Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.3, p. 521–533.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_521_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 521-533

Scattering Matrix for the Reflection-Transmission Problem in a Viscoelastic Medium.

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Sunto. – *In questo articolo si studia un problema di riflessione e trasmissione per onde di tipo armonico nel tempo, che si propagano in un solido viscoelastico, anisotropo, stratificato. Si assume che il mezzo occupi l'intero spazio e che le onde siano inviate dall'alto o dal basso con incidenza obliqua. La matrice di scattering è definita generalizzando la costruzione seguita nel caso scalare, cioè quando il solido è isotropo e l'incidenza delle onde è normale. Si discutono l'esistenza, l'unicità e alcune proprietà della matrice di scattering.*

Summary. – *The reflection-transmission problem of time-harmonic waves in a viscoelastic, anisotropic and stratified solid is examined. The medium is supposed to occupy the whole space. The waves are sent either from upwards or downwards with oblique incidence. The scattering matrix is defined by generalizing the procedure followed in the scalar case, namely, when the solid is isotropic and the wave incidence is normal. Existence, uniqueness and properties of the scattering matrix are discussed.*

1. – Introduction.

The aim of this paper is to analyze the reflection-transmission problem of time-harmonic waves in a solid occupying the whole space $\mathbb{R}^3$. The solid is supposed to be linear, viscoelastic, anisotropic and stratified along the direction of a suitably chosen z-axis. Furthermore, we assume that the medium behaves asymptotically like a homogeneous solid, namely the material parameters tend to constant values as $z \to \pm\infty$. The material parameters are allowed to present jump discontinuities at planes orthogonal to the z-axis. In this case the continuity of the displacement and of the traction vectors at planes of discontinuity is required.

Owing to the stratification, time harmonic waves coming from infinity produce reflected and transmitted waves. The oblique direction of the signal and the anisotropy of the medium make the reflection-transmission problem six dimensional. Indeed, the wave propagation is governed by a system of six complex valued first order differential equations for the traction-displacement vector. A system with this form has been introduced by Stroh [13] to study wave propagation in elastic media. It has been adopted by many authors (see e.g. [6], [10]) to investigate surface waves. Here, following [3], the Stroh formalism is used in order to obtain a theorem of existence and uniqueness for reflected and transmitted waves.

The system governing the wave propagation decouples into three systems of two first order differential equations, provided that the material is isotropic and the incidence is normal. The scattering problem for the decoupled systems has been investigated by several authors (see e.g. [1], [5], [9], [11], [14], [15]). In particular, by means of a suitable transformation of the dependent and independent variables, the problem can be reduced to a linear Schrödinger equation ([1]). It has been proved that this equation admits two independent solutions, called Jost solutions, in term of which the reflection and transmission coefficients and hence the scattering matrix, are defined ([9], [11]).

The main purpose of this paper is to generalize this procedure to the six dimensional case. The first step is to introduce six independent solutions related to the Jost solutions of the Schrödinger equation. The scattering matrix is then defined and existence and uniqueness are discussed.

In order to introduce the reflection and transmission matrices, we need a definition of incoming and outgoing waves. This is achieved in Section 3 looking at the sign of the energy flux $\mathcal{F}$. Specifically we extend the method followed in [2], where the reflection-transmission problem between two viscoelastic anisotropic half-spaces is investigated. The energy flux $\mathcal{F}$ is represented as a Hermitian quadratic form associated to a matrix Φ^+ (or Φ^-) defined by means of the solutions with a particular behaviour as $z \to +\infty$ ($z \to -\infty$). Such solutions are related to the Jost solutions.

Under the assumptions that the diagonal blocks of Φ^+ and Φ^- are definite positive and negative, in Section 4 we prove existence and uniqueness of the scattering matrix $\mathbb{S}$.

Finally, in the last section, we study the properties of the scattering matrix. It is well known ([9]) that the scattering matrix $\mathbb{S}$ of the Schrödinger equation is unitary and that the transmission coefficients, recovered for up-going and down-going waves, are equal. We show that in the six dimensional case these properties do not hold in general and a relation between the transmission coefficients is found. More precisely, if suitable conditions on material parameters hold, we prove the equality between the determinants of the transmission matrices. This property is satisfied, for instance, by an isotropic viscoelastic solid.

2. – Statement of the problem.

Let us consider a viscoelastic or elastic system, occupying the whole space $\mathbb{R}^3$. By neglecting the body force, the evolution of the solid is governed by the equation

$$\rho \frac{\partial^2 \boldsymbol{U}}{\partial t^2} = \nabla \cdot \mathcal{T} \,, \tag{2.1}$$

where ρ is the mass density, $\boldsymbol{U}$ is the displacement vector and $\mathcal{T}$ is the Cauchy stress tensor. In particular, $\mathcal{T}$ satisfies the linear constitutive equation

$$\mathcal{T}(\boldsymbol{x}, t) = \mathbb{G}_0(\boldsymbol{x})\nabla \boldsymbol{U}(\boldsymbol{x}, t) + \int_0^{+\infty} \mathbb{G}'(\boldsymbol{x}, s)\nabla \boldsymbol{U}(\boldsymbol{x}, t-s)ds \,, \tag{2.2}$$

where $\mathbb{G}_0$ and $\mathbb{G}'$ are fourth order tensors with the following properties ([7])

$$\mathbb{G}_{0ijkh} = \mathbb{G}_{0jikh} = \mathbb{G}_{0ijhk} \,, \qquad \mathbb{G}'_{ijkh} = \mathbb{G}'_{jikh} = \mathbb{G}'_{ijhk} \,.$$

Notice that $\mathbb{G}'$ is not required to be symmetric, namely it is not required that

$$\mathbb{G}'_{ijkh} = \mathbb{G}'_{khji} \,,$$

while the second law of thermodynamics guarantees the symmetry of $\mathbb{G}_0$ ([8]).

We denote by $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$ the unit vectors of the axes x, y, z and assume that the medium is stratified along the z-direction, namely we suppose that the material parameters ρ, $\mathbb{G}_0$ and $\mathbb{G}'$ depend only on the variable z.

In order to examine the behaviour of a wave propagating through the solid, obliquely with respect to z-direction, we look for a solution of (2.1) in the form

$$\boldsymbol{U}(\boldsymbol{x}, t) = \boldsymbol{u}(z)e^{i(\boldsymbol{k}_\parallel \cdot \boldsymbol{x} - \omega t)} \,,$$

where $\boldsymbol{k}_\parallel = k_x \boldsymbol{e}_x + k_y \boldsymbol{e}_y$ is a real valued vector and $\omega \in \mathbb{R}$ is the frequency.

Substitution into (2.2) shows that

$$\mathcal{T} = \mathbb{G}\nabla \boldsymbol{U} \,,$$

where the complex valued fourth order tensor $\mathbb{G}$ is defined by

$$\mathbb{G}(\boldsymbol{x}, \omega) = \mathbb{G}_0(\boldsymbol{x}) + \int_0^{+\infty} \mathbb{G}'(\boldsymbol{x}, s)e^{i\omega s}ds \,.$$

In the case of elastic solids, $\mathbb{G}'$ vanishes, so that $\mathbb{G}$ is real.

Let us introduce the traction at the planes $z =$ const, defined as $\mathcal{T}\boldsymbol{e}_z$ and the vector $\boldsymbol{t}$ satisfying

$$\mathcal{T}\boldsymbol{e}_z = \boldsymbol{t}(z)e^{i(\boldsymbol{k}_\parallel \cdot \boldsymbol{x} - \omega t)} \,.$$

Moreover, we denote by $\boldsymbol{w}$ the displacement-traction vector, i.e.

$$\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{t})^T .$$

We find that the equation (2.1) is equivalent to the linear system of first-order differential equations in the Stroh form ([13])

$$\boldsymbol{w}' = \mathbb{N}\boldsymbol{w} . \tag{2.3}$$

Denoting by $\mathbb{I}$ the identity matrix and by $\boldsymbol{a}\mathbb{G}\boldsymbol{b}$ the matrix whose elements are

$$[\boldsymbol{a}\mathbb{G}\boldsymbol{b}]_{ih} = \mathbb{G}_{ijkh} a_j b_k ,$$

we represent $\mathbb{N}$ by the block structure

$$\mathbb{N} = \begin{pmatrix} \mathbb{N}_1 & \mathbb{N}_2 \\ \mathbb{N}_3 & \mathbb{N}_4 \end{pmatrix} ,$$

where

$$\mathbb{N}_1 = \mathbb{N}_4^T = -i(\boldsymbol{e}_z\mathbb{G}\boldsymbol{e}_z)^{-1}(\boldsymbol{e}_z\mathbb{G}\boldsymbol{k}_\parallel) , \tag{2.4}$$

$$\mathbb{N}_2 = (\boldsymbol{e}_z\mathbb{G}\boldsymbol{e}_z)^{-1} , \tag{2.5}$$

$$\mathbb{N}_3 = -\rho\omega^2\mathbb{I} + (\boldsymbol{k}_\parallel\mathbb{G}\boldsymbol{k}_\parallel) - (\boldsymbol{k}_\parallel\mathbb{G}\boldsymbol{e}_z)(\boldsymbol{e}_z\mathbb{G}\boldsymbol{e}_z)^{-1}(\boldsymbol{e}_z\mathbb{G}\boldsymbol{k}_\parallel) \tag{2.6}$$

and we suppose that $\boldsymbol{e}_z\mathbb{G}\boldsymbol{e}_z$ is non-singular ([2]).

Notice that if $\boldsymbol{k}_\parallel = 0$ and the solid is isotropic, $\mathbb{N}$ assumes the form

$$\mathbb{N} = \begin{pmatrix} 0 & (\boldsymbol{e}_z\mathbb{G}\boldsymbol{e}_z)^{-1} \\ -\rho\omega^2\mathbb{I} & 0 \end{pmatrix} , \tag{2.7}$$

where

$$(\boldsymbol{e}_z\mathbb{G}\boldsymbol{e}_z)^{-1} = \mathrm{diag}\left(\frac{1}{\mu}, \frac{1}{\mu}, \frac{1}{2\mu+\lambda}\right) .$$

Then the system (2.3) splits into three systems of first order differential equations. For instance, the first component of the displacement and of the traction vectors satisfy

$$\begin{cases} u_x' = \dfrac{1}{\mu} t_x \\ t_x' = -\rho\omega^2 u_x \end{cases}$$

Each system leads to a scalar problem which can be analyzed by means of the techniques developed e.g. in [14] and [15].

The definition of the scattering matrix depends on the asymptotic behaviour of solutions of the system (2.3) and hence on the asymptotic behaviour of the matrix $\mathbb{N}$. We assume the existence of the limits of $\mathbb{N}$ as $z \to \pm\infty$, denoted by

$$\lim_{z\to-\infty} \mathbb{N} = \mathbb{N}^- , \qquad \lim_{z\to+\infty} \mathbb{N} = \mathbb{N}^+ .$$

The matrices $\mathbb{N}^+$ and $\mathbb{N}^-$ are supposed to be simple matrices. Notice that they are not required to be equal.

Furthermore, we suppose that the medium behaves asymptotically like a homogeneous solid, namely we suppose that there exist $a, b \in \mathbb{R}$ satisfying the conditions

$$\int_a^{+\infty} \|\mathbb{N} - \mathbb{N}^+\| dz < +\infty , \qquad \int_{-\infty}^{b} \|\mathbb{N} - \mathbb{N}^-\| dz < +\infty ,$$

where we denote by $\|\cdot\|$ any norm in the space of sixth order square matrices.

Let $(i\sigma_k^+, \boldsymbol{p}_k^+)$, $(i\sigma_k^-, \boldsymbol{p}_k^-)$, $k = 1, ..., 6$ be the eigenvalues and the corresponding independent eigenvectors of $\mathbb{N}^+$ and $\mathbb{N}^-$ respectively. Denote by $\mathbb{P}^+$ and $\mathbb{P}^-$ the matrices whose columns are $\boldsymbol{p}_1^+, ..., \boldsymbol{p}_6^+$ and $\boldsymbol{p}_1^-, ..., \boldsymbol{p}_6^-$, i.e.

$$\mathbb{P}^+ = [\boldsymbol{p}_1^+, ..., \boldsymbol{p}_6^+] , \qquad \mathbb{P}^- = [\boldsymbol{p}_1^-, ..., \boldsymbol{p}_6^-] .$$

We choose $\mathbb{P}^+$ and $\mathbb{P}^-$ such that

$$\det \mathbb{P}^+ = \det \mathbb{P}^- = 1 .$$

No additional condition is required to determine $\mathbb{P}^+$ and $\mathbb{P}^-$ univocally.

Under these assumptions ([4]), there exist two sets of six solutions $\{\boldsymbol{\varphi}_1^+, \dots, \boldsymbol{\varphi}_6^+\}$, $\{\boldsymbol{\varphi}_1^-, \dots, \boldsymbol{\varphi}_6^-\}$ of the system (2.3) with asymptotic behaviours

$$\lim_{z\to+\infty} \boldsymbol{\varphi}_k^+ e^{-i\sigma_k^+ z} = \boldsymbol{p}_k^+ , \tag{2.8}$$

$$\lim_{z\to-\infty} \boldsymbol{\varphi}_k^- e^{-i\sigma_k^- z} = \boldsymbol{p}_k^- ,$$

for every $k = 1, ..., 6$. Since $\{\boldsymbol{p}_k^+\}_{k=1,...,6}$ and $\{\boldsymbol{p}_k^-\}_{k=1,...,6}$ are bases of $\mathbb{C}^6$, the functions $\{\boldsymbol{\varphi}_k^+\}_{k=1,...,6}$ and $\{\boldsymbol{\varphi}_k^-\}_{k=1,...,6}$ are independent solutions of the system (2.3), i.e. any solution $\boldsymbol{w}$ of (2.3) can be written in the form

$$\boldsymbol{w} = \sum_{a=1}^{6} c_a^+ \boldsymbol{\varphi}_a^+ = \sum_{a=1}^{6} c_a^- \boldsymbol{\varphi}_a^-$$

with suitable coefficients $c_a^+, c_a^- \in \mathbb{C}$, $a = 1, ..., 6$.

The functions $\{\boldsymbol{\varphi}_k^+\}_{k=1,...,6}$ and $\{\boldsymbol{\varphi}_k^-\}_{k=1,...,6}$ will play an important role in the definition of the reflection and transmission matrices.

3. – Energy flux.

In order to represent the reflected and transmitted waves, we need a rule to distinguish the waves propagating in the increasing and decreasing z-direction. To this aim, we associate to every solution $\boldsymbol{w}$ of the system (2.3) the corresponding energy flux across a plane $z =$const, defined as

$$\mathcal{F}(\boldsymbol{w})(z) = -\,i\,\frac{\omega}{4}(\boldsymbol{u}^* \cdot \boldsymbol{t} - \boldsymbol{u} \cdot \boldsymbol{t}^*) = (\boldsymbol{u}^*, \boldsymbol{t}^*)\mathbb{J}(\boldsymbol{u}, \boldsymbol{t})^T = \boldsymbol{w}^{\dagger}\mathbb{J}\boldsymbol{w}\,,$$

where $\mathbb{J}$ is the matrix defined by

$$\mathbb{J} = -\frac{i\omega}{4}\begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

and the symbols * and † denote the conjugate and the conjugate transpose respectively.

It has been proved in [3] that the energy flux $\mathcal{F}$ is non-increasing if the solid is viscoelastic and that it is constant if the solid is elastic. This property is crucial to prove existence and uniqueness of the scattering matrix.

We say that a wave $\boldsymbol{w}$ is asymptotically up-going [down-going] when $z \to +\infty$ if there exists $Z_{\boldsymbol{w}}^+ > 0$ such that

$$\mathcal{F}(\boldsymbol{w})(z) > 0\,, \qquad [\mathcal{F}(\boldsymbol{w})(z) < 0] \qquad \forall z \geq Z_{\boldsymbol{w}}^+\,.$$

Of course, up-going [down-going] waves propagate in the direction of increasing [decreasing] z. Moreover, these definitions are generalized straightforwardly to the case when $z \to -\infty$.

Notice that, by letting $\boldsymbol{w} = \sum_{k=1}^{6} c_k \boldsymbol{\varphi}_k^+$, we have

$$\mathcal{F}(\boldsymbol{w}) = \sum_{h,k=1}^{6} c_k^* c_h (\boldsymbol{\varphi}_k^+)^{\dagger}\mathbb{J}\boldsymbol{\varphi}_h^+\,.$$

We denote by ϕ^+ the 6×6 matrix whose columns are the functions $\boldsymbol{\varphi}_1^+, \ldots, \boldsymbol{\varphi}_6^+$, i.e.

$$\phi^+ = (\boldsymbol{\varphi}_1^+, ..., \boldsymbol{\varphi}_6^+)$$

and we define

$$\Phi^+ = (\phi^+)^{\dagger}\mathbb{J}\phi^+\,.$$

Since ϕ^+ is non-singular, the matrices Φ^+ and $\mathbb{J}$ are congruent and hence they have the same number of positive, negative and zero eigenvalues ([12]). In particular, since the eigenvalues of $\mathbb{J}$ are

$$\lambda_{1,2,3} = \frac{\omega}{4}\,, \qquad \lambda_{4,5,6} = -\frac{\omega}{4}\,,$$

Φ^+ has three positive and three negative eigenvalues.

This property is not sufficient to guarantee that a linear combination of up-going [down-going] waves is an up-going [down-going] wave. Therefore, further conditions on the matrix Φ^+ are required.

For convenience, we represent Φ^+ in the form

$$\begin{pmatrix} \Phi_p^+ & \tilde{\Phi} \\ \tilde{\Phi}^\dagger & \Phi_n^+ \end{pmatrix}.$$

We assume that there exists $Z^+ > 0$ such that Φ_p^+ and Φ_n^+ are positive and negative definite, respectively, for all $z \geq Z^+$.

Under these assumptions, if

$$\boldsymbol{w} = \sum_{k=1}^{3} c_k \boldsymbol{\varphi}_k^+\,, \qquad \text{or} \qquad \boldsymbol{w} = \sum_{k=4}^{6} c_k \boldsymbol{\varphi}_k^+\,,$$

we obtain

$$\mathcal{F}(\boldsymbol{w})(z) = \sum_{h,k=1}^{3} c_k^* c_h \Phi_{p,kh}^+(z) > 0\,, \quad \text{or} \quad \mathcal{F}(\boldsymbol{w})(z) = \sum_{h,k=4}^{6} c_k^* c_h \Phi_{n,kh}^+(z) < 0\,,$$

for all $z \geq Z^+$ and for every triplet $(c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$. In particular, $\boldsymbol{\varphi}_1^+, \boldsymbol{\varphi}_2^+, \boldsymbol{\varphi}_3^+$ are up-going waves while $\boldsymbol{\varphi}_4^+, \boldsymbol{\varphi}_5^+, \boldsymbol{\varphi}_6^+$ are down-going waves. In addition, if $\Phi_p^+(z)$ and $\Phi_n^+(z)$ are positive and negative definite for all $z \geq Z^+$, then $Z_{\boldsymbol{w}}^+ = Z^+$ for every wave $\boldsymbol{w}$, up-going or down-going when $z \to +\infty$.

By the same procedure, we define the matrix Φ^- as

$$\Phi^- = (\phi^-)^\dagger \mathbb{J} \phi^- = \begin{pmatrix} \Phi_p^- & \tilde{\Phi} \\ \tilde{\Phi}^\dagger & \Phi_n^- \end{pmatrix}$$

and we assume that Φ_p^- and Φ_n^- are definitively positive and negative definite. Hence, $\boldsymbol{\varphi}_1^-, \boldsymbol{\varphi}_2^-, \boldsymbol{\varphi}_3^-$ and $\boldsymbol{\varphi}_4^-, \boldsymbol{\varphi}_5^-, \boldsymbol{\varphi}_6^-$ represent waves propagating respectively upward and downward.

4. – Scattering matrix.

This section is devoted to the definition of the scattering matrix. Firstly, we introduce the reflection and transmission matrices, as a consequence of the following result.

THEOREM 4.1. – *Suppose that Φ_p^+, Φ_p^- and Φ_n^+, Φ_n^- are positive definite and negative definite definitively. Then $\{\boldsymbol{\varphi}_k^+, \boldsymbol{\varphi}_{k+3}^-\}_{k=1,2,3}$ is a basis for the solutions of the system* (2.3).

PROOF. – To prove the independence of $\boldsymbol{\varphi}_1^+, ..., \boldsymbol{\varphi}_6^-$, we consider the sum

$$c_1\boldsymbol{\varphi}_1^+ + c_2\boldsymbol{\varphi}_2^+ + c_3\boldsymbol{\varphi}_3^+ + c_4\boldsymbol{\varphi}_4^- + c_5\boldsymbol{\varphi}_5^- + c_6\boldsymbol{\varphi}_6^- = 0$$

and we define

$$\boldsymbol{v} = \sum_{k=1}^{3} c_k\boldsymbol{\varphi}_k^+ = -\sum_{k=4}^{6} c_k\boldsymbol{\varphi}_k^- .$$

The assumptions on the matrices Φ^+ and Φ^- provide the inequalities

$$\mathcal{F}(\boldsymbol{v})(z) \geq 0\,, \qquad \forall z \geq Z^+\,,$$

$$\mathcal{F}(\boldsymbol{v})(z) \leq 0\,, \qquad \forall z \leq -Z^-\,.$$

Since $\mathcal{F}$ is non-increasing, we deduce that

$$0 \leq \mathcal{F}(Z^+) \leq \mathcal{F}(-Z^-) \leq 0$$

which implies

$$c_k = 0\,, \qquad \forall k = 1, ..., 6$$

and hence the independence of $\boldsymbol{\varphi}_1^+, \boldsymbol{\varphi}_2^+, \boldsymbol{\varphi}_3^+, \boldsymbol{\varphi}_4^-, \boldsymbol{\varphi}_5^-, \boldsymbol{\varphi}_6^-$. □

The independence of $\{\boldsymbol{\varphi}_k^+, \boldsymbol{\varphi}_{k+3}^-\}_{k=1,2,3}$ guarantees that there exist unique coefficients $\mathbb{R}_{kh}^-$ and $\mathbb{T}_{kh}^-$ satisfying the relations

$$\sum_{k=1}^{3} \boldsymbol{\varphi}_k^+ \mathbb{T}_{kh}^+ = \boldsymbol{\varphi}_h^- + \sum_{k=1}^{3} \boldsymbol{\varphi}_{k+3}^- \mathbb{R}_{kh}^-\,, \qquad h = 1, 2, 3\,. \tag{4.1}$$

The assumptions on the energy flux ensure that the left hand side of (4.1) represents a wave propagating in the positive z-direction when $z \to +\infty$, whereas the two terms of the right hand side can be identified respectively with up-going and down-going waves when $z \to -\infty$. Therefore, if we identify $\boldsymbol{\varphi}_h^-$, $h = 1, 2, 3$, with the generators of the up-going incident waves, then we can interpret $\mathbb{T}_{kh}^+$ and $\mathbb{R}_{kh}^-$, respectively, as the corresponding transmission and the reflection coefficients.

By a similar procedure, we show existence and uniqueness of the coefficients $\mathbb{T}_{kh}^-, \mathbb{R}_{kh}^-$ verifying the relations

$$\sum_{k=1}^{3} \boldsymbol{\varphi}_{k+3}^- \mathbb{T}_{kh}^- = \boldsymbol{\varphi}_{h+3}^+ + \sum_{k=1}^{3} \boldsymbol{\varphi}_k^+ \mathbb{R}_{kh}^+\,, \qquad h = 1, 2, 3\,. \tag{4.2}$$

The matrices $\mathbb{T}^{\pm}$ and $\mathbb{R}^{\pm}$ are called transmission and reflection matrices.

We observe that the basis of the Theorem 4.1 is related to the Jost solutions of the scalar Schrödinger equation.

Next we introduce the vectors $\boldsymbol{w}_1, \dots, \boldsymbol{w}_6$ defined by

$$\boldsymbol{w}_h = \sum_{k=1}^{3} \boldsymbol{\varphi}_k^+ \mathbb{T}_{kh}^+ , \qquad \boldsymbol{w}_{h+3} = \sum_{k=1}^{3} \boldsymbol{\varphi}_{k+3}^- \mathbb{T}_{kh}^- , \qquad h = 1, 2, 3 .$$

In view of (4.1) and (4.2), they can be written in the form

$$(\boldsymbol{w}_1, \dots, \boldsymbol{w}_6) = (\boldsymbol{\varphi}_1^+, \dots, \boldsymbol{\varphi}_6^+) \begin{pmatrix} \mathbb{T}^+ & \mathbb{R}^+ \\ 0 & \mathbb{I} \end{pmatrix} = (\boldsymbol{\varphi}_1^-, \dots, \boldsymbol{\varphi}_6^-) \begin{pmatrix} \mathbb{I} & 0 \\ \mathbb{R}^- & \mathbb{T}^- \end{pmatrix} . \tag{4.3}$$

Moreover, we define the scattering matrix $\mathbb{S}$ defined by

$$\mathbb{S} = \begin{pmatrix} \mathbb{T}^+ & \mathbb{R}^- \\ \mathbb{R}^+ & \mathbb{T}^- \end{pmatrix} .$$

The reflection-transmission problem consists in finding $\boldsymbol{w}_1, \dots, \boldsymbol{w}_6$ or the scattering matrix $\mathbb{S}$ from (2.3) and (4.3).

In view of the previous results, we have proved the following theorem.

THEOREM 4.2 (Existence and uniqueness). – *If Φ_p^+ and Φ_p^- are definitively positive definite and Φ_n^+ and Φ_n^- are definitively negative definite, there exists a unique scattering matrix for the reflection-transmission problem.*

It is worth noting that the uniqueness of the scattering matrix depends on the choice of the bases $\boldsymbol{\varphi}_1^-, \dots, \boldsymbol{\varphi}_6^-$ and $\boldsymbol{\varphi}_1^+, \dots, \boldsymbol{\varphi}_6^+$.

5. – Properties of the scattering matrix.

In this section, we examine some properties of the scattering matrix. It is well known from the analysis of the scalar Schrödinger equation, that the transmission coefficients T^+ and T^- are non-zero ([5]). This condition is generalized by the following result.

THEOREM 5.1. – *The matrices $\mathbb{T}^+$ and $\mathbb{T}^-$ are non-singular.*

PROOF. – If $\det \mathbb{T}^+ = 0$, then there exists a non-trivial triplet $(a_1, a_2, a_3) \in \mathbb{C}^3$ such that

$$\sum_{h,k=1}^{3} \boldsymbol{\varphi}_k^+ \mathbb{T}_{kh}^+ a_h = \mathbf{0} ,$$

which, in view of the equations (4.1), leads to the equality

$$\sum_{h=1}^{3} \varphi_h^- a_h + \sum_{h,k=1}^{3} \varphi_{k+3}^- \mathbb{R}_{kh}^- a_h = 0\,.$$

Since the functions $\{\varphi_h^+\}_{h=1,\ldots,6}$ are independent, we obtain

$$a_h = 0 \qquad\qquad h = 1, 2, 3\,,$$

which is a contradiction.

By repeating the same arguments, one can prove that $\mathbb{T}^-$ is non-singular. □

The scattering matrix $\mathbb{S}$ for the scalar Schrödinger equation is unitary and the relations $T^+(\omega) = T^-(\omega)$, $(T^+)^*(\omega) = T^+(-\omega)$, $(R^\pm)^*(\omega) = R^\pm(-\omega)$ hold, ([9]). In the general case these properties cannot be extended to the scattering problem for viscoelastic solids and we are only able to prove the following weaker result.

THEOREM 5.2. – *Let* $(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_6)$ *be the solution of the system* $\boldsymbol{w}' = \mathbb{N}\boldsymbol{w}$ *defined by* (4.3). *If the condition*

$$\frac{d}{dz}\det(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_6) = 0 \tag{5.4}$$

is satisfied, then

$$\det \mathbb{T}^+ = \det \mathbb{T}^-\,. \tag{5.5}$$

Moreover, the equalities

$$\sum_{k=1}^{6} \sigma_k^- = 0\,, \qquad\qquad \sum_{k=1}^{6} \sigma_k^+ = 0$$

provide a necessary condition for (5.4). *Conversely, if the trace of the matrix* $\mathbb{N}_1$ *defined by* (2.4) *vanishes, then* (5.4) *holds.*

PROOF. – The system (4.3) yields the relations

$$\det(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_6) = \det(\varphi_1^+, \ldots, \varphi_6^+)\det \mathbb{T}^+ = \det(\varphi_1^-, \ldots, \varphi_6^-)\det \mathbb{T}^-\,. \tag{5.6}$$

By differentiating (5.6) with respect to z, in view of the condition (5.4) and of the Theorem 5.1, we find that

$$\frac{d}{dz}\det \phi^+ = 0\,, \qquad\qquad \frac{d}{dz}\det \phi^- = 0\,.$$

Thus $\det\phi^+$ and $\det\phi^-$ are constant.

On the other hand, from the Binet formula, we obtain

$$\det \phi^+ = \det(\varphi_1^+ e^{-i\sigma_1^+ z}, \ldots, \varphi_6^+ e^{-i\sigma_6^+ z})\, e^{i\sum_k \sigma_k^+ z}\,,$$

so that, by letting $z \to +\infty$ and keeping the asymptotic conditions (2.8) into account, we deduce

$$\sum_{k=1}^{6} \sigma_k^+ = 0\,, \qquad \det \phi^+ = \det \mathbb{P}^+ = 1\,.$$

The same argument leads to

$$\sum_{k=1}^{6} \sigma_k^- = 0\,, \qquad \det \phi^- = \det \mathbb{P}^- = 1\,.$$

By substituting in (5.6), we prove (5.5).

Suppose now that the trace of $\mathbb{N}_1$ vanishes. If we denote by $\eta_{j_1,\dots,j_6}$ the Levi-Civita's symbol, we have

$$\det(\boldsymbol{w}_1, \dots, \boldsymbol{w}_6) = \sum_{j_1,\dots,j_6=1}^{6} \eta_{j_1,\dots,j_6} w_{j_1 1} \dots w_{j_6 6}\,.$$

By differentiating with respect to z, we obtain

$$\frac{d}{dz} \det[\boldsymbol{w}_1, \dots, \boldsymbol{w}_6] = \sum_{j_1,\dots,j_6=1}^{6} \eta_{j_1,\dots,j_6} (w'_{j_1 1} w_{j_2 2} \dots w_{j_6 6} + \dots + w_{j_1 1} \dots w_{j_5 5} w'_{j_6 6})\,.$$

Since $\boldsymbol{w}_1, \dots, \boldsymbol{w}_6$ are solutions of (2.3), in view of the symmetry properties (2.4)-(2.6), we obtain the relation

$$\frac{d}{dz} \det(\boldsymbol{w}_1, \dots, \boldsymbol{w}_6) = 2(\mathrm{tr}\mathbb{N}_1) \sum_{j_1,\dots,j_6=1}^{6} \eta_{j_1,\dots,j_6} w_{j_1 1} \dots w_{j_6 6}\,,$$

where $\mathrm{tr}\mathbb{N}_1$ denote the trace of $\mathbb{N}_1$. Therefore the Wronskian of $\boldsymbol{w}_1, \dots, \boldsymbol{w}_6$ is independent of z. □

The following example shows that in general $\mathbb{T}^+ \neq \mathbb{T}^-$ and that $\mathbb{S}$ is not unitary. Consider two viscoelastic, homogeneous and isotropic half-spaces in welded contact at the plane $z = 0$. Restrict attention to waves that hit normally the plane $z = 0$, i.e. $\boldsymbol{k}_\parallel = 0$. In this case, the matrix $\mathbb{N}$ assumes the form (2.7), where the density ρ and the Lamè coefficients λ, μ are given by

$$\rho(z) = \begin{cases} \rho^+ & z > 0 \\ \rho^- & z < 0 \end{cases} \qquad \lambda(z) = \begin{cases} \lambda^+ & z > 0 \\ \lambda^- & z < 0 \end{cases}$$

$$\mu(z) = \begin{cases} \mu^+ & z > 0 \\ \mu^- & z < 0 \end{cases}$$

Denote by $i\sigma_L^\pm$ and $i\sigma_T^\pm$ the eigenvalues of $\mathbb{P}^\pm$. By straightforward calculations one finds the representations

$$\mathbb{T}^{\pm} = \zeta^{\pm}\mathrm{diag}\left(\frac{2\rho^{\mp}\sigma_L^{\mp}}{\rho^{-}\sigma_L^{+}+\rho^{+}\sigma_L^{-}}, \frac{2\rho^{\mp}\sigma_T^{\mp}}{\rho^{-}\sigma_T^{+}+\rho^{+}\sigma_T^{-}}, \frac{2\rho^{\mp}\sigma_T^{\mp}}{\rho^{-}\sigma_T^{+}+\rho^{+}\sigma_T^{-}}\right),$$

$$\mathbb{R}^{\pm} = \mathrm{diag}\left(\frac{\rho^{\mp}\sigma_L^{\pm}-\rho^{\pm}\sigma_L^{\mp}}{\rho^{-}\sigma_L^{+}+\rho^{+}\sigma_L^{-}}, \frac{\rho^{\mp}\sigma_T^{\pm}-\rho^{\pm}\sigma_T^{\mp}}{\rho^{-}\sigma_T^{+}+\rho^{+}\sigma_T^{-}}, \frac{\rho^{\mp}\sigma_T^{\pm}-\rho^{\pm}\sigma_T^{\mp}}{\rho^{-}\sigma_T^{+}+\rho^{+}\sigma_T^{-}}\right),$$

where ζ^- and ζ^+ are given by

$$\zeta^{+} = \frac{\sigma_L^{+}(\sigma_T^{+})^2(\rho^{+})^3}{\sigma_L^{-}(\sigma_T^{-})^2(\rho^{-})^3}, \qquad \zeta^{-} = \frac{\sigma_L^{-}(\sigma_T^{-})^2(\rho^{-})^3}{\sigma_L^{+}(\sigma_T^{+})^2(\rho^{+})^3}.$$

Therefore, $\mathbb{T}^+ \neq \mathbb{T}^-$ and $\mathbb{S}\mathbb{S}^\dagger \neq \mathbb{I}$.

Notice that (5.5) holds since the trace of $\mathbb{N}_1$ vanishes.

The same condition is also satisfied when we consider waves such that $\boldsymbol{k}_\parallel \neq 0$ (oblique incidence) in an isotropic solid. Indeed, in this case it is easily verified that the diagonal elements of the matrix $\mathbb{N}_1$ vanish.

Acknowledgment. The author thanks the Referee for her/his careful reading and many valuable suggestions.

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Pervenuta in Redazione
il 19 aprile 2005 e in forma rivista il 9 aprile 2006