}(\beta) + F_0(a)\chi_{(\beta=1/3)}(\beta) + E_0\overline{a}^{1/3}\chi_{\langle 1/3,+\infty\rangle}(\beta)\ ,$$

where $F_0(a) \approx E_0\overline{a^{1/3}}$ when $\xi \approx 0$, $F_0(a) \approx E_0\overline{a}^{1/3}$ when $\frac{1}{\xi} \approx 0$, $\xi := 2\int\limits_{-1}^{1}\sqrt{W}$, $C_0 := (3/4)^{2/3}$, $E_0 := C_0\xi^{2/3}$. Formula (4) was conjectured in [1], p. 814. and it is to be understood in the following way: if $\beta \in \langle 0, 1/3\rangle$, then the internally created fast scale $\varepsilon^{1/3}$ is shorter than the externally imposed fast scale $\varepsilon^{\beta}$, so that oscillation on the scale $\varepsilon^{\beta}$ is not relevant to computation of asymptotic energy. In the case $\beta > 1/3$, however, the scale $\varepsilon^{\beta}$ is shorter, and thus oscillation in $a(\varepsilon^{-\beta}s)$ becomes relevant. On the other hand, in the critical case $\beta = 1/3$ "locking" of the internally created and the externally imposed scale induces an additional conflict to the minimizers, which can be best explained as an impossibility of function to be both $\varepsilon^{1/3}$-periodic and $O(\varepsilon^{1/3})$-periodic at the same time. To provide at least a partial insight into this situation, we formulate and solve asymptotic problem for $F_0(a)$ in terms of $W$. In all cases, asymptotic energy is independent of boundary conditions. Second, we want to describe geometric properties of the minimizers of $I_a^{\varepsilon}$ as $\varepsilon \longrightarrow 0$. In the case $\beta = 0$ an interpretation of geometry of minimizers is deduced from the convergence of $\varepsilon$-blowups (2) of minimizers in the space of Young measures on micropatterns. By contrast, when $\beta > 0$, we offer a weaker result, which, in our opinion, still gives good enough information in this respect as $\varepsilon \longrightarrow 0$.

Other variants of the functional (1) were considered in [3] and [9] (see also references therein).

This note is organized as follows. In Section 2 we fix the notation, and we recall some well-known results which we will use. In Section 3 we derive the main results (cf. Theorem 3.2, Theorem 3.4). Finally, in Section 4 we interpret our results in terms of geometric properties of the minimizers.

## 2. – Some preliminaries

Throughout the note we work on the unit interval $\langle 0, 1\rangle \subseteq \boldsymbol{R}$, but all the proofs can be carried out if we consider any bounded open interval $\omega \subseteq \boldsymbol{R}$ endowed with Lebesgue measure (denoted by $\lambda$). We consider the set $K$ of all Borel measurable mappings $x : \boldsymbol{R} \longrightarrow [-\infty, +\infty]$ (modulo equivalence $\lambda$-almost everywhere), which can be made compact and metrizable topological space by defining a pullback topology on $K$ with respect to weak-star topology on $\mathrm{L}^{\infty}(\boldsymbol{R}; [-1, 1])$ via mapping $x \mapsto \frac{2}{\pi}\arctan(x)$ (cf. [1], p. 778, 806 for details). By $\mathrm{C}(K)$ we denote the Banach space of all continuous real functions on $K$. A $K$-valued Young measure on $\langle 0, 1\rangle$ (or *Young measure on micropatterns*) is a map $\boldsymbol{\nu} \in \mathrm{L}^{\infty}_{w*}(\langle 0, 1\rangle; \mathcal{M}(K))$

(where by $\mathrm{L}^\infty_{w*}(\langle 0,1\rangle;\mathcal{M}(K))$ we denote the dual of $\mathrm{L}^1(\langle 0,1\rangle;\mathrm{C}(K))$, cf. [2] for details), $\boldsymbol{\nu}: s \mapsto \nu_s$, such that $\nu$ is a probability measure for almost every $\in \langle 0,1\rangle$. The set of all $K$-valued Young measures is denoted by $YM(\langle 0,1\rangle;K)$ and it is always endowed with the weak-star topology of $\mathrm{L}^\infty_{w*}(\langle 0,1\rangle;\mathcal{M}(K))$. $\mathcal{I}(K)$ denotes the class of all probability measures on $K$ which are invariant with respect to action of the group of functional translations on $K$ (cf. [1], p. 778, p. 795). As usual, $\mathrm{H}^2_{per}\langle 0,1\rangle$ denotes the set of all $\mathrm{H}^2_{loc}(\boldsymbol{R})$ functions, extended by periodicity out of $\langle 0,1\rangle$. By $Sx$ we denote a set of all discontinuities for some $x \in K$, while $|Sx|$ denotes cardinality of the set $Sx$. If $a$ is periodic function, $\overline{a}$ denotes average of $a$ over its period. By $\lceil\sigma\rceil$ ($\lfloor\sigma\rfloor$, resp.) we denote the smallest integer greater or equal to $\sigma \in \boldsymbol{R}$ (the largest integer below $\sigma \in \boldsymbol{R}$, resp.). We say that $a \in K$ is simple function if $a(s) = \sum_{k=1}^{N} \alpha_k \chi_{\omega_k}(s)$, where $\omega_k$ are pairwise disjoint measurable sets. If $M > 0$ and $\varepsilon > 0$ are given, we set $\varepsilon_{\star,M} := \lceil \varepsilon^{-\beta} M^{-1}\rceil^{-1/\beta}$, $\varepsilon_{\star\star,M} := \lfloor \varepsilon^{-\beta} M^{-1}\rfloor^{-1/\beta}$, $\rho_{\varepsilon,\star\star,M} := \varepsilon^{\beta}_{\star\star,M} M^{-1}\varepsilon^{-\beta}$, $\rho_{\varepsilon,\star,M} := \varepsilon^{\beta}_{\star,M} M^{-1}\varepsilon^{-\beta}$. Then $\rho_{\varepsilon,\star,M} \nearrow 1$, $\rho_{\varepsilon,\star\star,M} \searrow 1$ as $\varepsilon \longrightarrow 0$. If $M = 1$, we define $\varepsilon_\star := \varepsilon_{\star,1}$, $\varepsilon_{\star\star} := \varepsilon_{\star\star,1}$, $\rho_{\varepsilon,\star\star} := \rho_{\varepsilon,\star\star,1}$, $\rho_{\varepsilon,\star} := \rho_{\varepsilon,\star,1}$. In the following we use the term "sequence" also to denote families labeled by the continuous parameter $\varepsilon$, which tends to 0.

DEFINITION 2.1 [$\Gamma$-convergence]. – *Let $X$ be a metric space. A sequence of functions $F^\varepsilon : X \longrightarrow [0,+\infty]$ $\Gamma$-converges to $F$ on $X$, and we write $F^\varepsilon \xrightarrow{\Gamma} F$, if the following is fulfilled:*

(i) *Lower-bound inequality: for every $x \in X$ and a sequence $(x^\varepsilon)$ in $X$ such that $x^\varepsilon \longrightarrow x$ it holds* $\liminf_\varepsilon F^\varepsilon(x^\varepsilon) \geq F(x)$, *and*

(ii) *Upper-bound inequality: For any $y$ in $X$ there exists a sequence $(y^\varepsilon)$ in $X$ such that $y^\varepsilon \longrightarrow y$ and* $\limsup_\varepsilon F^\varepsilon(y^\varepsilon) \leq F(y)$.

The proof of the following Proposition can be found in chapters 6 and 7 in [4]:

PROPOSITION 2.1. – *If the points $x^\varepsilon$ minimize $F^\varepsilon$ for every $\varepsilon$, and $F^\varepsilon \xrightarrow{\Gamma} F$ as $\varepsilon \longrightarrow 0$, then every cluster point $x$ of the sequence $(x^\varepsilon)$ minimizes $F$. In particular, there holds $\lim\limits_{\varepsilon\longrightarrow 0} F^\varepsilon(x^\varepsilon) = F(x)$.*

We introduce the following classes of functions:

DEFINITION 2.2. – *Let $\omega \subset \boldsymbol{R}$ be a fixed interval. A function $x : \omega \longrightarrow \boldsymbol{R}$ is said to be of the class $\mathcal{S}(\omega)$ if $x$ is piecewise affine continuous function on $\omega$ such that $x'(\tau) \in \{-1,1\}$ for almost every $\tau \in \omega$. A function $x : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ belongs to the class $\mathcal{S}_{per}(\omega)$ if $x$ can be extended from $\omega$ to $\boldsymbol{R}$ by periodicity in such a way that there holds $x \in \mathcal{S}(J)$ for any interval $J \subseteq \boldsymbol{R}$.*

For a given bounded open interval $\omega \subseteq \boldsymbol{R}$ we define $f_a^{\varepsilon,\omega}, f_a^{\omega} : \mathrm{L}^1(\omega) \longrightarrow [0, +\infty]$ by

$$f_a^{\varepsilon,\omega}(v) := \begin{cases} \displaystyle\fint_{\omega} \Big( \varepsilon^{2/3} v''^2(\tau) + \varepsilon^{-2/3} W(v'(\tau)) + a(\tau) v^2(\tau) \Big) d\tau, & \text{if } v \in \mathrm{H}^2(\omega), \\ +\infty, & \text{otherwise}, \end{cases} \tag{5}$$

$$f_a^{\omega}(x) := \begin{cases} \dfrac{\xi}{\lambda(\omega)} |S_\omega(x')| + \displaystyle\fint_{\omega} a(\tau) x^2(\tau) d\tau, & \text{if } x \in \mathcal{S}(\omega), \\ +\infty, & \text{otherwise}, \end{cases} \tag{6}$$

where, for $\omega = \langle \gamma_1, \gamma_2 \rangle$ we define $S_\omega(x') := Sx' \cap [\gamma_1, \gamma_2\rangle$. In particular, for a given $h > 0$ we set $f_a^{\varepsilon,h} := f_a^{\varepsilon,\langle 0,h\rangle}$, $f_a^h(x) := f_a^{\langle 0,h\rangle}$. When no confusion is possible we write $f_a^\varepsilon$ ($f_a$, resp.) instead of $f_a^{\varepsilon,\omega}$ ($f_a^\omega$, resp.). Note that functionals $f_a^{\varepsilon,h}$ and $f_a^h$ also depend on $\xi > 0$. When such a dependance is not essential to our consideration, we avoid labeling which includes $\xi$. However, if $\xi = 1$ we write $\varphi_a^h$ in stead of $f_a^h$ (cf. Theorem 3.4). Also, in this note we frequently use a version of the Modica-Mortola theorem in one dimension (cf. [6], [1], Proposition 3.3):

PROPOSITION 2.2. – *Suppose $a \in \mathrm{L}^1_{loc}(\boldsymbol{R})$. Set $a_s^\varepsilon(\tau) := a(s + \varepsilon^{1/3-\beta}\tau)$, $\tau \in \boldsymbol{R}$, where $\beta \in \langle 0, 1/3\rangle$. Then for every bounded open interval $\omega \subseteq \boldsymbol{R}$ there holds:*

$$f_{a_s^\varepsilon}^{\varepsilon,\omega} \xrightarrow{\Gamma} f_{a(s)}^{\omega} \quad on \ \mathrm{L}^1(\omega) \ \ (a.e. \ s \in \boldsymbol{R}), \tag{7}$$

$$f_a^{\varepsilon,\omega} \xrightarrow{\Gamma} f_a^{\omega} \quad on \ \mathrm{L}^1(\omega) \ \ as \ \ \varepsilon \longrightarrow 0 . \tag{8}$$

*If $a_n \longrightarrow a$ in $\mathrm{L}^1_{loc}(\boldsymbol{R})$ as $n \longrightarrow +\infty$, then $f_{a_n}^\omega \xrightarrow{\Gamma} f_a^\omega$ on $\mathrm{L}^1(\omega)$.*

## 3. – Main results.

In this section we lay out our main results. To begin with, we note that an attempt of rewriting (3) in terms of $\varepsilon$-blowups (2) eventually results in representation

$$\varepsilon^{-2/3} I_a^\varepsilon(v) = \int_0^1 f_{\tilde{a}_s^\varepsilon}^\varepsilon (R_s^\varepsilon v) ds ,$$

where $v \in \mathrm{H}^2_{per}\langle 0, 1\rangle$ and $\tilde{a}_s^\varepsilon(\tau) := a(\varepsilon^{-\beta}s + \varepsilon^{1/3-\beta}\tau)$, $s, \tau \in \boldsymbol{R}$. Clearly, if $\beta \in \langle 0, 1/3]$, the sequence $(\tilde{a}_s^\varepsilon)$ does not converge weakly in $\mathrm{L}^1_{loc}(\boldsymbol{R})$ as $\varepsilon \longrightarrow 0$ (hence the sequence $(f_{\tilde{a}_s^\varepsilon}^\varepsilon)$ does not $\Gamma$-converge for a.e. $s \in \boldsymbol{R}$). Thus $\varepsilon$-blowup (2) is not always suitable. Herein we propose technical improvement of calculations from [1] in order to capture asymptotic behavior of (3). In particular, if $\beta \in \langle 0, 1/3\rangle$, then a different $\varepsilon$-blowup is used. The case $\beta > 1/3$ is much simpler. For $M > 0$ and $\varepsilon \in \langle 0, 1\rangle$ we set

$$I^{\varepsilon,\star}_{a,M}(w) := \int_0^{M\varepsilon^\beta} \Big(\varepsilon^2 w''^2(s) + W(w'(s)) + a(\varepsilon^{-\beta}s)w^2(s)\Big)ds\ , \quad w \in \mathrm{H}^2\langle 0, M\varepsilon^\beta\rangle\ ,$$

$$\mathcal{E}^{\varepsilon,\star}_{a,M}(\beta) := \min_{w\in \mathrm{H}^2\langle 0,M\varepsilon^\beta\rangle} \varepsilon^{-2/3} I^{\varepsilon,\star}_{a,M}(w)\ , \quad \mathcal{E}^{\varepsilon,\star}_{a,M,per}(\beta) := \min_{w\in \mathrm{H}^2_{per}\langle 0,M\varepsilon^\beta\rangle} \varepsilon^{-2/3} I^{\varepsilon,\star}_{a,M}(w)\ .$$

First we obtain the following simple estimate:

PROPOSITION 3.1. – *Let $\beta > 0$. Then there holds*

$$\liminf_{\varepsilon\longrightarrow 0} \mathcal{E}^\varepsilon_a(\beta) \geq \sup_{M>0} \frac{M}{\lceil M\rceil} \liminf_{\varepsilon\longrightarrow 0} \mathcal{E}^{\varepsilon,\star}_{a,M}(\beta)\ . \tag{9}$$

PROOF. – Put $N := \varepsilon^{-\beta}_{\star\star,\lceil M\rceil}$. Consider $v \in \mathrm{H}^2\langle 0, 1\rangle$. For $j = 1, \dots, N$ we set $v_j(s) := v(s + (j-1)\lceil M\rceil\varepsilon^\beta)$, $s \in \langle 0, \lceil M\rceil\varepsilon^\beta\rangle$. Since $\lceil M\rceil \in \boldsymbol{N}$, 1-periodicity of $a$ and $\rho^{-1}_{\varepsilon,\star\star,\lceil M\rceil} \in \langle 0, 1\rangle$ imply

$$\begin{aligned}\varepsilon^{-2/3} I^\varepsilon_a(v) &\geq \rho^{-1}_{\varepsilon,\star\star,\lceil M\rceil} \sum_{j=1}^N \varepsilon^\beta_{\star\star,\lceil M\rceil} \varepsilon^{-2/3} I^{\varepsilon,\star}_{a,\lceil M\rceil}(v_j)\\ &\geq \rho^{-1}_{\varepsilon,\star\star,\lceil M\rceil} \min_{w\in \mathrm{H}^2\langle 0,\lceil M\rceil\varepsilon^\beta\rangle} \varepsilon^{-2/3} I^{\varepsilon,\star}_{a,\lceil M\rceil}(w)\\ &\geq \frac{\lfloor \varepsilon^{-\beta}\lceil M\rceil^{-1}\rfloor}{\varepsilon^{-\beta}\lceil M\rceil^{-1}} \frac{M}{\lceil M\rceil} \min_{w\in \mathrm{H}^2\langle 0,M\varepsilon^\beta\rangle} \varepsilon^{-2/3} I^{\varepsilon,\star}_{a,M}(w)\ .\end{aligned}$$

Thus, by passing to the limit as $\varepsilon \longrightarrow 0$, we infer (9). □

To proceed, we sketch the proof the lower bound when $\beta \in \langle 0, 1/3\rangle$.

THEOREM 3.1 [the subcritical case: the lower bound]. – *Let $\beta \in \langle 0, 1/3\rangle$. Then* $\liminf_{\varepsilon\longrightarrow 0} \mathcal{E}^\varepsilon_a(\beta) \geq E_0 \overline{a^{1/3}}$.

PROOF. – Let $\omega \subset\subset \langle 0, 1\rangle$ be an open interval and let $v \in \mathrm{H}^2\langle 0, \varepsilon^\beta\rangle$. Consider $v_\star(s) := \varepsilon^{-\beta}v(\varepsilon^\beta s)$, $s \in \langle 0, \varepsilon^\beta\rangle$, and $\varepsilon$-blowup $R^{\varepsilon,\star}_s v(\tau) := \varepsilon^{-(1/3-\beta)}v(s + \varepsilon^{1/3-\beta}\tau)$, $\tau \in \langle -r, r\rangle$. For $s \in \omega$ we set $x_s(\tau) := R^{\varepsilon,\star}_s v_\star(\tau)$, $\tau \in \langle -r, r\rangle$. Then we have $v'_\star(s) = v'(\varepsilon^\beta s)$, $x'_s(\tau) = v'_\star(s + \varepsilon^{1/3-\beta}\tau)$, $x''_s(\tau) = \varepsilon^{1/3-\beta}v''_\star(s + \varepsilon^{1/3-\beta}\tau)$, $v''_\star(s) = \varepsilon^\beta v''(\varepsilon^\beta s)$. Let

$$\mathcal{I}^\varepsilon_\omega(v) := \int_\omega \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(\varepsilon^{-\beta}s)v^2(s)\Big)ds\ . \tag{10}$$

Then, similarly as in [1], p. 781, by Fubini's Theorem it follows

$$\varepsilon^{-2/3}\int_{-r}^{r} \mathcal{I}^{\varepsilon}_{\varepsilon^{\beta}\omega+\varepsilon^{1/3}\tau}(v)d\tau = \int_{\omega} f^{\varepsilon}_{a^{\varepsilon}_{s}}(R^{\varepsilon,\star}_{s} v_{\star})ds \ , \tag{11}$$

where $f^{\varepsilon}_{a^{\varepsilon}_{s}} : \mathrm{H}^2\langle -r, r\rangle \longrightarrow [0, +\infty\rangle$ is defined as in section 2. In particular, for every $v \in \mathrm{H}^2_{per}\langle 0, \varepsilon^{\beta}\rangle$ there holds

$$\varepsilon^{-2/3} I^{\varepsilon,\star}_{a,1}(v) = \int_{0}^{1} f^{\varepsilon}_{a^{\varepsilon}_{s}}(R^{\varepsilon,\star}_{s} v_{\star})ds \ . \tag{12}$$

Let

$$I^{\varepsilon,r,\star}_{a}(v) := \int_{-r}^{r} \mathcal{I}^{\varepsilon}_{\varepsilon^{\beta}\langle 0,1\rangle+\varepsilon^{1/3}\tau}(v)d\tau \ . \tag{13}$$

Suppose $F^{\varepsilon}_{a}, F_{a} : YM(\langle 0,1\rangle; K) \longrightarrow [0, +\infty]$ are defined as follows:

$$F^{\varepsilon}_{a}(\boldsymbol{\nu}) := \begin{cases} \int_{0}^{1} \langle \nu_s, f^{\varepsilon}_{a^{\varepsilon}_{s}} ds, & \text{if } \nu_s = \delta_{R^{\varepsilon,\star}_{s} v_{\star}} \text{ for some } v_{\star} \in \mathrm{H}^2_{per}\langle 0,1\rangle\rangle \\ +\infty, & \text{otherwise} \ , \end{cases} \tag{14}$$

$$F_{a}(\boldsymbol{\nu}) := \begin{cases} \int_{0}^{1} \langle \nu_s, f_{a(s)}\rangle ds, & \text{if } \nu_s \in \mathcal{I}(K) \text{ for a.e. } s \in \langle 0,1\rangle \\ +\infty, & \text{otherwise} \ . \end{cases}$$

Then (7) and Theorem 3.4 in [1] give $F^{\varepsilon}_{a} \xrightarrow{\Gamma} F_{a}$. It can be verified (see comments in [1], section 6.1, p. 813) that the convergence is preserved if $v_{\star}$ in (14) satisfies $v_{\star} \in \mathrm{H}^2(\Omega)$ for some open interval $\Omega$ such that $\langle 0,1\rangle \subset\subset \Omega$. In particular, by Proposition 2.1, (11), (12), (13) and by Theorem 3.12 in [1], there holds $\lim\limits_{\varepsilon\longrightarrow 0} \mathcal{E}^{\varepsilon,\star}_{a,1}(\beta) = \lim\limits_{\varepsilon\longrightarrow 0} \mathcal{E}^{\varepsilon,\star}_{a,1,per}(\beta) = E_0\overline{a^{1/3}}$. To sum up, we note that for $r > 0$ there holds $\lim\limits_{\varepsilon\longrightarrow 0} \min\limits_{v\in\mathrm{H}^2\langle 0,\varepsilon^{\beta}\rangle} \varepsilon^{-2/3} I^{\varepsilon,r,\star}_{a}(v) = \lim\limits_{\varepsilon\longrightarrow 0} \mathcal{E}^{\varepsilon,\star}_{a,1}(\beta)$, and we apply Proposition 3.1. $\square$

Next, we establish the upper bound in the case $\beta \in \langle 0, 1/3\rangle$.

THEOREM 3.2 [the subcritical case: the upper bound]. – *Let* $\beta \in \langle 0, 1/3\rangle$. *Then* $\limsup\limits_{\varepsilon\longrightarrow 0} \mathcal{E}^{\varepsilon}_{a,per}(\beta) \le E_0\overline{a^{1/3}}$.

PROOF. – Consider arbitrary $\Delta \in \langle 0,1\rangle$, $\delta \in \langle 0,\Delta\rangle$, $M > 0$, $F^M := \{\sigma \in \langle 0,1\rangle : a(\sigma) > M\}$ and open intervals $J^{\delta} := \langle \delta, 1-\delta\rangle$, $J_{j-1} := \langle j-1, j\rangle$, $J^{\delta}_{j-1} := J^{\delta} + j - 1$, $E^{\delta}_{j} := J_{j-1}\backslash J^{\delta}_{j-1}, j \in \boldsymbol{N}$, $E^{\delta} := \bigcup_{j=1}^{\varepsilon_{\star}^{-\beta}} E^{\delta}_{j}$. By Theorem 3.4 in [1] for every $\eta > 0$ there exists $\overline{M}_\eta > 0$ and a sequence of functions $(\overline{v}^\varepsilon_\star)$ (which depends on $\eta$ and $M$) such that $\overline{v}^\varepsilon_\star \in \mathrm{H}^2_{per}\langle 0,1\rangle$ and with properties $\|\overline{v}^\varepsilon_\star\|_{\mathrm{L}^\infty(\boldsymbol{R})} \le \overline{M}_\eta \varepsilon^{1/3-\beta}$,

$$\limsup_{\varepsilon \longrightarrow 0} \int_0^1 f^\varepsilon_{a^\varepsilon_s}(R^{\varepsilon,\star}_s \overline{v}^\varepsilon_\star) ds \le E_0 \overline{a^{1/3}} + \eta + O(\overline{M}^2_\eta) \int_{F^M} a(s) ds \,. \tag{15}$$

Set $v^\varepsilon_\star(s) := \rho^{-1}_{\varepsilon,\star} \overline{v}^\varepsilon_\star(\rho_{\varepsilon,\star} s)$, $s \in \boldsymbol{R}$. We consider the sequence $w^\varepsilon : \langle 0,1\rangle \longrightarrow \boldsymbol{R}$ defined by $w^\varepsilon_\star(s) := v^\varepsilon_\star(s)$, if $s \in J^\delta$ ($w^\varepsilon_\star(s) := \tilde{v}^\varepsilon_\star(s)$, if $s \in \langle 0,1\rangle \backslash J^\delta$, resp.), where $\tilde{v}^\varepsilon_\star : \langle 0,1\rangle \backslash J^\delta \longrightarrow \boldsymbol{R}$ is chosen in such a way that $w^\varepsilon_\star \in \mathrm{H}^2_{per}\langle 0,1\rangle$ and on each of the connected components in the domain $\tilde{v}^\varepsilon_\star$ has the following properties: derivative of $\tilde{v}^\varepsilon_\star$ takes alternately the values 1 and $-1$ on consecutive intervals of order $\varepsilon^{1/3-\beta}$ (except the first and the last one, which have length of order $\overline{M}_\eta \varepsilon^{1/3-\beta}$), apart from transition layers of order $\varepsilon^{1-\beta}$ at the end of each such interval, where the second derivative is of order $\varepsilon^{-(1-\beta)}$. The value of $\tilde{v}^\varepsilon_\star$ is of order $\varepsilon^{1/3-\beta}$ (except in the first and the last interval, where it is of order $\overline{M}_\eta \varepsilon^{1/3-\beta}$ (cf. Figure 1). In particular, there holds $\|w^\varepsilon_\star\|_{\mathrm{L}^\infty(\boldsymbol{R})} \le \overline{M}_\eta \varepsilon^{1/3-\beta}$. Let $X_\varepsilon$ ($Y_\varepsilon$, resp.) denotes the set of all points in $\langle 0,1\rangle$ with property that $w^\varepsilon_\star$ is of order $\overline{M}_\eta \varepsilon^{1/3-\beta}$ ($\varepsilon^{1/3-\beta}$, resp.). By construction there holds $\lambda(X_\varepsilon) = O(\overline{M}_\eta)\varepsilon^{1/3-\beta}$, $\lambda(Y_\varepsilon) \le O(1)\delta$. Set $w^\varepsilon(s) := \varepsilon^\beta_\star w^\varepsilon_\star(\varepsilon^{-\beta}_\star s)$, $a_\varepsilon(s) := a(\rho_{\varepsilon,\star} s)$, $s \in \boldsymbol{R}$. Then $w^\varepsilon \in \mathrm{H}^2_{per}\langle 0, \varepsilon^\beta_\star\rangle$ and therefore $w^\varepsilon \in \mathrm{H}^2_{per}\langle 0,1\rangle$. Note that there exists $\varepsilon_0(\delta) > 0$ such that for every $\varepsilon \in \langle 0, \varepsilon_0(\delta)]$ there holds $\langle 0,1\rangle = \bigcup_{j=1}^{\varepsilon^{-\beta}_\star} \varepsilon^\beta_\star J_{j-1}$, $J^\delta_{j-1} \subset \rho^{-1}_{\varepsilon,\star} J_{j-1}$. Since $\overline{v}^\varepsilon_\star$ and $a$ are

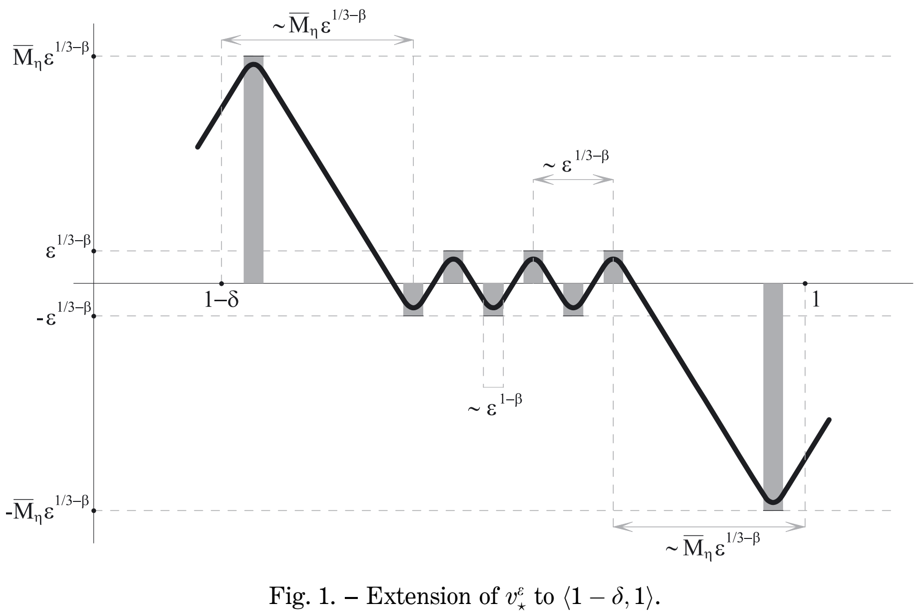


Fig. 1. – Extension of $v^\varepsilon_\star$ to $\langle 1-\delta, 1\rangle$.

1-periodic, it results

$$I_a^\varepsilon(w^\varepsilon) \le \rho_{\varepsilon,\star}^{-1}\int_0^1 f_{a_s^\varepsilon}^\varepsilon(R_s^{\varepsilon,\star}\overline{v}_\star^\varepsilon)ds + \varepsilon_\star^\beta\varepsilon^{-2/3}\int_{E^\delta}\left(\varepsilon^2\varepsilon_\star^{-2\beta}w_\star^{\varepsilon''2} + W(w_\star^{\varepsilon'}) + a_\varepsilon\varepsilon_\star^{2\beta}w_\star^{\varepsilon 2}\right).$$

Let $Q^\varepsilon$ ($L^\varepsilon$, resp.) denotes the set of all points in the domain where $w_\star^\varepsilon$ is quadratic (linear, resp.). For every $j = 1, \dots, \varepsilon_\star^{-\beta}$ there holds

$$\begin{aligned}&\varepsilon^{-2/3}\int_{E_j^\delta\cap Q^\varepsilon}\left(\varepsilon^2\varepsilon_\star^{-2\beta}w_\star^{\varepsilon''2} + W(w_\star^{\varepsilon'}) + a_\varepsilon\varepsilon_\star^{2\beta}w_\star^{\varepsilon 2}\right)\\ &\le O(1)\varepsilon^{1/3-\beta} + 2\|a_\varepsilon\|_{\mathrm{L}^1(E_j^\delta\cap Q^\varepsilon\cap X^\varepsilon)}\overline{M}_\eta^2 + O(1)\|a_\varepsilon\|_{\mathrm{L}^1(E_j^\delta\cap Q^\varepsilon\cap Y^\varepsilon)}.\end{aligned}$$

On the other hand, estimate on the set $L^\varepsilon$ reads

$$\begin{aligned}&\varepsilon^{-2/3}\int_{E_j^\delta\cap L^\varepsilon}\left(\varepsilon^2\varepsilon_\star^{-2\beta}w_\star^{\varepsilon''2} + W(w_\star^{\varepsilon'}) + a_\varepsilon\varepsilon_\star^{2\beta}w_\star^{\varepsilon 2}\right)\\ &\le 2\|a_\varepsilon\|_{\mathrm{L}^1(E_j^\delta\cap L^\varepsilon\cap X^\varepsilon)}\overline{M}_\eta^2 + O(1)\|a_\varepsilon\|_{\mathrm{L}^1(E_j^\delta\cap L^\varepsilon\cap Y^\varepsilon)}.\end{aligned}$$

Since $a$ is 1-periodic, there exists $\varepsilon_1(\delta) > 0$, $\varepsilon_0(\delta) \ge \varepsilon_1(\delta)$, such that for every $\varepsilon \in \langle 0, \varepsilon_1(\delta)\rangle$ and every $j \in \boldsymbol{N}$ there holds $\int_{\rho_{\varepsilon,\star}E_j^\delta} a \le 2M\Delta + 2\int_{F^M} a$. Hence, for arbitrary $M > 0$ and $\delta \in \langle 0, \Delta\rangle$ we recover

$$\limsup_{\varepsilon\longrightarrow 0}\varepsilon_\star^\beta\varepsilon^{-2/3}\int_{E^\delta}\left(\varepsilon^{2-2\beta}w_\star^{\varepsilon''2} + W(w_\star^{\varepsilon'}) + a_\varepsilon\varepsilon^{2\beta}w_\star^{\varepsilon 2}\right) \le O(\overline{M}_\eta^2)\left(M\Delta + \int_{F^M} a\right).$$

In effect, we get

$$\limsup_{\varepsilon\longrightarrow 0}\varepsilon^{-2/3}I_a^\varepsilon(w^\varepsilon) \le E_0\overline{a^{1/3}} + \eta + O(\overline{M}_\eta^2)\left(M\Delta + \int_{F^M} a\right). \tag{16}$$

At last, we pass to the limit in (16) (first as $\delta \longrightarrow 0$ and $\Delta \longrightarrow 0$, then as $M \longrightarrow +\infty$ and finally as $\eta \longrightarrow 0$), getting the upper bound. □

Now we consider the case $\beta = 1/3$. As a preparation, we minimize $f_\alpha^M$ for arbitrary $\alpha > 0$ in the case when $M > 0$ is large enough.

PROPOSITION 3.2. – *For every $\alpha > 0$ there exists $M_0(\alpha) > 0$ such that for every $M \ge M_0(\alpha)$ there exists (unique up to a translation) $M$-periodic sawtooth function $\overline{x}_{M,\alpha} \in \mathcal{S}_{per}\langle 0, M\rangle$ which minimizes $f_\alpha^M$ on $\mathcal{S}_{per}\langle 0, M\rangle$. Furthermore, there holds*

$$\lim_{M\longrightarrow+\infty}\min_{x\in\mathcal{S}_{per}\langle 0,M\rangle} f_\alpha^M(x) = \lim_{M\longrightarrow+\infty}\min_{x\in\mathcal{S}\langle 0,M\rangle} f_\alpha^M(x) = E_0\alpha^{1/3}. \tag{17}$$

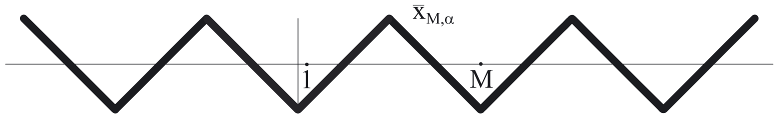


Fig. 2. – The symmetric sawtooth function $\overline{x}_{M,\alpha} \in \mathcal{S}_{per}\langle 0, M\rangle$.

PROOF. – Consider $n \in \boldsymbol{N}$ and $x \in \mathcal{S}_{per}\langle 0, M\rangle$ such that $|Sx' \cap [0, M\rangle| = 2n$, where $Sx' = \{t_0, t_1, \ldots, t_{2n-1}, t_{2n}\}$ and $0 \le t_0 < t_1 < \ldots t_{2n-1} < t_{2n} = M$. Set $h_k := t_k - t_{k-1}$, $p_k := \fint_{t_{k-1}}^{t_k} x(\tau)d\tau$, $k = 1, \ldots, 2n$. We introduce a function $g_\alpha : \langle 0, +\infty\rangle \times \boldsymbol{R} \longrightarrow \langle 0, +\infty\rangle$ defined by $g_\alpha(h, p) := \frac{\xi}{h} + \frac{\alpha}{12} h^2 + \alpha p^2$. Then there holds $f_\alpha^M(x) = \sum_{k=1}^{2n} \frac{h_k}{M} g_\alpha(h_k, p_k)$, where $\sum_{k=1}^{2n} h_k = M$. Moreover, we obtain

$$f_\alpha^M(x) \ge \sum_{k=1}^{2n} \frac{h_k}{M} g_\alpha(h_k, 0) = 2n\frac{\xi}{M} + \frac{\alpha}{12M} \sum_{k=1}^{2n} h_k^3 \ ,$$

where equality is achieved by $M$-periodic sawtooth function shown in Figure 2. It is easy to see that the solution of the minimization problem for the function $(h_1, \ldots, h_{2n}) \mapsto \sum_{k=1}^{2n} h_k^3$ with the constraint $\sum_{k=1}^{2n} h_k = M$ is given by $h_j = \frac{M}{2n}$, $j = 1, \ldots, 2n$. Thus, the minimum of $f_\alpha^M$ is achieved by (unique up to a translation) sawtooth function $\overline{x}_{M,\alpha} \in \mathcal{S}_{per}\langle 0, M\rangle$ such that $t_k = k\frac{M}{2n}$, $p_k = 0$, $k = 1, \ldots, 2n$. In particular, there holds $f_\alpha^M(\overline{x}_{M,\alpha}) = 2\xi\frac{n}{M} + \frac{\alpha}{48}\left(\frac{M}{n}\right)^2$. Note that we can set $\lambda := \frac{n}{M}$ and minimize $\lambda \mapsto 2\xi\lambda + \frac{\alpha}{48}\lambda^{-2}$ over all $\lambda > 0$ to conclude that the optimal $\lambda$ is $\lambda_0 := \left(\frac{\alpha}{48\xi}\right)^{1/3}$. If $M > 0$ is fixed, then it can easily be checked that the optimal $n_\star \in \boldsymbol{N}$ is $n_\star := \lceil \lambda_0 M \rceil$ (or $n_\star := \lfloor \lambda_0 M \rfloor$). By $M_0(\alpha)$ we denote the smallest $M > 0$ such that $\lfloor \lambda_0 M \rfloor \ge 1$. Then for every $M \ge M_0(\alpha)$ there exists $\overline{x}_{M,\alpha} \in \mathcal{S}_{per}\langle 0, M\rangle$ which minimizes $f_a^M$ on $\mathcal{S}_{per}\langle 0, M\rangle$. A short computation gives

$$\lim_{M \longrightarrow +\infty} f_\alpha^M(\overline{x}_{M,\alpha}) = \lim_{M \longrightarrow +\infty} \left(2\xi\frac{n_\star}{M} + \frac{\alpha}{48}\frac{M^2}{n_\star^2}\right) = 2\xi\lambda_0 + \frac{\alpha}{48}\lambda_0^{-2} = E_0\alpha^{1/3} \ .$$

Consider now $u_M \in \mathcal{S}\langle 0, M\rangle$ such that there holds $\min_{x \in \mathcal{S}\langle 0, M\rangle} f_\alpha^M(x) = f_\alpha^M(u_M)$ and $v_M \in \mathcal{S}_{per}\langle 0, M\rangle$ shown in Figure 3. Then $|Sv_M' \cap [0, M\rangle| \le |Su_M' \cap [0, M\rangle| + 3$, $|v_M(\tau)| \le |u_M(\tau)|$, $\tau \in \langle 0, M\rangle$, so that

$$(18) \qquad \min_{x \in \mathcal{S}_{per}\langle 0, M\rangle} f_\alpha^M(x) \le f_\alpha^M(v_M) \le f_\alpha^M(u_M) + \frac{3\xi}{M} \ , \ f_\alpha^M(u_M) \le \min_{x \in \mathcal{S}_{per}\langle 0, M\rangle} f_\alpha^M(x) \ .$$

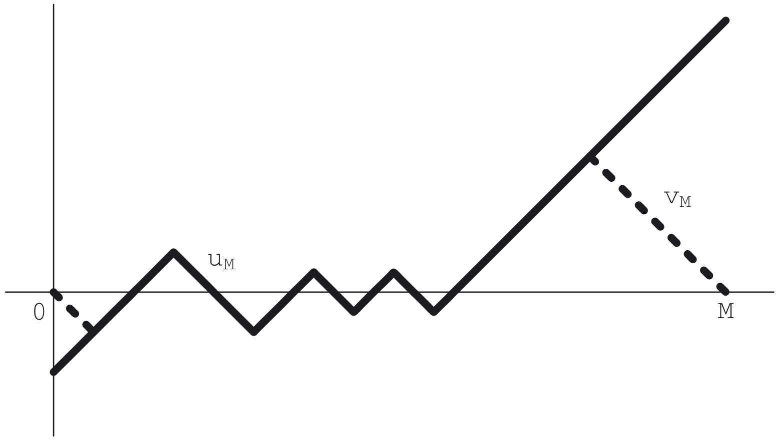


Fig. 3. – Construction of $v_M \in \mathcal{S}_{per}\langle 0, M\rangle$.

To furnish the proof, we pass to the limit as $M \longrightarrow +\infty$ in (18). $\square$

At this point we turn our attention to recovering the lower and the upper bound in the critical case $\beta = 1/3$.

THEOREM 3.3 [the critical case]. – *There holds*

$$\liminf_{\varepsilon \longrightarrow 0} \mathcal{E}_a^{\varepsilon}(1/3) \geq \sup_{M>0} \frac{M}{\lceil M \rceil} \min_{x \in \mathcal{S}\langle 0,M\rangle} f_a^M(x) \,. \tag{19}$$

$$\limsup_{\varepsilon \longrightarrow 0} \mathcal{E}_{a,per}^{\varepsilon}(1/3) \leq \inf_{M>0} \min_{x \in \mathcal{S}_{per}\langle 0,M\rangle} f_a^M(x) \,. \tag{20}$$

PROOF. – First, we obtain the lower bound (19). Let $M > 0$. Since for $v \in \mathrm{H}^2\langle 0, M\varepsilon^{1/3}\rangle$ there holds $\varepsilon^{-2/3} I_{a,M}^{\varepsilon,\star}(v) = f_a^{\varepsilon,M}(v_\star)$, where $v_\star(s) := \varepsilon^{-1/3} v(\varepsilon^{1/3} s)$, by Proposition 2.1 and (8) there exists a sequence $(u_{\varepsilon,\star})$ which satisfy $u_{\varepsilon,\star} \in \mathrm{H}^2\langle 0, M\rangle$,

$$\lim_{\varepsilon \longrightarrow 0} \mathcal{E}_{a,M}^{\varepsilon,\star}(1/3) = \lim_{\varepsilon \longrightarrow 0} f_a^{\varepsilon,M}(u_{\varepsilon,\star}) = \min_{x \in \mathcal{S}\langle 0,M\rangle} f_a^M(x) \,. \tag{21}$$

Then we infer (19) by an application of Proposition 3.1.

We prove the upper bound (20) in two steps.

Step 1. By Proposition 3.6 in [1] there exists a sequence of 1-Lipschitz functions $(w_{\varepsilon,\star})$ such that $w_{\varepsilon,\star} \in \mathrm{H}^2_{per}\langle 0, M\rangle$, $\lim_{\varepsilon \longrightarrow 0} \mathcal{E}_{a,M,per}^{\varepsilon,\star}(1/3) = \lim_{\varepsilon \longrightarrow 0} f_a^{\varepsilon,M}(w_{\varepsilon,\star}) = \min_{x \in \mathcal{S}_{per}\langle 0,M\rangle} f_a^M(x)$. Consider $w_\varepsilon(s) := \varepsilon^{1/3} w_{\varepsilon,\star}(\varepsilon^{-1/3} s)$, $s \in \langle 0, M\varepsilon^{1/3}\rangle$. Then there

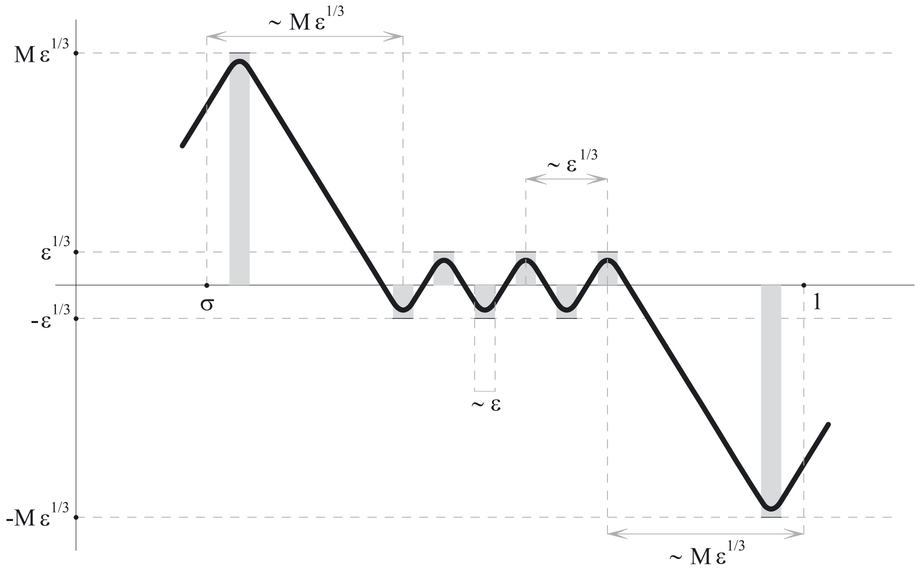


Fig. 4. – Construction of $v_\varepsilon$ on $\langle\sigma,1\rangle$, $\sigma\in\langle 1-2M\varepsilon^{1/3},1\rangle$.

holds $w_\varepsilon\in \mathrm{H}^2_{per}\langle 0,M\varepsilon^{1/3}\rangle$ and

$$\varepsilon^{-2/3}I^{\varepsilon}_{a}(w_\varepsilon)\le \rho^{-1}_{\varepsilon,\star,M}\varepsilon^{-2/3}I^{\varepsilon,\star}_{a,M}(w_\varepsilon)=\rho^{-1}_{\varepsilon,\star,M}f^{\varepsilon,M}_{a}(w_{\varepsilon,\star})\ . \tag{22}$$

Step 2. Since $\|w_{\varepsilon,\star}\|_{\mathrm{L}^\infty(\boldsymbol{R})}\le M$, we get $\|w_\varepsilon\|_{\mathrm{L}^\infty(\boldsymbol{R})}\le M\varepsilon^{1/3}$. To achieve 1-periodicity of the minimizer we adjust behavior of $w_\varepsilon$ near the right edge of the interval $\langle 0,1\rangle$, so as to get $v_\varepsilon\in\mathrm{H}^2_{per}\langle 0,1\rangle$ (cf. Figure 4). Then there holds $\|v_\varepsilon\|_{\mathrm{L}^\infty(\boldsymbol{R})}\le M\varepsilon^{1/3}$, $\|v'_\varepsilon\|_{\mathrm{L}^\infty(\boldsymbol{R})}\le O(1)$. Let $Q^\varepsilon$ ($L^\varepsilon$, resp.) denotes the set of all points in $[\sigma,1]$ where $v_\varepsilon$ is quadratic (linear, resp.). Then $\lambda(Q^\varepsilon)\le O(1)\varepsilon$, $\lambda(L^\varepsilon)\le 2M\varepsilon^{1/3}$. Similarly as in the proof of Theorem 3.2 for sufficiently large ball $B\subset\boldsymbol{R}$ there holds

$$\varepsilon^{-2/3}\int\limits_{Q^\varepsilon}\Big(\varepsilon^2 v''^2_\varepsilon(s)+W(v'_\varepsilon(s))+a(\varepsilon^{-1/3}s)v^2_\varepsilon(s)\Big)ds$$

$$\le O(1)\varepsilon^{1/3}+\|W\|_{\mathrm{L}^\infty(B)}O(1)\varepsilon^{1/3}+M^2\varepsilon^{1/3}\int\limits_{\varepsilon^{-1/3}Q^\varepsilon}a(s)ds\ ,$$

$$\varepsilon^{-2/3}\int\limits_{L^\varepsilon}\big(\varepsilon^2 v''^2_\varepsilon(s)+W(v'_\varepsilon(s))+a(\varepsilon^{-1/3}s)v^2_\varepsilon(s)\big)ds$$

$$\le M^2\varepsilon^{1/3}\int\limits_{\varepsilon^{-1/3}L^\varepsilon}a(s)ds\ .$$

Consequently, we estimate

$$\varepsilon^{-2/3} I_a^\varepsilon(v_\varepsilon) \le \varepsilon^{-2/3} I_a^\varepsilon(w_\varepsilon) + O(1)\varepsilon^{1/3} + O(M^3)\varepsilon^{1/3}\|a\|_{\mathrm{L}^1\langle 0,1\rangle} \ . \tag{23}$$

By passing to the limit as $\varepsilon \longrightarrow 0$ in (23) and (22) we get (20). □

We can now derive the formula for asymptotic energy in the critical case $\beta = 1/3$ in terms of asymptotic behavior when $\xi \longrightarrow 0$ and $\xi \longrightarrow +\infty$.

THEOREM 3.4. – *There holds*

$$\lim_{\xi \longrightarrow 0} \xi^{-2/3} \mathcal{E}_a(1/3) = \lim_{\xi \longrightarrow 0} \xi^{-2/3} \mathcal{E}_{a,per}(1/3) = C_0 \overline{a^{1/3}} \ , \tag{24}$$

$$\lim_{\xi \longrightarrow +\infty} \xi^{-2/3} \mathcal{E}_a(1/3) = \lim_{\xi \longrightarrow +\infty} \xi^{-2/3} \mathcal{E}_{a,per}(1/3) = C_0 \overline{a}^{1/3} \ . \tag{25}$$

PROOF. – First, we consider the proof of (24). Let $u \in \mathcal{S}\langle 0, M\rangle$ ($z \in \mathcal{S}_{per}\langle 0, M\rangle$, resp.) minimizes $f_a^M$ on $\mathcal{S}\langle 0, M\rangle$ ($\mathcal{S}_{per}\langle 0, M\rangle$, resp.). Thus (19) for $M \in \boldsymbol{N}$ gives $\mathcal{E}_a(1/3) \ge f_a^M(u)$. Set $\xi_\star := \lceil \xi^{-1/3} \rceil^{-3}$, $y_s(\tau) := \xi_\star^{-1/3} z(s + \xi_\star^{1/3}\tau)$, $b_s^\xi(\tau) := a(s + \xi_\star^{1/3}\tau)$, $\tau \in \boldsymbol{R}$, $s \in \boldsymbol{R}$. Then there holds $\xi_\star^{-1/3} \in \boldsymbol{N}$, $\xi^{1/3} \ge \xi_\star^{1/3}$, $y_s \in \mathcal{S}_{per}\langle 0, M\xi_\star^{-1/3}\rangle$, $b_s^\xi \longrightarrow a(s)$ in $\mathrm{L}^1_{loc}(\boldsymbol{R})$ as $\xi \longrightarrow 0$ (a.e. $s \in \boldsymbol{R}$). By construction in Figure 3 we estimate

$$\begin{aligned}
\xi^{-2/3} f_a^M(u) \ge& \xi^{1/3} \frac{|Sz' \cap [0, M\rangle|}{M} + \xi^{-2/3} \int_0^M a(s) z^2(s) ds - \frac{3\xi^{1/3}}{M} \\
\ge& \sum_{j=1}^{\xi_*^{-1/3}} \xi_*^{1/3} \int_0^M \left( \frac{|Sy_s' \cap [(j-1)M, jM\rangle|}{M} + \int_0^M b_s^\xi y_s^2 \right) ds - \frac{3\xi^{1/3}}{M} \\
\ge& \int_0^M \min_{y \in \mathcal{S}\langle 0, M\rangle} \varphi_{b_s^\xi}^M(y) ds - \frac{3\xi^{1/3}}{M} \ .
\end{aligned}$$

In the limit as $\xi \longrightarrow 0$ and as $M \longrightarrow +\infty$, we have

$$\liminf_{M \longrightarrow +\infty} \liminf_{\xi \longrightarrow 0} \xi^{-2/3} \min_{x \in \mathcal{S}\langle 0, M\rangle} f_a^M(x) \ge \liminf_{M \longrightarrow +\infty} \int_0^1 \min_{y \in \mathcal{S}\langle 0, M\rangle} \varphi_{a(s)}^M ds \ ,$$

and Fatou's Lemma (combined with Proposition 3.2) yields the lower bound associated to (24). On the other hand, for a given $M \in \boldsymbol{N}$ we consider functionals

$\Phi_a^{\xi,M}, \Phi_a^M : YM(\langle 0,M\rangle;K) \longrightarrow [0,+\infty]$ defined by

$$\Phi_a^{\xi,M}(\boldsymbol{v}) := \begin{cases} \displaystyle\int_0^M \langle v_s, \varphi^M_{b_s^\xi}\rangle ds, & \text{if } v_s = \delta_{\,R_s^\xi x} \text{ for some x} \in \ \mathcal{S}_{\text{per}}\langle 0, \mathrm{M}\xi^{\ 1/\ 3}\rangle \\ +\infty, & \text{otherwise}, \end{cases}$$

$$\Phi_a^{M}(\boldsymbol{v}) := \begin{cases} \displaystyle\int_0^M \langle v_s, \varphi^M_{a(s)}\rangle ds, & \text{if } v_{\text{s}} \in \ \mathcal{I}(\text{K}) \text{ for a.e. } s \in \ \langle 0, M\rangle \\ +\infty, & \text{otherwise}, \end{cases}$$

where $R_s^\xi x$ is $\xi$-blowup of $x$ at point $s$ defined by (2). Similarly as in Theorem 3.1, we can write $\xi^{-2/3} f_a^{M\xi^{1/3}}(x) = \Phi_a^{\xi,M}(\delta_{R_s^\xi x})$, where $x \in \mathcal{S}_{per}\langle 0, M\xi^{1/3}\rangle$. Then there holds $\Phi_a^{\xi,M} \xrightarrow{\Gamma} \Phi_a^M$ on $YM(\langle 0,M\rangle;K)$ as $\xi \longrightarrow 0$. Therefore by (20) and Theorem 3.4 in [1] it results $\limsup\limits_{\xi\longrightarrow 0} \xi^{-2/3}\mathcal{E}_{a,per}(1/3) \le \min\limits_{\boldsymbol{v}} \Phi_a^M(\boldsymbol{v}) = C_0\overline{a^{1/3}}$.

To prove the lower bound in (25) (the upper bound in (25), resp.), we note that by (19) ((20), resp.) there holds

$$\xi^{-2/3}\mathcal{E}_a(1/3) \ge \frac{M\xi^{1/3}}{\lceil M\xi^{1/3}\rceil} \min_{z\in\mathcal{S}\langle 0,M\xi^{1/3}\rangle} \xi^{-2/3} f_a^{M\xi^{1/3}}(z)$$

$$\left(\xi^{-2/3}\mathcal{E}_{a,per}(1/3) \le \min_{z\in\mathcal{S}_{per}\langle 0,M\xi^{1/3}\rangle} \xi^{-2/3} f_a^{M\xi^{1/3}}(z)\ , \ \text{ resp.}\right).$$

For a given $z \in \mathcal{S}\langle 0, M\xi^{1/3}\rangle$ we define $\tilde{z} \in \mathcal{S}_{per}\langle 0, M\xi^{1/3}\rangle$ according to Figure 3, and we set $\tilde{y}_s(\tau) = \xi^{-1/3}\tilde{z}(s+\xi^{1/3}\tau)$, $s,\tau \in \boldsymbol{R}$ (For $\overline{x} \in \mathcal{S}_{per}\langle 0,M\rangle$ which minimizes $f_{\overline{a}}^M$ we define $\overline{z} \in \mathcal{S}_{per}\langle 0, M\xi^{1/3}\rangle$ by $\overline{z}(s) := \xi^{1/3}\overline{x}(\xi^{-1/3}s)$, and we set $\overline{y}_{\xi,s}(\tau) := \xi^{-1/3}\overline{z}(s+\xi^{1/3}\tau)$, $s,\tau\in\boldsymbol{R}$, resp.). Set $c_s^\xi(\tau) := c(s+\xi^{1/3}\tau)$. Then $c_s^\xi \longrightarrow \overline{a}$ in $\mathrm{L}^1_{loc}(\boldsymbol{R})$ as $\xi \longrightarrow +\infty$ (a.e. $s \in \boldsymbol{R}$), $y_s \in \mathcal{S}\langle 0,M\rangle$ ($\overline{y}_{\xi,s} \in \mathcal{S}_{per}\langle 0,M\rangle$, $\overline{y}_{\xi,s} \longrightarrow \overline{x}$ uniformly in $s \in \langle 0,M\rangle$ and $\tau \in \langle 0,M\rangle$ as $\xi \longrightarrow +\infty$, resp.), and there holds

$$\xi^{1/3}\frac{|S\tilde{z}' \cap [0,M\xi^{1/3}\rangle|}{M\xi^{1/3}} + \xi^{-2/3}\int_0^{M\xi^{1/3}} a(s)\tilde{z}^2(s)ds = \int_0^M \varphi^M_{c_s^\xi}(\tilde{y}_s)ds\ ,$$

$$\min_{z\in\mathcal{S}\langle 0,M\xi^{1/3}\rangle} \xi^{-2/3} f_a^{M\xi^{1/3}}(z) \ge \int_0^M \min_{y\in\mathcal{S}\langle 0,M\rangle} \varphi^M_{c_s^\xi}(y)ds - \frac{3}{M}$$

$$\left(\xi^{-2/3} f_a^{M\xi^{1/3}}(\overline{z}) = \int_0^M \varphi^M_{c_s^\xi}(\overline{y}_{\xi,s})ds\ , \ \text{ resp.}\right).$$

In the limit as $\xi \longrightarrow +\infty$, we get $\liminf\limits_{\xi\longrightarrow+\infty} \xi^{-2/3}\mathcal{E}_a(1/3) \ge \min\limits_{y\in\mathcal{S}\langle 0,M\rangle} \varphi^M_{\overline{a}}(y) - \frac{3}{M}$

$\left(\limsup_{\xi \longrightarrow +\infty} \xi^{-2/3}\mathcal{E}_{a,per}(1/3) \leq \varphi^M_{\overline{a}}(\overline{x}),\ \text{resp.}\right)$, which, as we consider the limit as $M \longrightarrow +\infty$, gives the lower bound in (25) (the upper bound in (25), resp.). $\square$

In the next corollary we present some further properties of the rescaled asymptotic energy in the critical case.

COROLLARY 3.1. – *Set* $\mathcal{F}(a,\xi) := \xi^{-2/3} \lim_{M \longrightarrow +\infty} \min_{x \in \mathcal{S}_{per}\langle 0,M\rangle} f^M_a(x)$, $a_\lambda(s) := a(\lambda s)$, $s \in \boldsymbol{R}$, $\lambda > 0$. *Then there holds* $\mathcal{F}(a,\xi) = \xi^{-2/3}\mathcal{E}_a(1/3) = \xi^{-2/3}\mathcal{E}_{a,per}(1/3)$,

(i) $\lim_{\xi \longrightarrow 0} \mathcal{F}(a,\xi) = C_0\overline{a^{1/3}}$, $\lim_{\xi \longrightarrow +\infty} \mathcal{F}(a,\xi) = C_0\overline{a}^{1/3}$.

(ii) *if* $\xi_n, \xi_\infty > 0$, $\xi_n \longrightarrow \xi_\infty$, *and* $a_n \rightharpoonup a$ *in* $\mathrm{L}^1_{loc}(\boldsymbol{R})$ *as* $n \longrightarrow +\infty$, *then* $\lim_{n \longrightarrow +\infty} \mathcal{F}(a_n,\xi_n) = \mathcal{F}(a,\xi_\infty)$.

*Furthermore, if* $\xi = \xi(\lambda)$, *then there holds:*

(iii) $\lim_{\lambda \longrightarrow +\infty} \lambda\xi^{1/3}(\lambda) = 0 \left(\lim_{\lambda \longrightarrow +\infty} \lambda\xi^{1/3}(\lambda) = +\infty,\ \textit{resp.}\right)$ *implies*

$\lim_{\lambda \longrightarrow +\infty} \mathcal{F}(a_\lambda,\xi(\lambda)) = C_0\overline{a^{1/3}}\ ,\ \left(\lim_{\lambda \longrightarrow +\infty} \mathcal{F}(a_\lambda,\xi(\lambda)) = C_0\overline{a}^{1/3}\ ,\ \text{resp.}\right)$.

(iv) $\lim_{\lambda \longrightarrow 0} \lambda\xi^{1/3}(\lambda) = 0 \left(\lim_{\lambda \longrightarrow 0} \lambda\xi^{1/3}(\lambda) = +\infty,\ \textit{resp.}\right)$ *implies*

$\lim_{\lambda \longrightarrow 0} \mathcal{F}(a_\lambda,\xi(\lambda)) = C_0\overline{a^{1/3}}\ ,\ \left(\lim_{\lambda \longrightarrow 0} \mathcal{F}(a_\lambda,\xi(\lambda)) = C_0\overline{a}^{1/3}\ ,\ \text{resp.}\right)$.

PROOF. – Consider $M \in \boldsymbol{N}$. By construction described in Figure 3, we have

$$\min_{x \in \mathcal{S}_{per}\langle 0,M\rangle} f^M_a(x) - \frac{3\xi}{M} \leq \mathcal{E}_a(1/3) = \mathcal{E}_{a,per}(1/3) \leq \min_{x \in \mathcal{S}_{per}\langle 0,M\rangle} f^M_a(x)\ . \tag{26}$$

As we pass to the limit in (26) as $M \longrightarrow +\infty$, it results $\mathcal{F}(a,\xi) = \xi^{-2/3}\mathcal{E}_a(1/3) = \xi^{-2/3}\mathcal{E}_{a,per}(1/3)$. Thus ($i$) holds. Next, we note that there holds: if $\xi_n \longrightarrow \xi_\infty$ and $a_n \rightharpoonup a$ in $\mathrm{L}^1_{loc}(\boldsymbol{R})$ as $n \longrightarrow +\infty$, then $f^{M,\xi_n}_{a_n} \xrightarrow{\Gamma} f^{M,\xi_\infty}_a$ as $n \longrightarrow +\infty$ on $\mathrm{L}^1\langle 0,M\rangle$ for every $M \in \boldsymbol{N}$. Suppose that $\overline{x}_n$ ($\overline{y}_n$, resp.) minimizes $f^{M,\xi_n}_{a_n}$ on $\mathcal{S}_{per}\langle 0,M\rangle$ ($\mathcal{S}\langle 0,M\rangle$, resp.), while $\overline{x}_\infty$ ($\overline{y}_\infty$, resp.) minimizes $f^{M,\xi_\infty}_a$ on $\mathcal{S}_{per}\langle 0,M\rangle$ ($\mathcal{S}\langle 0,M\rangle$, resp.). Set $M_k := 2^k$, $k \in \boldsymbol{N}$. Then for every $k \in \boldsymbol{N}$ ($M \in \boldsymbol{N}$, resp.) there holds

$$\mathcal{F}(a_n,\xi_n) = \inf_{k \in \boldsymbol{N}} \xi_n^{-2/3} f^{M_k,\xi_n}_{a_n}(\overline{x}_n) \leq \xi^{-2/3} f^{M_k,\xi_n}_{a_n}(\overline{x}_n)$$

$$\left(\mathcal{F}(a_n,\xi_n) \geq \sup_{M \in \boldsymbol{N}} \xi_n^{-2/3} f^{M,\xi_n}_{a_n}(\overline{y}_n) \geq \xi_n^{-2/3} f^{M,\xi_n}_{a_n}(\overline{y}_n)\ ,\ \text{resp.}\right)\ .$$

We pass to the limit as $n \longrightarrow +\infty$, and then as $k \longrightarrow +\infty$ ($M \longrightarrow +\infty$, resp.), getting (ii). Verification of (iii) and (iv) is similar to the proof of Theorem 3.4 and it is left to the interested reader. $\square$

In the end, we briefly explain how to approach the case $\beta > 1/3$.

THEOREM 3.5 [the supercritical case]. – *Let* $\beta > 1/3$. *Then* $\lim_{\varepsilon \longrightarrow 0} \mathcal{E}_a^\varepsilon(\beta) = \lim_{\varepsilon \longrightarrow 0} \mathcal{E}_{a,per}^\varepsilon(\beta) = E_0 \overline{a}^{1/3}$.

PROOF. – As pointed out in [1], p. 814, the claim easily follows by direct application of results in [1], where the case $\beta = 0$ was studied. We consider $\varepsilon$-blowup (2). Then we can write $\varepsilon^{-2/3} I_a^\varepsilon(v) = \int_0^1 f_{\tilde{a}_s^\varepsilon}^\varepsilon (R_s^\varepsilon v) ds$, $v \in \mathrm{H}^2_{per}\langle 0, 1\rangle$, where the only distinction in comparison to the case $\beta = 0$ is that $\tilde{a}_s^\varepsilon : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ is now defined by $\tilde{a}_s^\varepsilon(\tau) := a(\varepsilon^{-\beta} s + \varepsilon^{1/3-\beta}\tau)$. By a version of the McShane lemma there holds $\tilde{a}_s^\varepsilon \rightharpoonup \overline{a}$ in $\mathrm{L}^1_{loc}(\boldsymbol{R})$ (a.e. $s \in \langle 0, 1\rangle$) as $\varepsilon \longrightarrow 0$. Thus we arrive at the conclusion that there holds $f_{\tilde{a}_s^\varepsilon}^\varepsilon \xrightarrow{\Gamma} f_{\overline{a}}$ (a.e. $s \in \langle 0, 1\rangle$) as $\varepsilon \longrightarrow 0$, which, as shown in [1], Theorem 3.4, is sufficient to deduce the claim of the theorem. $\square$

## 4. – Ending Remarks

REMARK 4.1. – The present analysis, in our view, shows that minimizers $v^\varepsilon \in \mathrm{H}^2_{per}\langle 0, 1\rangle$ of $I_a^\varepsilon$ develop a sawtooth-like behavior. Consider a subsequence $\varepsilon_k := \left(\frac{1}{k}\right)^{1/\beta}$ and $a \in \mathrm{L}^\infty_{per}\langle 0, 1\rangle$ ($a \neq$ const.). Set $L_0 := (48\xi)^{1/3}$. Let $\beta \in \langle 0, 1/3\rangle$. By Theorem 1.1 in [7] and an approximation of $a$ by simple functions we can establish the error estimate $\mathcal{E}_{a,per}^{\varepsilon_k}(\beta) = E_0 \overline{a^{1/3}} + O(\varepsilon_k^{2/3-\beta})$. We define $v^{\varepsilon_k, j}(s) := v^{\varepsilon_k}(s + (j-1)\varepsilon_k^\beta)$, $s \in \langle 0, \varepsilon_k^\beta\rangle$, $j \in \mathbf{Z}$. Since we can write ${\varepsilon_k}^{-2/3} I_a^{\varepsilon_k}(v^{\varepsilon_k}) = \sum_{j=1}^{\varepsilon_k^{-\beta}} \varepsilon_k^\beta {\varepsilon_k}^{-2/3} I_a^{\varepsilon_k,\star}(v^{\varepsilon_k, j})$, for every $j = 1, \ldots, \varepsilon_k^{-\beta}$ we get ${\varepsilon_k}^{-2/3} I_a^{\varepsilon_k,\star}(v^{\varepsilon_k,j}) = E_0 \overline{a^{1/3}} + O(\varepsilon_k^{2/3-2\beta})$, so that for every $j \in \mathbf{Z}$ (by periodicity of $v^{\varepsilon_k}$) there holds $\lim_{k \longrightarrow +\infty} \mathcal{E}_a^{\varepsilon_k,\star}(\beta) = \lim_{k \longrightarrow +\infty} {\varepsilon_k}^{-2/3} I_a^{\varepsilon_k,\star}(v^{\varepsilon_k,j}) = E_0 \overline{a^{1/3}}$. In particular, by (12) and Corollary 3.13 in [1] for every $j \in \mathbf{Z}$ there holds $\delta_{R_s^{\varepsilon_k,\star} v_\star^{\varepsilon_k,j}} \xrightarrow{*} \varepsilon_{\overline{x}^s}$ (a.e. $s \in \langle 0, 1\rangle$) as $k \longrightarrow +\infty$, where $\varepsilon_{\overline{x}^s}$ is the unique probability measure, invariant with respect to translations, supported on the orbit of $L_0(a(s))^{-1/3}$-periodic sawtooth function $\overline{x}^s$ with zero average and two corners per period (see [1], p. 778, p. 790). We believe that the later convergence provides a fairly good interpretation of geometric properties of the minimizing sequence $(v^{\varepsilon_k})$. In the language adopted from [1], p. 763, we conclude that every minimizer $v^{\varepsilon_k}$ for $\varepsilon_k \approx 0$ in the neighborhood of almost every point $s \in \langle 0,1\rangle$ resembles a periodic sawtooth function with minimal period $L_0(a(\varepsilon_k^{-\beta}s))^{-1/3}\varepsilon_k^{1/3}$. If $\beta = 1/3$, similar, but more careful, analysis reveals that minimizers $v^{\varepsilon_k}$ satisfy $v_\star^{\varepsilon_k,j} \longrightarrow \overline{x}_a$ in $\mathrm{L}^1\langle 0,1\rangle$, where $j \in \mathbf{Z}$ and $\overline{x}_a \in \mathcal{S}\langle 0,1\rangle$ minimizes $f_a^1$ (note that minimizers of $f_a^1$ by no means necessarily belong to the class $\mathcal{S}_{per}\langle 0,1\rangle$). In the case $\beta > 1/3$ the interpretation can be directly deduced from the convergence $\delta_{R_s^\varepsilon v^\varepsilon} \xrightarrow{*} \varepsilon_{\overline{y}}$, (a.e. $s \in \langle 0,1\rangle$) as $\varepsilon \longrightarrow 0$, where $\overline{y}$ is $L_0(\overline{a})^{-1/3}$-periodic sawtooth function with zero average and two corners per period.

REMARK 4.2. – As a further example of variational problem which involves multiple small scales, we can consider the functional defined by

$$\mathcal{J}^\varepsilon_{a,\beta,\gamma}(v) := \int_0^1 \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(\varepsilon^{-\beta}s, \varepsilon^{-\gamma}s)v^2(s)\Big)ds\ , \tag{27}$$

where $v \in \mathrm{H}^2_{per}\langle 0,1\rangle$, $a \in \mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$ is Carathéodory function, and $\beta,\gamma > 0$. Then we can adapt calculations in Section 3 so as to compute rescaled asymptotic energy $\mathcal{E}_a(\beta,\gamma)$ associated to (27) as $\varepsilon \longrightarrow 0$. It can be shown that $\mathcal{E}_a(\beta,\gamma)$ is equal to $E_0 \int_0^1\int_0^1 a^{1/3}(\tau_1,\tau_2)d\tau_1 d\tau_2$, when $\beta,\gamma \in \langle 0,1/3\rangle$, while it is equal to $E_0\overline{a}^{1/3}$, when $\beta,\gamma > 1/3$. On the other hand, if $\beta \in \langle 0,1/3\rangle$ and $\gamma > 1/3$, $\mathcal{E}_a(\beta,\gamma)$ takes value $E_0 \int_0^1 \Big[\int_0^1 a(\tau_1,\tau_2)d\tau_2\Big]^{1/3} d\tau_1$.

*Acknowledgments*. This research would have been impossible without help and encouragement of prof. Stefan Müller. The author wishes to thank the staff of the Max Planck Institute for Mathematics in the Sciences, Leipzig, for their effort to make his visits comfortable.

## REFERENCES

[1] G. ALBERTI - S. MÜLLER, *A new approach to variational problems with multiple scales*, Comm. Pure Appl. Math., **54** (2001), 761-825.

[2] J. M. BALL, *A version of the fundamental theorem for Young measures*, in PDE's and Continuum Models of Phase Transitions (M. Rascle and al., eds.), Lecture Notes in Physics, **344**, Springer, Berlin 1989.

[3] R. CHOKSI, *Scaling laws in microphase separation of diblock copolymers*, J. Nonlinear Sci., **11** (2001), 223-236.

[4] G. DALMASO, *An Introduction to Γ-convergence, Progress in Nonlinear Differential Equations*, Birkhäuser, Boston 1993.

[5] R. V. KOHN - S. MÜLLER, *Branching of twins near an austensite-twinned-martensite interface*, Philosophical Magazine A, **66** (1992), 697-715.

[6] L. Modica - S. Mortola, *Un esempio di $\Gamma$-convergenca*, Boll. Un. Mat. Ital. (5), **14-B** (1977), 285-299.

[7] S. Müller, *Singular perturbations as a selection criterion for minimizing sequences*, Calc. Var., **1** (1993), 169-204.

[8] T. Ohta - K. Kawasaki, *Equilibrium morphology of block copolymer melts*, Macromolecules, **19** (1986), 2621-2632.

[9] A. Raguž, *Relaxation of Ginzburg-Landau functional with* 1*-Lipschitz penalizing term in one dimension by Young measures on micropatterns*, Asymptotic Anal., **41** (3,4) (2005), 331-361.

[10] L. C. Young, *Lectures on the calculus of variations and optimal control theory*, Chelsea, 1980.

Department of Mathematics
University of Zagreb, Bijenicka 30, 10 000 Zagreb, Croatia
E-mail: andrija@math.hr

*Pervenuta in Redazione*
*il 31 marzo 2006*