Antonio Aizpuru, Francisco J. García-Pacheco

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L^2-Summand Vectors and Complemented Hilbertizable Subspaces.

A. Aizpuru - F. J. García-Pacheco

Sunto. – *In questo articolo, mostriamo una condizione necessaria e sufficiente affinché uno spazio di Banach reale abbia un sottospazio infinito dimensionale il quale sia hilbertizzabile e complementato, usando tecniche relazionate con vettori* L^2*-summand.*

Summary. – *In this paper, we show a necessary and sufficient condition for a real Banach space to have an infinite dimensional subspace which is hilbertizable and complemented using techniques related to* L^2*-summand vectors.*

1. – Introduction.

This article is aimed to characterize whether an infinite dimensional real Banach space has an infinite dimensional, hilbertizable, complemented subspace. We will use an inductive process based upon the concept of "L^2-summand".

Definition 1 [Behrends, 1976]. – *Let X be a real Banach space. Let M be a closed subspace of X. Then, M is said to be an* L^2*-summand subspace of X if there exists another closed subspace N of X verifying* $X = (M \oplus N)_2$, *in other words,* $\|m+n\|^2 = \|m\|^2 + \|n\|^2$ *for every* $m \in M$ *and every* $n \in N$. *The linear projection* π_M *of X onto M that fixes the elements of M and maps the elements of N to* $\{0\}$ *is called the* L^2*-summand projection of X onto M.*

We refer the reader to [2] and [3] for a wider perspective about L^2-summand subspaces.

Definition 2 [Carlson and Hicks, 1978]. – *Let X be a real Banach space. Let* $e \in X$. *Then, e is said to be an* L^2 *-summand vector of X if* $\mathbb{R}e$ *is an* L^2*-summand subspace of X. The set of all* L^2*-summand vectors of X is denoted by* L^2_X.

The references [5], [4], and [1] show a wide perspective of L^2-summand vectors. To finish the introduction, we will recall the most relevant and important results about L^2 -summand vectors. We will begin by a characterization of Hilbert spaces in terms of L^2 -summand vectors (see [5]).

THEOREM 1 [Carlson and Hicks, 1978]. – *Let X be a real Banach space. The following conditions are equivalent:*

1. *The space X is Hilbert.*
2. *The set L_X^2 is dense in X.*

The previous result was improved to obtain the next characterization (see [4]).

THEOREM 2 [Becerra-Guerrero and Rodríguez-Palacios, 2002]. – *Let X be a real Banach space. The following conditions are equivalent:*

1. *The space X is Hilbert.*
2. *The set L_X^2 has nonempty interior.*

Finally, the next result determines the linear structure of the set of L^2 - summand vectors (see [1]).

THEOREM 3 [Aizpuru and García-Pacheco, 2006]. – *Let X be a real Banach space. Then, the set L_X^2 is an L^2-summand subspace of X.*

2. – Results.

In this section we will state and prove our main results. We will begin by the following lemma, upon which we will base the whole inductive process.

LEMMA 1. – *Let X be a real Banach space. Consider an element $e \in \ker\left(\pi_{\mathsf{L}_X^2}\right)$. There exists an equivalent norm $\lfloor\cdot\rfloor$ on X such that*

1. *$\mathsf{L}_X^2 + \mathbb{R}e$ is contained in L_Y^2, and*
2. *$\ker\left(\pi_{\mathsf{L}_Y^2}\right) + \mathbb{R}e$ is contained in $\ker\left(\pi_{\mathsf{L}_X^2}\right)$;*

where Y denotes the space X endowed with the norm $\lfloor\cdot\rfloor$.

PROOF. – Consider a closed maximal subspace F of $\ker\left(\pi_{\mathsf{L}_X^2}\right)$ such that $F \oplus \mathbb{R}e = \ker\left(\pi_{\mathsf{L}_X^2}\right)$. Take $M = \mathsf{L}_X^2 + F$. We have that M is a closed maximal subspace of X verifying that $M \oplus \mathbb{R}e = X$ and L_X^2 is contained in M. Consider the equivalent norm on X defined, for every $y \in X$, by the formula $\lfloor y\rfloor = \sqrt{\|m\|^2 + \|\delta e\|^2}$, being m in M, δ a real number, and $y = m + \delta e$.

1. It is not difficult to check that e is an L^2-summand vector of Y and $M = \ker(\pi^Y_{\mathbb{R}e})$. Consider an element $u \in \mathsf{L}^2_X$. Let n and γ be in $\ker(\pi^X_{\mathbb{R}u})$ and $\mathbb{R}$, respectively. We can write $n = m + \delta e$ with $m \in M$ and $\delta \in \mathbb{R}$. Then, since $e \in \ker\left(\pi_{\mathsf{L}^2_X}\right) \subseteq \ker(\pi^X_{\mathbb{R}u})$ and $m = n - \delta e \in \ker(\pi^X_{\mathbb{R}u})$, we have that
$$\begin{aligned}\lfloor n + \gamma u\rfloor^2 &= \|m + \gamma u\|^2 + \|\delta e\|^2\\ &= \left(\|m\|^2 + \|\gamma u\|^2\right) + \|\delta e\|^2\\ &= \left(\|m\|^2 + \|\delta e\|^2\right) + \|\gamma u\|^2\\ &= \lfloor n\rfloor^2 + \lfloor \gamma u\rfloor^2.\end{aligned}$$
Notice that $\ker(\pi^Y_{\mathbb{R}u}) = \ker(\pi^X_{\mathbb{R}u})$. Observe also that $\mathsf{L}^2_Y = \mathsf{L}^2_X + \mathbb{R}e + \mathsf{L}^2_F$.

2. It will be enough to prove that $\ker\left(\pi_{\mathsf{L}^2_Y}\right)$ is contained in F. Let $g \in \ker\left(\pi_{\mathsf{L}^2_Y}\right)$. First, we will see that $g \in \ker\left(\pi_{\mathsf{L}^2_X}\right)$. We can write $g = v + h$ with $v \in \mathsf{L}^2_X$ and $h \in \ker\left(\pi_{\mathsf{L}^2_X}\right)$. We can also write $h = f + \lambda e$ with $f \in F$ and $\lambda \in \mathbb{R}$. Then, $\lfloor h\rfloor^2 = \|f\|^2 + \|\lambda e\|^2$ and $\lfloor g\rfloor^2 = \|v + f\|^2 + \|\lambda e\|^2$. Therefore,
$$\begin{aligned}\lfloor h\rfloor^2 &= \lfloor g - v\rfloor^2\\ &= \lfloor g\rfloor^2 + \lfloor v\rfloor^2\\ &= \|v + f\|^2 + \|\lambda e\|^2 + \lfloor v\rfloor^2\\ &= \|v\|^2 + \|f\|^2 + \|\lambda e\|^2 + \|v\|^2\\ &= 2\|v\|^2 + \lfloor h\rfloor^2.\end{aligned}$$
Thus, $v = 0$ and $g = h \in \ker\left(\pi_{\mathsf{L}^2_X}\right)$. Now, let us see that $g \in F$. On the one hand, we have that $\lfloor g\rfloor^2 = \|f\|^2 + \|\lambda e\|^2 = \lfloor f\rfloor^2 + \lfloor \lambda e\rfloor^2$. On the other hand, we have that $f = g - \lambda e$, therefore $\lfloor f\rfloor^2 = \lfloor g\rfloor^2 + \lfloor \lambda e\rfloor^2$. Then, $2\lfloor \lambda e\rfloor^2 = 0$ and this implies that $\lambda = 0$ and $g = f \in F$. Finally, observe that $\ker\left(\pi_{\mathsf{L}^2_Y}\right) = \ker\left(\pi_{\mathsf{L}^2_F}\right)$.

Notice that the previous lemma directly implies the following corollary.

COROLLARY 1. – *Let X be an infinite dimensional, real Banach space. Then, there exists three sequences* $\left(\|\cdot\|_n\right)_{n\geq 0}$, $(\mathsf{H}_n)_{n\geq 0}$, *and* $(\mathsf{E}_n)_{n\geq 0}$ *verifying*

1. $\|\cdot\|_n$ *is an equivalent norm on X for every* $n \geq 0$,
2. H_n *is a Hilbert subspace of* X_n *and* E_n *is a closed subspace of* X_n *for every* $n \geq 0$,
3. $X_n = (\mathsf{H}_n \oplus \mathsf{E}_n)_2$ *for every* $n \geq 0$,

4. $\mathsf{H}_n \subsetneq \mathsf{H}_{n+1}$ *and* $\mathsf{E}_{n+1} \subsetneq \mathsf{E}_n$ *for every* $n \geq 0$,
5. $\|\cdot\|_{n+1} = \|\cdot\|_n$ *on* H_n *and on* E_{n+1} *for every* $n \geq 0$, *and*
6. $\ker\left(\pi_{\mathbb{R}u}^{X_n}\right) = \ker\left(\pi_{\mathbb{R}u}^{X_{n+1}}\right)$ *for every* $u \in \mathsf{H}_n$ *and every* $n \geq 0$;

where X_n *denotes the space* X *endowed with the norm* $\|\cdot\|_n$. *We will call the* 3-*tupla* $\left((\|\cdot\|_n)_{n\geq 0}, (\mathsf{H}_n)_{n\geq 0}, (\mathsf{E}_n)_{n\geq 0}\right)$ *as canonical sequence of Hilbert spaces for* X.

Before stating the main result, we will prove the following proposition, which will be really helpful.

PROPOSITION 1. – *Let* X *be an infinite dimensional, real Banach space. Consider* $\left((\|\cdot\|_n)_{n\geq 0}, (\mathsf{H}_n)_{n\geq 0}, (\mathsf{E}_n)_{n\geq 0}\right)$ *to be a canonical sequence of Hilbert spaces for* X. *Then:*

1. *If* $\operatorname{cl}\left(\bigcup_{n\geq 0} \mathsf{H}_n\right)$ *is reflexive, then* $X = \operatorname{cl}\left(\bigcup_{n\geq 0} \mathsf{H}_n\right) + \bigcap_{n\geq 0} \mathsf{E}_n$.
2. *The map* $\|\cdot\|_\infty = \lim_{n\to\infty} (\|\cdot\|_n)$ *is well defined and is a norm on* $\bigcup_{n\geq 0} \mathsf{H}_n + \bigcap_{n\geq 0} \mathsf{E}_n$ *such that* $\|v+e\|_\infty^2 = \|v\|_\infty^2 + \|e\|_\infty^2$ *for every* $v \in \bigcup_{n\geq 0} \mathsf{H}_n$ *and every* $e \in \bigcap_{n\geq 0} \mathsf{E}_n$. *This norm which will be called canonical sequence norm.*
3. *Every* $u \in \bigcup_{n\geq 0} \mathsf{H}_n$ *is an* L^2-*summand vector of* $\bigcup_{n\geq 0} \mathsf{H}_n + \bigcap_{n\geq 0} \mathsf{E}_n$ *in the norm* $\|\cdot\|_\infty$.
4. *If* $\|\cdot\|_\infty$ *and* $\|\cdot\|$ *are equivalent norms on* $\bigcup_{n\geq 0} \mathsf{H}_n + \bigcap_{n\geq 0} \mathsf{E}_n$, *then we have that* $\operatorname{cl}\left(\bigcup_{n\geq 0} \mathsf{H}_n\right) \cap \bigcap_{n\geq 0} \mathsf{E}_n = \{0\}$.

PROOF. –

1. Consider $x \in X$. For every $n \geq 0$, there exist $u_n \in \mathsf{H}_n$ and $g_n \in \mathsf{E}_n$ such that $x = u_n + g_n$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded and $\operatorname{cl}\left(\bigcup_{n\geq 0} \mathsf{H}_n\right)$ is reflexive, we can take a subsequence $\left(u_{n_j}\right)_{j\geq 0}$ which is ω-convergent to some $u \in \operatorname{cl}\left(\bigcup_{n\geq 0} \mathsf{H}_n\right)$. It is clear that $\left(g_{n_j}\right)_{j\geq 0}$ is ω-convergent to $x - u$. Fix $n \geq 0$. Let $j_0 \geq 0$ such that $n_{j_0} \geq n$. Then, for every $j \geq j_0$ we have that $g_{n_j} \in \mathsf{E}_{n_j} \subseteq \mathsf{E}_{n_{j_0}} \subseteq \mathsf{E}_n$. Therefore, $x - u \in \mathsf{E}_n$. Since n is arbitrary, we have that $x - u \in \bigcap_{n\geq 0} \mathsf{E}_n$.
2. First we will see that $\|\cdot\|_\infty$ is well defined. If $y \in \bigcup_{n\geq 0} \mathsf{H}_n$, then there exists $m \geq 0$ such that $y \in \mathsf{H}_m$, therefore $\|y\|_n = \|y\|_m$ for every $n \geq m$ (since $\|\cdot\|_{n+1} = \|\cdot\|_n$ on H_n for every $n \geq 0$), and hence $\|y\|_\infty = \|y\|_m$. If $y \in \bigcap_{n\geq 0} \mathsf{E}_n$,

then $\|y\|_n = \|y\|_0$ for every $n \geq 0$ (since $\|\cdot\|_{n+1} = \|\cdot\|_n$ on E_{n+1} for every $n \geq 0$), therefore $\|y\|_\infty = \|y\|_0$. If $y = u + g$ with $u \in \bigcup_{n\geq 0} \mathsf{H}_n$ and $g \in \bigcap_{n\geq 0} \mathsf{E}_n$, then take $m \geq 0$ such that $u \in \mathsf{H}_m$, so for every $n \geq m$ we have that $\|y\|_n = \sqrt{\|u\|_n^2 + \|g\|_n^2} = \sqrt{\|u\|_m^2 + \|g\|_0^2}$, and thus $\|y\|_\infty = \sqrt{\|u\|_m^2 + \|g\|_0^2}$. Now, it is not difficult to check that $\|\cdot\|_\infty$ is a norm on $\bigcup_{n\geq 0} \mathsf{H}_n + \bigcap_{n\geq 0} \mathsf{E}_n$ such that $\|v + e\|_\infty^2 = \|v\|_\infty^2 + \|e\|_\infty^2$ for every $v \in \bigcup_{n\geq 0} \mathsf{H}_n$ and every $e \in \bigcap_{n\geq 0} \mathsf{E}_n$.

3. Let $m \geq 0$ and take $u \in \mathsf{H}_m$. Consider a real number δ and let h be in $\ker(\pi_{\mathbb{R}u}^{X_m}) \cap \left(\bigcup_{n\geq 0} \mathsf{H}_n + \bigcap_{n\geq 0} \mathsf{E}_n\right)$. We have that $\ker(\pi_{\mathbb{R}u}^{X_m}) = \ker(\pi_{\mathbb{R}u}^{X_n})$ for every $n \geq m$, therefore $\|h + \delta u\|_n^2 = \|h\|_n^2 + \|\delta u\|_n^2$ for every $n \geq m$, and hence $\|h + \delta u\|_\infty^2 = \|h\|_\infty^2 + \|\delta u\|_\infty^2$.

4. Let x be in $\mathrm{cl}\left(\bigcup_{n\geq 0} \mathsf{H}_n\right) \cap \bigcap_{n\geq 0} \mathsf{E}_n$ and consider a sequence $(x_n)_{n\in\mathbb{N}}$ in $\bigcup_{n\geq 0} \mathsf{H}_n$ which converges to x. For every $n \in \mathbb{N}$, we have that $\|x_n + x\|_\infty^2 = \|x_n\|_\infty^2 + \|x\|_\infty^2$. So, if $n \to \infty$ then we deduce that $4\|x\|_\infty^2 = 2\|x\|_\infty^2$, therefore $x = 0$.

Finally, we are ready to state and prove the main result in this paper.

THEOREM 4. – *Let X be an infinite dimensional, real Banach space. The following are equivalent:*

1. *The space X has an infinite dimensional, closed subspace which is hilbertizable and complemented.*
2. *The space X has a canonical sequence of Hilbert spaces whose canonical sequence norm is equivalent to the original norm.*

PROOF. – Assume that 1 holds. Then, there exists an equivalent norm $\|\cdot\|_\infty$ on X such that $X_\infty = (\mathsf{H} \oplus \mathsf{E})_2$, where X_∞ denotes the space X endowed with the norm $\|\cdot\|_\infty$, H is an infinite dimensional, separable Hilbert subspace of X_∞, and E is a closed subspace of X_∞. Let $(e_n)_{n\geq 0}$ be a countable orthonormal system in H. For $n = 0$, we take $\|\cdot\|_0 = \|\cdot\|_\infty$, $\mathsf{H}_0 = \{0\}$, and $\mathsf{E}_0 = X$. For $n \in \mathbb{N}$, we take $\|\cdot\|_n = \|\cdot\|_\infty$, $\mathsf{H}_n = \mathrm{span}\{e_0, \ldots, e_{n-1}\}$, and $\mathsf{E}_n = \mathrm{span}\{e_0, \ldots, e_{n-1}\}^\perp \oplus \mathsf{E}$. It is not difficult to check that $\left((\|\cdot\|_n)_{n\geq 0}, (\mathsf{H}_n)_{n\geq 0}, (\mathsf{E}_n)_{n\geq 0}\right)$ is a canonical sequence of Hilbert spaces such that the respective canonical sequence norm is equivalent to the original norm. Conversely, assume that 2 holds. Let $\left((\|\cdot\|_n)_{n\geq 0}, (\mathsf{H}_n)_{n\geq 0}, (\mathsf{E}_n)_{n\geq 0}\right)$ be a canonical sequence of Hilbert spaces such that the respective canonical sequence norm is equivalent to the original norm. First, observe that $\|\cdot\|_\infty$ can be extended to a norm on $\mathrm{cl}\Big(\bigcup_{n\geq 0} \mathsf{H}_n\Big) + \bigcap_{n\geq 0} \mathsf{E}_n$, which is equivalent to the original norm of X. Indeed, since the function $\|\cdot\|_\infty$ is Lipschitz on $\bigcup_{n\geq 0} \mathsf{H}_n + \bigcap_{n\geq 0} \mathsf{E}_n$ in the norm of X, it has a unique continuous extension to $\mathrm{cl}\Big(\bigcup_{n\geq 0} \mathsf{H}_n\Big) + \bigcap_{n\geq 0} \mathsf{E}_n$; this extension is easily seen to be a norm which is equivalent to the norm of X. Finally, let us see that $X = \mathrm{cl}\Big(\bigcup_{n\geq 0} \mathsf{H}_n\Big) \oplus \Big(\bigcap_{n\geq 0} \mathsf{E}_n\Big)$. By the paragraph 3 of Proposition 1, every $u \in \bigcup_{n\geq 0} \mathsf{H}_n$ is an L^2-summand vector of $\bigcup_{n\geq 0} \mathsf{H}_n$. Therefore, by Theorem 1, we deduce that $\mathrm{cl}\Big(\bigcup_{n\geq 0} \mathsf{H}_n\Big)$ is a Hilbert space for $\|\cdot\|_\infty$ (and thus, reflexive). By the paragraph 1 of Proposition 1, $X = \mathrm{cl}\Big(\bigcup_{n\geq 0} \mathsf{H}_n\Big) + \Big(\bigcap_{n\geq 0} \mathsf{E}_n\Big)$. And, by the paragraph 4 of Proposition 1, $\mathrm{cl}\Big(\bigcup_{n\geq 0} \mathsf{H}_n\Big) \cap \Big(\bigcap_{n\geq 0} \mathsf{E}_n\Big) = \{0\}$.

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A. Aizpuru: Departamento de Matemáticas
Universidad de Cádiz, Puerto Real, Spain 11510, The European Union
E-mail: antonio.aizpuru@uca.es

F. J. García-Pacheco: Department of Mathematical Sciences
Kent State University, Kent, Ohio 44242, The United States
E-mail: fgarcia@math.kent.edu

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