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On some Orthogonality Relations in Real Normed Spaces and Characterizations of Inner Products.

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Sunto. – *Usando alcuni funzionali soddisfacenti condizioni molto più generali degli usuali assiomi dei prodotti scalari e considerando alcune deboli versioni delle relazioni di ortogonalità negli spazi reali normati, troviamo nuove caratterizzazioni dei prodotti scalari nei casi delle ortogonalità di James e Pitagora, ma non nel caso dell'ortogonalità di Birkhoff.*

Summary. – *Using some functionals which fulfil much more general requirements than the usual axioms of inner products and by considering some weak versions of orthogonal relations in real normed spaces we find new characterizations of inner products in the cases of James and Pythagoras orthogonalities but we show that this is not the case when Birkhoff orthogonality is postulated.*

In a real normed space $(E, \| \ \|)$ the mappings $\rho'_{\pm}$ associated to a norm $\| \ \|$ and defined by

$$\rho'_{\pm}(x,y) = \lim_{t \to 0^{\pm}} \frac{\|x+ty\|^2 - \|x\|^2}{2t}$$

generalize inner products and play a crucial role in many geometrical aspects formulated in normed spaces (see, e.g. [1], [2], [3]).

The basic properties of $\rho'_{\pm}$ to be used frequently are the following

(a) $\rho'_{\pm}(x,x) = \|x\|^2$ and $|\rho'_{\pm}(x,y)| \le \|x\| \, \|y\|$
(b) $\rho'_{-}(x,y) \le \rho'_{+}(x,y)$
(c) $\rho'_{\pm}(ax,y) = \rho'_{\pm}(x,ay) = a\rho'_{\pm}(x,y)$, $a \ge 0$
(d) $\rho'_{+}(ax,y) = \rho'_{+}(x,ay) = a\rho'_{-}(x,y)$, $a \le 0$
(e) $\rho'_{\pm}(x,ax+y) = a\|x\|^2 + \rho'_{\pm}(x,y)$, $a \in \mathbb{R}$
(f) $\rho'_{+}(x,y) = \rho'_{+}(y,x)$ for all x, y in E if and only if E is an inner product space.

In this paper we will use the classical orthogonality relations in a real normed space $(E, \| \ \|)$, namely,

- x is *Birkhoff orthogonal* to y ($x \perp_B y$) if $\|x\| \leq \|x + \lambda y\|$ for every λ in $\mathbb{R}$;
- x is *James orthogonal* to y ($x \# y$) if $\|x - y\| = \|x + y\|$;
- x is *Pythagoras orthogonal* to y ($x \perp^P y$) if $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Our aim in this paper is to consider such orthogonal relations in situations where we use functionals which satisfy weaker conditions than those quoted above for ρ'_+ and to show in which cases the required properties force the derivability of the norm from an inner product. Precisely, let $(E, \| \ \|)$ be a real normed space and let F be a function from $E \times E$ into $\mathbb{R}$ such that:

(i) $F(x, x) \neq 0 = F(0, 0)$ whenever $x \neq 0$;

(ii) There exists $\varepsilon > 0$ and a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all x, y in E and $\lambda > 0$:

$$|F(x, x + \lambda y) - F(x, x) - f(\lambda)F(x, y)| \leq \varepsilon;$$

(iii) There exists $\delta > 0$ such that for all x, y in E and $\lambda > 0$

$$|F(\lambda x, y) - F(x, \lambda y)| \leq \delta.$$

The set of conditions (i), (ii) and (iii) is a weaker requirement than that of satisfying all properties (a), (e) and (c), and (ii) and (iii) represent a Hyers-Ulam stability conditions for (e) and (c).

Remark. – Conditions (i), (ii) and (iii) are independent even in dimension 1. To see this consider the following exemples:

(a) The function $F(x, y) = 0$ for all x, y satisfies (ii) and (iii) but not (i).

(b) Given $\delta > 0$, the function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $F(x, y) = x^2$, if $x = y$ and $F(x, y) = \mathrm{Min}\,(xy, \delta)$ if $x \neq y$ satisfies (i) and (iii) but does not satisfy (ii) because the existence of a function f such that

$$|\mathrm{Min}\,(x^2 + \lambda xy, \delta) - x^2 - f(\lambda)\,\mathrm{Min}\,(xy, \delta)| \leq \varepsilon,$$

is a contradiction since the term on the left tends to infinity when x tends to infinity.

(c) The function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $F(x, y) = y$ satisfy (i), (ii) with $f(\lambda) = \lambda$, for all λ, but (iii) fails.

We begin with the technical but crucial result:

Lemma. – *Let $(E, \| \ \|)$ be a real normed space with dim $E \geq 2$. If a functional $F : E \times E \to \mathbb{R}$ satisfies* (i), (ii), (iii) *and the condition*

$$F(x, \lambda x)(F(x, \lambda x) - \lambda\|x\|^2) = 0, \tag{1}$$

for all x in E and λ in $\mathbb{R}^+$, then necessarily

$$F(x, \lambda y) = F(\lambda x, y) = \lambda F(x, y), \tag{2}$$

and

$$F(x, x + \lambda y) = \|x\|^2 + \lambda F(x, y), \tag{3}$$

for all x, y in E and λ in $\mathbb{R}^+$.

PROOF. – Assume that F satisfies (i), (ii), (iii) as well as condition (1). By (1) with $\lambda = 1$ and using (i) we deduce $F(x, x) = \|x\|^2$ for all x in E. Therefore by (ii) with $x = y$, for any $\lambda > 0$ we have

$$|F(x, (1 + \lambda)x) - (1 + f(\lambda))\|x\|^2| \leq \varepsilon$$

and hence

$$\lim_{\|x\| \to \infty} \frac{F(x, (1 + \lambda)x)}{\|x\|^2} = 1 + f(\lambda) \tag{4}$$

for every $x \in E$, $x \neq 0$. It follows that the inequality $F(x, (1 + \lambda)x) > 0$ is satisfied for $\|x\|$ large enough, thus by (1) we also have

$$\lim_{\|x\| \to \infty} \frac{F(x, (1 + \lambda)x)}{\|x\|^2} = \lim_{\|x\| \to \infty} \frac{(1 + \lambda)\|x\|^2}{\|x\|^2} = 1 + \lambda. \tag{5}$$

From (4) and (5) we get $f(\lambda) = \lambda$ for every $\lambda > 0$.

Next, for $b > 0$ the substitution $y = bz$ into (ii) imply

$$|\lambda b F(x, z) - \lambda F(x, bz)| \leq |\lambda b F(x, z) + \|x\|^2 - F(x, x + (\lambda b)z)|$$
$$+ |F(x, x + \lambda(bz)) - \|x\|^2 - \lambda F(x, bz)| \leq 2\varepsilon,$$

i.e, $|bF(x, z) - F(x, bz)| \leq 2\varepsilon/\lambda$ and letting λ to tend to infinity

$$F(x, bz) = bF(x, z), \tag{6}$$

for all $b > 0$. Consequently by (iii) we will obtain

$$|F(bx, z) - bF(x, z)| = |F(bx, z) - F(x, bz)| \leq \delta,$$

and dividing by b and taking limit when b tends to infinity:

$$\lim_{b \to \infty} \frac{F(bx, z)}{b} = F(x, z),$$

i.e., for all $a > 0$

$$F(ax, z) = \lim_{b \to \infty} \frac{F(bax, z)}{b} = \lim_{b \to \infty} a \frac{F(bax, z)}{ba} = aF(x, z). \tag{7}$$

Thus (6) and (7) prove (2). Finally, taking $x = nu$ and $y = nv$ in (ii):

$$|F(nu, nu + \lambda nv) - n^2\|u\|^2 - \lambda F(nu, nv)| \leq \varepsilon$$

but using (6) and (7):

$$|F(u, u + \lambda v) - \|u\|^2 - \lambda F(u, v)| \leq \varepsilon/n^2,$$

so letting n to tend to infinity we obtain (3). $\square$

THEOREM 1. – *Let* $(E, \|,\|)$ *be a real normed space with dim* $E \geq 2$*. A functional* $F : E \times E \to \mathbb{R}$ *satisfies* (i), (ii), (iii) *and the Pythagorean identity*

$$\text{(iv)} \qquad \|x\|^2 = \left\| \frac{F(y,x)}{\|y\|^2} y \right\|^2 + \left\| x - \frac{F(y,x)}{\|y\|^2} y \right\|^2,$$

for all x, y *in E,* $y \neq 0$*, if and only if E is an inner product space whose inner product associated to the norm* $\| \ \|$ *is, precisely, F.*

PROOF. – Assume that F satisfies (i), (ii), (iii) and (iv). Then

$$\|x\|^2\|y\|^4 = [F(y,x)]^2\|y\|^2 + \left\| \|y\|^2 x - F(y,x)y \right\|^2,$$

and for $x = \lambda y$, $\lambda > 0$ we obtain

$$F(y, \lambda y)(F(y, \lambda y) - \lambda\|y\|^2) = 0,$$

which is condition (1), so we can apply the previous lemma and we induce the validity of both (2) and (3). Bearing this in mind and introducing $y = u$ and $x = u + \lambda v$, $\lambda > 0$ into (iv) we get

$$\begin{aligned} \|u + \lambda v\|^2\|u\|^4 &= [F(u, u + \lambda v)]^2\|u\|^2 + \| \|u\|^2(u + \lambda v) - F(u, u + \lambda v)u\|^2 \\ &= (\|u\|^2 + \lambda F(u,v))^2\|u\|^2 \\ &\quad + \left\| \|u\|^2(u + \lambda v) - \|u\|^2 u - \lambda F(u,v)u \right\|^2 \end{aligned}$$

whence

$$\text{(8)} \qquad \|u\|^4(\|u + \lambda v\|^2 - \|u\|^2)$$
$$= \lambda^2[F(u,v)]^2\|u\|^2 + 2\lambda F(u,v)\|u\|^4 + \lambda^2\left\| \|u\|^2 v - F(u,v)u \right\|^2,$$

and dividing by 2λ and taking limits when $\lambda \to 0+$ we obtain

$$\text{(9)} \qquad \rho'_+(u,v) = F(u,v).$$

Finally, using (9), (8) with $\lambda = 1$ and (iv)

$$\begin{aligned} \|u\|^4(\|u+v\|^2 - \|u\|^2) &= [\rho'_+(u,v)]^2\|u\|^2 + 2\rho'_+(u,v)\|u\|^4 + \|\|u\|^2 v - \rho'_+(u,v)u\|^2 \\ &= [\rho'_+(u,v)]^2\|u\|^2 + 2\rho'_+(u,v)\|u\|^4 \\ &\quad + \left\{\|v\|^2\|u\|^4 - [\rho'_+(u,v)]^2\|u\|^2\right\} \\ &= \left(2\rho'_+(u,v) + \|v\|^2\right)\|u\|^4, \end{aligned}$$

i.e.,

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2\rho'_+(u,v),$$

so ρ'_+ is symmetric in u and v and therefore it must be an inner product. The converse is immediate. □

REMARK. – If (i) or (ii) do not hold we may still have (iii) and (iv) for a norm derivable from an inner product but different from F. The following examples show why this is possible.

EXAMPLE 1. – $F \equiv 0$ satisfies (ii), (iii) and (iv) but does not verify (i).

EXAMPLE 2. – In the euclidean space $(\mathbb{R}^2, \|\ \|)$, if χ denotes the characteristic function of the real set $[0,1] \cap \mathbb{Q}$, define $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by means of

$$F(x,y) = \chi(|x \cdot y|/\|x\|\|y\|)(x \cdot y).$$

Then it is easy to see that F satisfies (i), (iii) for any $\delta > 0$ and (iv). If F would satisfy (ii) then $x = y$ yields $f(\lambda) = \lambda$, for all $\lambda > 0$ and substituting $x = (3n, 4n)$, $y = (1,0)$ one arrives to an inequality where the right hand side is ε and in the left hand side the term may tend to infinity when n grows.

Now we turn our attention to James orthogonality

THEOREM 2. – *Let* $(E, \|\ \|)$ *be a real normed space with* $\dim E \geq 2$. *A functional* $F : E \times E \to \mathbb{R}$ *satisfies* (i), (ii), (iii) *and the James identity*

(v) $$\|x\| = \left\| x - 2\frac{F(y,x)}{\|y\|^2} y \right\|, \text{ for all } x, y \text{ in } E, y \neq 0,$$

if and only if E is an i.p.s. whose inner product is F.

PROOF. – Assume that F satisfies the above mentioned conditions.
Using (v) with $x = \lambda y$, $\lambda > 0$, and having in mind (i) we obtain

$$F(y, \lambda y) = 0 \quad \text{or} \quad F(y, \lambda y) = \lambda \|y\|^2,$$

i.e., (1) holds and by the Lemma we have (2) and (3). Introducing $y = u$ and $x = u + \lambda v$, $\lambda > 0$, into (v) and using (3)

$$\|u + \lambda v\| = \left\| u + \lambda\left(2\frac{F(u,v)}{\|u\|^2}u - v\right)\right\|, \tag{10}$$

whence

$$\frac{\|u + \lambda v\|^2 - \|u\|^2}{2\lambda} = \frac{\left\| u + \lambda\left(2\,\frac{F(u,v)}{\|u\|^2}\,u - v\right)\right\|^2 - \|u\|^2}{2\lambda}$$

So, taking limit when λ tends to zero from the right:

$$\rho'_+(u,v) = \rho'_+\left(u, 2\frac{F(u,v)}{\|u\|^2}u - v\right) = 2\frac{F(u,v)}{\|u\|^2}\,\|u\|^2 - \rho'_-(u,v)$$

and therefore $F(u,v) = (\rho'_+(u,v) + \rho'_-(u,v))/2$.

Substituying in (10) the last expression, changing λ by $\frac{1}{\mu}$ and taking limit when μ tends to zero one obtains:

$$\|v\| = \left\| v - \frac{\rho'_+(u,v) + \rho'_-(u,v)}{\|u\|^2}u \right\|, \text{ for all } u, v \text{ in } E,\ u \neq 0$$

and by Theorem 2 in [4] pag. 166, the norm satisfy property (i) of Theorem 1 pag. 163 where $\eta'(u,v)$ exists when $\rho'_+(u,v) = \rho'_-(u,v)$ and in this case $\eta'(u,v) = \dfrac{\rho'_+(u,v)}{\|u\|}$, and then E is an i.p.s.

The converse is obvious. $\square$

NOTE. – Theorems 1 and 2 may be useful also for showing that a real normed space is not an i.p.s., e.g., consider $E = c_0$ (i.e. the vector space of all real sequences convergent to zero) endowed with the norm $\|x\| = \sup\{|x_n| n \geq 1\}$ for $x = (x_n)$ in c_0. Take $F = \rho'_+$ so (i), (ii) and (iii) are obviously satisfied and consider $x = (1/n)$ and $y = (1/n^2)$ in c_0. Then $\|x\| = \|y\| = 1$ and $\rho'_+(y,x) = 1$ so (iv) would imply the contradiction $\|x - y\| = 0$, i.e., $x = y$ which is impossible. Therefore c_0 is not an i.p.s. Analogously condition (v) may be used to arise the same conclusion.

THEOREM 3. – *In a real normed space* $(E, \|\ \|)$ *with* $\dim E \geq 2$, *let* $F : E \times E \to \mathbb{R}$ *a functional satisfying* (i), (ii), (iii). *Then, the Birkhoff inequalities:*

$$\rho'_-\left(\frac{F(y,x)}{\|y\|^2}y, x - \frac{F(y,x)}{\|y\|^2}y\right) \leq 0 \leq \rho'_+\left(\frac{F(y,x)}{\|y\|^2}y, x - \frac{F(y,x)}{\|y\|^2}y\right) \tag{vi}$$

hold for all x, y in E, $y \neq 0$, if and only if $\rho'_-(x,y) \leq F(x,y) \leq \rho'_+(x,y)$ for all x, y in E.

PROOF. – Assume that F satisfies (i), (ii), (iii) and (vi).

By the substitution $x = \lambda y$, $\lambda > 0$, into (vi) using (c), (e) and (a) we obtain precisely condition (1)

$$F(y,\lambda y)\left(\lambda - \frac{F(y,\lambda y)}{\|y\|^2}\right) = 0$$

so F satisfy the hypothesis of the Lemma, i.e., (2) and (3) hold. Introducing $y = u$ and $x = u + \lambda v$, $\lambda > 0$ in (vi) and using (3), we have

$$\rho'_-\left(\left(1 + \lambda\frac{F(u,v)}{\|u\|^2}\right)u, \lambda v - \lambda\frac{F(u,v)}{\|u\|^2}u\right) \leq 0$$

$$\leq \rho'_+\left(\left(1 + \lambda\frac{F(u,v)}{\|u\|^2}\right)u, \lambda v - \lambda\frac{F(u,v)}{\|u\|^2}u\right)$$

and hence, assuming that λ is near enough to zero so that $1 + \lambda\dfrac{F(u,v)}{\|x\|^2}$ is a positive number, we get

$$\lambda\left(1 + \lambda\frac{F(u,v)}{\|u\|^2}\right)\big(-F(u,v) + \rho'_-(u,v)\big) \leq 0$$

$$\leq \lambda\left(1 + \lambda\frac{F(u,v)}{\|u\|^2}\right)\big(-F(u,v) + \rho'_+(u,v)\big)$$

whence it follows that the inequalities $\rho'_-(u,v) \leq F(u,v) \leq \rho'_+(u,v)$ hold for all u, v in E, $u \neq 0$; on the other hand it is easy to verify that the above inequalities still hold for $u = 0$.

Reciprocally, assume that F satisfies (i), (ii), (iii) and $\rho'_-(x,y) \leq F(x,y) \leq \rho'_+(x,y)$ for all x, y in E.

Let x, y be in E, $y \neq 0$. If $F(y,x) = 0$, then (vi) is trivially satisfied. If $F(y,x) \neq 0$, then using (e), (c), (d) and (a), (vi) is equivalent to

$$F(y,x)\rho'_-(y,x) \leq [F(y,x)]^2 \leq F(y,x)\rho'_+(y,x)$$

i.e. $\rho'_-(y,x) \leq F(y,x) \leq \rho'_+(y,x)$ □

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