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Deficient Coerciveness Estimate for an Abstract Differential Equation with a Parameter Dependent Boundary Conditions.

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Sunto. – *In questo lavoro si considera un problema differenziale astratto di tipo ellittico, in cui sia l'equazione che le condizioni ai limiti possono contenere un parametro spettrale.*

Prima si prova che questo operatore è un isomorfismo tra appropriati spazi funzionali e poi si dimostra una stima coerciva con difetto.

I risultati ottenuti sono applicati allo studio di alcune classi di problemi ellittici, anche possibilmente degeneri.

Summary. – *In this paper we consider an abstract elliptic differential problem where the equation and the boundary conditions may contain a spectral parameter.*

We first prove that this problem generates an isomorphism between appropriate spaces and we establish a more precise estimate called coerciveness estimate with defect.

The results obtained are applied to study some classes of elliptic, and also possibly degenerate, problems.

1. – Introduction.

The problem

$$\begin{cases} Au = f \text{ on } \Omega \\ B_j u = g_j \text{ on } \partial\Omega, \quad j = 1, m \end{cases}$$

where Ω is a smooth bounded open set of $\mathbb{R}^n$, is a regular elliptic boundary value problem if A is a strongly elliptic differential operator of order m and the boundary operators B_j are normal and satisfy the so-called Sapiro-Lopatinski conditions or complementing conditions [12, 14].

Many works are devoted to the study of regular boundary value problems for elliptic partial differential equations depending on a spectral parameter [1, 2, 15, 18]. Such parameter may appear in both the equation and the boundary conditions. In particular, an optimal estimate which has an explicit dependence with respect to the spectral parameter has been established. Such an estimate is called a coerciveness estimate.

However, nonregular problems, not satisfying Sapiro-Lopatinski conditions, are less studied. In the papers and monographs [3, 4, 6, 15, 17, 18] sufficient conditions ensuring that coerciveness estimates still hold are given. We quote in particular [18] where a number of such problems is considered.

In [6], the author considered the following problem

$$(1) \qquad \begin{cases} L_0(\lambda)u &= u''(x) - (A+\lambda I)^2 u(x) = f(x) \quad x \in [0,1] \\ L_{10}(\lambda)u &= \lambda(a_1 u(0) + \beta_1 u(1)) + \delta_1 u'(0) + \gamma_1 u'(1) = f_1 \\ L_{20}(\lambda)u &= a_2 u(0) + \beta_2 u(1) = f_2 \,. \end{cases}$$

Under some conditions on the operator A, he obtained a coerciveness estimate. One of these conditions is that $\theta_1 + \theta_2 \neq 0$, where

$$\theta_1 = \begin{vmatrix} a_1 & \beta_1 \\ a_2 & \beta_2 \end{vmatrix} \quad \text{and} \quad \theta_2 = \begin{vmatrix} \delta_1 & \gamma_1 \\ a_2 & -\beta_2 \end{vmatrix}.$$

In this paper we prove that we still have coerciveness, but with defect, when $\theta_1 + \theta_2 = 0$. Coerciveness with defect means that the greatest exponent of the modulus of the spectral parameter in the estimate is less than the exponent of the spectral parameter appearing in the equation.

Moreover, we prove that the operator generated by our problem is an isomorphism between appropriate spaces. Then, we apply the abstract results to some classes of boundary value problems for elliptic and quasielliptic partial differential equations in a cylinder.

More precisely, in section 2, we give some background preliminaries. The principal boundary value problem for abstract differential equations is studied in section 3. We prove the isomorphism between appropriate spaces and we establish a deficient coercive estimate for the solution in which, of course, the exponent of the spectral parameter is not optimal, as in the case $\theta_1 + \theta_2 \neq 0$. In section 4, we show that the obtained abstract results apply to some boundary value problems for elliptic partial differential equations in a cylinder and degenerate elliptic operators with nonstandard boundary conditions.

2. – Preliminaries.

Let H be a Hilbert space, A a closed linear operator in H and D_A its domain. We denote by $B(H)$ the space of bounded operators acting in H, endowed with the usual operator norm, and by $L^p(0,1;H)$ the Banach space of strongly measurable functions $x \longrightarrow u(x) : (0,1) \longrightarrow H$, whose p^{th} power norms are summable, with the norm $\|u\|_{0,p}^{p} = \int_0^1 \|u(x)\|_H^p dx, p \in (1, \infty)$.

The vector-valued Sobolev space is defined as

$$W_p^n(0,1; D(A^2), H) = \{u : A^2 u \in L^p(0,1; H);\ u^{(n)} \in L^p(0,1; H)\},$$

the norm in this space is given by

$$\|u\|_{W_p^n(0,1;D(A^2),H)} = \left\|A^2 u\right\|_{L^p(0,1;H)} + \left\|u^{(n)}\right\|_{L^p(0,1;H)}.$$

The space $D(A)$ is defined by

$$D(A) = \left\{u \in D_A; \|u\|_{D(A)}^2 = \|u\|_H^2 + \|Au\|_H^2 < \infty\right\}$$

and it is precisely the domain of A equipped with the Hilbertian graph norm.

Let $-A$ be the generator of the analytic semigroup e^{-tA} for $t > 0$, decreasing at infinity, and strongly continuous for $t \geq 0$. We define the interpolation space [14, p. 96]:

$$(H, D(A^m))_{\theta,p} = \left\{u \in H; \|u\|_{m,\theta}^p = \int_0^\infty t^{m(1-\theta)p-1} \left\|A^m e^{-tA} u\right\|_H^p dt < \infty\right\},$$
$$0 < \theta < 1, m \in \mathbb{N}, 1 < p < \infty.$$

$\|.\|_{m,\theta}$ is the norm in $(H, D(A^m))_{\theta,p}$.

Let $(Fu)(\sigma) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-i\sigma x} u(x) dx$ be the Fourier transform of u.We recall the following.

Definition 1. – *The mapping $T : \mathbb{R} \longrightarrow B(H)$, is said to be a Fourier multiplier of the type (p, p) if for all $f \in L^p(\mathbb{R}, H)$ we have*
$\left\|F^{-1} TFf\right\|_{L^p(\mathbb{R},H)} \leq C \|f\|_{L^p(\mathbb{R},H)}$.

Theorem 2 (Mikhlin-Schwartz) [7]. – *If the mapping*

$$\begin{array}{rccc} T: & \mathbb{R} & \mapsto & B(H) \\ & \sigma & \mapsto & T(\sigma) \end{array}$$

is continuously differentiable, and the inequalities

$$\|T(\sigma)\|_{B(H)} \leq C \text{ and } \left\|\frac{\partial}{\partial\sigma} T(\sigma)\right\|_{B(H)} \leq \frac{C}{|\sigma|} \text{ for } \sigma \neq 0$$

hold for all $\sigma \in \mathbb{R}$, then $T(\sigma)$is a Fourier multiplier of type (p, p).

3. – Coerciveness with defect of the principal problem.

Consider in $L^p(0,1;H)$ the following problem

$$
(2) \qquad \begin{cases} L_0(\lambda)u &= u''(x) - (A+\lambda I)^2 u(x) = f(x), \quad x \in (0,1), \\ L_{10}(\lambda)u &= \lambda(a_1 u(0) + \beta_1 u(1)) + \delta_1 u'(0) + \gamma_1 u'(1) = f_1, \\ L_{20}(\lambda)u &= a_2 u(0) + \beta_2 u(1) = f_2, \end{cases}
$$

where $a_1,\ a_2,\ \beta_1,\ \beta_2,\ \delta_1,\ \gamma_1 \in \mathbb{C}$; $f_1 \in (H, D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}$, $f_2 \in (H, D(A^2))_{1-\frac{1}{2p},p}$, $f \in L^p(0,1;H)$ and $u = u(x)$ is the unknown function.

Let $\mathcal{L}_0(\lambda)$ denote the operator

$$
\begin{array}{rcl} \mathcal{L}_0(\lambda) : u & \mapsto & (L_0(\lambda)u, L_{10}(\lambda)u, L_{20}(\lambda)u) \\ W_p^2(0,1;D(A^2),H) & \mapsto & L^p(0,1;H) \times (H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p} \times (H,D(A^2))_{1-\frac{1}{2p},p}. \end{array}
$$

We shall show that this operator is an isomorphism between the above mentioned spaces. In addition, we have a non-coerciveness estimate for some λ.

THEOREM 3. – *Suppose that*

1. *A is a closed linear operator with a dense domain in H, satisfying* $\|(A-sI)^{-1}\| \le \dfrac{C}{1+|s|}$, *for* $|\arg s| \ge \dfrac{\pi}{2}$ *and* $Re s \to -\infty$.

2. $\theta_1 = -\theta_2$, $\theta_2 \neq 0$, *with*

$$
\theta_1 = \begin{vmatrix} a_1 & \beta_1 \\ a_2 & \beta_2 \end{vmatrix},\ \theta_2 = \begin{vmatrix} \delta_1 & \gamma_1 \\ a_2 & -\beta_2 \end{vmatrix}.
$$

3. *There exists* $\delta \in (0, \dfrac{\pi}{2})$ *such that* $\left|\arg(1+\beta_1\gamma_1^{-1})\right| \le \delta$ *with* $\gamma_1 \neq 0$ *and* $|\arg(1 - a_1\delta_1^{-1})| \le \delta$ *with* $\delta_1 \neq 0$.

Then for all λ *such that* $|\arg\lambda| \le \dfrac{\pi}{2} - \delta$ *and* $|\lambda|$ *large enough, the operator*

$$
\mathcal{L}_0(\lambda) : u \longmapsto (L_0(\lambda)u,\ L_{10}(\lambda)u,\ L_{20}(\lambda)u)
$$

is an isomorphism from $W_p^2(0,1;D(A^2),H)$ *onto*

$$
L^p(0,1;H) \times (H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p} \times (H,D(A^2))_{1-\frac{1}{2p},p}.
$$

Moreover, for these λ *the following deficient coerciveness estimate holds:*

$$
\begin{aligned} &\|u''\|_{L^p(0,1;H)} + \left\|A^2u\right\|_{L^p(0,1;H)} + |\lambda|\|Au\|_{L^p(0,1;H)} + |\lambda|^2\|u\|_{L^p(0,1;H)} \\ &\quad \le C\{|\lambda|\|f\|_{L^p(0,1;H)} + \|f_1\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} \\ &\quad + |\lambda|^{2-\frac{1}{p}}\|f_1\|_H + \|f_2\|_{(H,D(A^2))_{1-\frac{1}{2p},p}} + |\lambda|^{3-\frac{1}{p}}\|f_2\|_H\}. \end{aligned}
$$

PROOF. – The operator $\mathcal{L}_0(\lambda)$ acts continuously from $W_p^2(0,1;H(A^2),H)$ into

$$L^p(0,1;H) \times (H, D(A^2))_{\frac{1}{2}-\frac{1}{2p},p} \times (H, D(A^2))_{1-\frac{1}{2p},p}\,;$$

this is a consequence of the continuity of the operator $u \mapsto L_0(\lambda)u$ and the Trace Theorem [12]. Indeed, we can show that $\|L_0(\lambda)u\|_{L^p(0,1;H)} \leq C(\lambda)\|u\|_{W_p^2(0,1;D(A^2),H)}$ and from the fact that $u(0)$, $u(1)$ belong to $(H, D(A^2))_{1-\frac{1}{2p}}$ and $u'(0)$, $u'(1)$ belong to $(H, D(A^2))_{\frac{1}{2}-\frac{1}{2p}}$ it follows that $u \mapsto L_{10}u$ and $u \mapsto L_{20}u$ act continuously from $W_p^2(0,1;D(A^2),H)$ to $(H, D(A^2))_{\frac{1}{2}-\frac{1}{2p}}$ and from $W_p^2(0,1;D(A^2),H)$ to $(H, D(A^2))_{1-\frac{1}{2p}}$, respectively.

Now, let $f_1 \in (H, D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}$, $f_2 \in (H, D(A^2))_{1-\frac{1}{2p},p}$, $f \in L^p(0,1;H)$. We shall show that the problem (2) has a solution $u \in W_p^2(0,1;D(A^2),H)$. We shall seek for this solution u under the form $u(x) = u_1(x) + u_2(x)$, where u_1 is the restriction to $[0,1]$ of $\widetilde{u}_1(x)$, $\tilde{u}_1(x)$ is the solution of the equation

$$L_0(\lambda)\widetilde{u}_1(x) = \tilde{f}(x), \quad x \in \mathbb{R}, \tag{3}$$

with

$$\tilde{f}(x) = \begin{cases} f(x) & x \in [0,1], \\ 0 & \text{otherwise}, \end{cases}$$

and $u_2(x)$ is the solution of the homogeneous problem

$$\begin{cases} L_0(\lambda)u_2(x) &= 0, \quad x \in [0,1], \\ L_{10}(\lambda)u_2(x) &= f_1 - L_{10}(\lambda)u_1(x), \\ L_{20}(\lambda)u_2(x) &= f_2 - L_{20}(\lambda)u_1(x). \end{cases} \tag{4}$$

By applying the Fourier transform to the equation (3) we obtain

$$\big((i\sigma)^2 - (A + \lambda I)^2\big)Fu = Ff, \sigma \in \mathbb{R}.$$

Since $(i\sigma)^2 - (A + \lambda I)^2 = -(A - (-i\sigma - \lambda))(A - (i\sigma - \lambda))$ and $\sigma \in \mathbb{R}$, this operator is invertible. This follows from the fact that $|\arg \lambda| \leq \frac{\pi}{2} - \varepsilon$, and by geometric arguments we can show that $i\sigma - \lambda$ and $-i\sigma - \lambda$ are in the resolvent $\rho(A)$ of A; notice also the estimate $|(i\sigma - \lambda)| \geq C(|\sigma| + |\lambda|)$. So, we can write

$$\widetilde{u}_1 = F^{-1}\big((i\sigma)^2 - (A + \lambda I)^2\big)^{-1}Ff, \sigma \in \mathbb{R}.$$

Hence $\widetilde{u}_1''$ is given by

$$\widetilde{u}_1'' = F^{-1}(i\sigma)^2\big((i\sigma)^2 - (A + \lambda I)^2\big)^{-1}Ff.$$

Setting

$$T_k(\sigma, \lambda) = \lambda^{2-k}(i\sigma)^k\big((i\sigma)^2 - (A + \lambda I)^2\big)^{-1},\ k = 0, 1, 2,$$

we obtain that $\|T_k(\sigma,\lambda)\| \leq c$, for $|\arg\lambda| \leq \frac{\pi}{2} - \varepsilon$ and $\left\|\frac{\partial}{\partial\sigma} T_k(\sigma,\lambda)\right\| \leq \frac{c}{|\sigma|}$.

By Mikhlin-Schwartz theorem [7] we conclude that $T_k(.,\lambda)$ are $(p,p)-$Fourier multipliers. Hence

$$\sum_{k=0}^{2} |\lambda|^{2-k} \|\widetilde{u}_1^{(k)}\|_{L^p(\mathbb{R};H)} = \sum_{k=0}^{2} \|F^{-1} T_k(\sigma,\lambda) F\widetilde{f}\|_{L^p(\mathbb{R};H)} \leq \|\widetilde{f}\|_{L^p(\mathbb{R};H)}.$$

Similarly, we can estimate $\|A^2\widetilde{u}_1\|_{L^p(\mathbb{R};H)}$, $|\lambda|\|A\widetilde{u}_1\|_{L^p(\mathbb{R};H)}$ and $|\lambda|\|\widetilde{u}_1'\|_{L^p(\mathbb{R};H)}$.

Finally, equation (3) has a unique solution $\widetilde{u}_1 \in W_p^2(\mathbb{R}; D(A^2), H)$ satisfying

$$\|\widetilde{u}_1''\|_{L^p(\mathbb{R};H)} + |\lambda|\|\widetilde{u}_1'\|_{L^p(\mathbb{R};H)} + |\lambda|^2\|\widetilde{u}_1\|_{L^p(\mathbb{R};H)} + \|A^2\widetilde{u}_1\|_{L^p(\mathbb{R};H)} + |\lambda|\|A\widetilde{u}_1\|_{L^p(\mathbb{R};H)}$$
$$\leq C\|\widetilde{f}\|_{L^p(\mathbb{R};H)}.$$

So, for u_1 we get the classical bound

$$\|u_1''\|_{L^p(0,1;H)} + \|A^2u_1\|_{L^p(0,1;H)} + |\lambda|\|Au_1\|_{L^p(0,1;H)} + |\lambda|^2\|u_1\|_{L^p(0,1;H)} \leq C\|f\|_{L^p(\mathbb{R};H)}.$$

Following [10, 17, 18], we write u_2, the solution of problem (4), in the form

$$u_2(x) = e^{-xA_\lambda} g_1 + e^{-(1-x)A_\lambda} g_2, \tag{5}$$

where $A_\lambda = A + \lambda I$ and $g_1, g_2 \in (H, D(A^2))_{1-\frac{1}{2p},p}$.

It suffices to calculate g_1 and g_2. Replacing (5) into the boundary conditions we obtain a system of equations in g_1 and g_2. The determinant of this system is given by

$$D(\lambda) = \theta_2 A + R(\lambda) = \theta_2 A(I + A^{-1}R(\lambda))$$

where

$$\|R(\lambda)\|_{B(H)} \to 0 \quad \text{when} \quad |\lambda| \to \infty.$$

By condition (2), $D(\lambda)$ is invertible and $(I + A^{-1}R(\lambda))^{-1} = I + T(\lambda)$ with $T(\lambda) = \sum_{k=1}^{\infty} (-A^{-1}R(\lambda))^k$. The series is convergent in the norm of $B(H)$, and thus,

$$D^{-1}(\lambda) = \theta_2^{-1} A^{-1}(I + T(\lambda)).$$

Hence, we deduce

$$\begin{aligned} g_1 &= D^{-1}(\lambda)\{(\beta_2 I + R_{11}(\lambda))(f_1 - L_{10}(\lambda)u_1) - (\lambda\beta_1 I + \gamma_1 A_\lambda + R_{12}(\lambda)) \\ &\quad (f_2 - L_{20}(\lambda)u_1)\}, \\ g_2 &= D^{-1}(\lambda)\{(\lambda a_1 I - \delta_1 A_\lambda - R_{21}(\lambda))(f_2 - L_{20}(\lambda)u_1) - (a_2 I + R_{22}(\lambda)) \\ &\quad (f_1 - L_{10}(\lambda)u_1)\}, \end{aligned} \tag{6}$$

where $\|R_{ij}(\lambda)\|_{B(H)} \to 0$ when $|\lambda| \to \infty$, since all $R_{ij}(\lambda)$ contain the function e^{-A_λ} which converges to 0 when $|\lambda| \to \infty$.

Using condition (2) and well known properties of interpolation spaces, we can see that g_1 and g_2 are in $(H, D(A^2))_{1-\frac{1}{2p},p}$. Thus, problem (4) has a unique solution $u_2(x)$ given by:

$$\begin{aligned} u_2(x) =& e^{-xA_\lambda}\theta_2^{-1}A^{-1}(I+T(\lambda))(\beta_2 I + R_{11}(\lambda))(f_1 - L_{10}(\lambda)u_1) \\ &- e^{-xA_\lambda}\theta_2^{-1}A^{-1}(I+T(\lambda))\gamma_1 A_\mu(I + A_\mu^{-1}\gamma_1^{-1}R_{12}(\lambda))(f_2 - L_{20}(\lambda)u_1) \\ &- e^{-(1-x)A_\lambda}\theta_2^{-1}A^{-1}(I+T(\lambda))\delta_1 A_{\mu'}(I + A_{\mu'}^{-1}\delta_1^{-1}R_{21}(\lambda))(f_2 - L_{20}(\lambda)u_1) \\ &+ e^{-(1-x)A_\lambda}\theta_2^{-1}A^{-1}(I+T(\lambda))(-\alpha_2 I - R_{22}(\lambda))(f_1 - L_{10}(\lambda)u_1), \end{aligned}$$

where $A_\mu = A + \lambda(1 + \beta_1\gamma_1^{-1})$ and $A_{\mu'} = A + \lambda(1 - \alpha_1\delta_1^{-1})$. This completes the proof that $\mathcal{L}_0(\lambda)$ is an isomorphism. To establish the deficient coerciveness, we estimate $\|u_2''\|_{L^p(0,1;H)}$ and to this end we write u_2'' in the form $u_2''(x) = v_1(x) + v_2(x) + v_3(x) + v_4(x)$, where

$$\begin{aligned} v_1(x) =& A_\lambda^2 e^{-xA_\lambda}\theta_2^{-1}A^{-1}(I+T(\lambda))(\beta_2 I + R_{11}(\lambda))(f_1 - L_{10}(\lambda)u_1) \\ v_2(x) =& -A_\lambda^2 e^{-xA_\lambda}\theta_2^{-1}A^{-1}(I+T(\lambda))\gamma_1 A_\mu(I + A_\mu^{-1}R_{12}(\lambda))(f_2 - L_{20}(\lambda)u_1) \\ v_3(x) =& A_\lambda^2 e^{-(1-x)A_\lambda}\theta_2^{-1}A^{-1}(I+T(\lambda))(-\alpha_2 I - R_{22}(\lambda))(f_1 - L_{10}(\lambda)u_1) \\ v_4(x) =& -A_\lambda^2 e^{-(1-x)A_\lambda}\theta_2^{-1}A^{-1}(I+T(\lambda))\delta_1 A_{\mu'}(I + A_{\mu'}^{-1}R_{21}(\lambda))(f_2 - L_{20}(\lambda)u_1). \end{aligned}$$

Since (with $h_1 = f_1 - L_{10}(x)u_1$)

$$\|v_1\|_{L^p(0,1;H)} \le C\|A_\lambda^2 e^{-xA_\lambda}A^{-1}h_1\|_{L^p(0,1;H)} \le C\|(A^2 + 2\lambda A + \lambda^2)e^{-xA_\lambda}A^{-1}h_1\|_{L^p(0,1;H)},$$

we need only to estimate in $L^p(0,1;H)$ each term of the expression

$$(A^2 + 2\lambda A + \lambda^2)e^{-xA_\lambda}A^{-1}(f_1 - L_{10}(\lambda)u_1).$$

We have

1.
$$\left(\int_0^1 \|A^2 e^{-xA_\lambda}A^{-1}h_1\|_H^p dx\right)^{\frac{1}{p}} = \left(\int_0^1 \|Ae^{-xA_\lambda}h_1\|_H^p dx\right)^{\frac{1}{p}} \le C\|h_1\|_{(H,D(A))1-\frac{1}{p},p},$$

but

$$(H, D(A))_{1-\frac{1}{p},p} = (H, D(A^2))_{\frac{1}{2}-\frac{1}{2p},p},$$

and so we get

$$\left(\int_0^1 \|A^2 e^{-xA_\lambda}A^{-1}h_1\|_H^p dx\right)^{\frac{1}{p}} \le C\|h_1\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}}.$$

2.

$$\left(\int_0^1 \|2\lambda A e^{-xA_\lambda} A^{-1} h_1\|_H^p dx\right)^{\frac{1}{p}} \leq |\lambda| c \|h_1\|_H \left(\int_0^1 e^{-xpRe\lambda} dx\right)^{\frac{1}{p}} \leq C|\lambda|^{1-\frac{1}{p}} \|h_1\|_H.$$

3.

$$\left(\int_0^1 \|\lambda^2 e^{-xA_\lambda} A^{-1} h_1\|_H^p dx\right)^{\frac{1}{p}} \leq |\lambda^2| \left(\int_0^1 e^{-xpRe\lambda} dx\right)^{\frac{1}{p}} \|A^{-1}\|_{B(H,H)} \|h_1\|_H$$

$$\leq C|\lambda^2||\lambda|^{-\frac{1}{p}} \|h_1\|_H = C|\lambda|^{2-\frac{1}{p}} \|h_1\|_H.$$

Hence

$$\|v_1\|_{L^p(0,1;H)} \leq C\left\{|\lambda|^{2-\frac{1}{p}} \|h_1\|_H + |\lambda|^{1-\frac{1}{p}} \|h_1\|_H + \|h_1\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}}\right\}.$$

Similarly we can estimate v_2, v_3, and v_4. We obtain

$$\begin{aligned}
&\|u_2''\|_{L^p(0,1;H)} + \|A^2 u_2\|_{L^p(0,1;H)} + |\lambda| \|A u_2\|_{L^p(0,1;H)} + |\lambda|^2 \|u_2\|_{L^p(0,1;H)} \\
&\leq C\{\|f_1\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} + \|L_{10}(\lambda)u_1\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} + \|f_2\|_{(H,D(A^2))_{1-\frac{1}{2p},p}} \\
&\quad + \|L_{20}(\lambda)u_1\|_{(H,D(A^2))_{1-\frac{1}{2p},p}} + |\lambda| \|f_2\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} \\
&\quad + |\lambda| \|L_{20}(\lambda)u_1\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} + |\lambda|^{2-\frac{1}{p}} \|f_1\|_H + |\lambda|^{1-\frac{1}{p}} \|f_1\|_H \\
&\quad + |\lambda|^{1-\frac{1}{p}} \|L_{10}(\lambda)u_1\|_H + |\lambda|^{2-\frac{1}{p}} \|L_{10}(\lambda)u_1\|_H + |\lambda|^{2-\frac{1}{p}} \|f_2\|_H \\
&\quad + |\lambda|^{3-\frac{1}{p}} \|f_2\|_H + |\lambda|^{2-\frac{1}{p}} \|L_{20}(\lambda)u_1\|_H + |\lambda|^{3-\frac{1}{p}} \|L_{20}(\lambda)u_1\|_H\}.
\end{aligned}$$

Using the reiteration theorem and the Young inequality, we obtain

$$|\lambda| \|f_2\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} \leq C\left\{|\lambda|^{2-\frac{1}{p}} \|f_2\|_H + \|f_2\|_{(H,D(A^2))_{1-\frac{1}{2p},p}}\right\}.$$

Therefore, we get the estimate for the solution of the problem (4)

$$\begin{aligned}
&\|u_2''\|_{L^p(0,1;H)} + \|A^2 u_2\|_{L^p(0,1;H)} + |\lambda| \|A u_2\|_{L^p(0,1;H)} + |\lambda|^2 \|u_2\|_{L^p(0,1;H)} \\
&\leq C\{\|f_1\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} + \|L_{10}(\lambda)u_1\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} + \|f_2\|_{(H,D(A^2))_{1-\frac{1}{2p},p}} \\
&\quad + \|L_{20}(\lambda)u_1\|_{(H,D(A^2))_{1-\frac{1}{2p},p}} + |\lambda| \|L_{20}(\lambda)u_1\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} \\
&\quad + |\lambda|^{2-\frac{1}{p}} \|f_1\|_H + |\lambda|^{1-\frac{1}{p}} \|f_1\|_H + |\lambda|^{1-\frac{1}{p}} \|L_{10}(\lambda)u_1\|_H \\
&\quad + |\lambda|^{2-\frac{1}{p}} \|L_{10}(\lambda)u_1\|_H + |\lambda|^{2-\frac{1}{p}} \|f_2\|_H \\
&\quad + |\lambda|^{3-\frac{1}{p}} \|f_2\|_H + |\lambda|^{2-\frac{1}{p}} \|L_{20}(\lambda)u_1\|_H + |\lambda|^{3-\frac{1}{p}} \|L_{20}(\lambda)u_1\|_H\}.
\end{aligned}$$

It remains to estimate the terms containing boundary operators. For example, we consider the term $\|L_{10}(\lambda)u_1\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}}$, for which we have

$$\|L_{10}(\lambda)u_1\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} = \left\|\lambda(a_1u_1(0)+\beta_1u_1(1))+\delta_1u_1'(0)+\gamma_1u_1'(1)\right\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}}$$

$$\leq C\Big[|\lambda|\|u_1(0)\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}}+|\lambda|\|u_1(1)\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} + \left\|u_1'(0)\right\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}}+\left\|u_1'(1)\right\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}}\Big].$$

By the trace theorem [11, p. 87], we deduce

$$\|u_1(0)\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} = \|u_1(0)\|_{(H,D(A))_{1-\frac{1}{p},p}} \leq C\Big\{\|Au_1\|_{L^p(0,1;H)}+\left\|u_1'\right\|_{L^p(0,1;H)}\Big\},$$

so that

$$|\lambda|\|u_1(0)\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} \leq C\Big\{|\lambda|\|Au_1\|_{L^p(0,1;H)}+|\lambda|\left\|u_1'\right\|_{L^p(0,1;H)}\Big\}$$

and, as it was seen previously,

$$|\lambda|\|Au_1\|_{L^p(0,1;H)} \leq C\|f\|_{L^p(0,1;H)},$$

$$|\lambda|\left\|u_1'\right\|_{L^p(0,1;H)} \leq C\|f\|_{L^p(0,1;H)}.$$

Therefore

$$|\lambda|\|u_1(0)\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} \leq C\|f\|_{L^p(0,1;H)},$$

which implies

$$\|L_{10}(\lambda)u_1\|_{(H,D(A^2))_{\frac{1}{2}-\frac{1}{2p},p}} \leq C\|f\|_{L^p(0,1;H)}.$$

The other terms can be similarly estimated. This completes the proof of the theorem. □

4. – Applications.

4.1 – *Application* 1. – Consider in $L^p(0,1;L^2(0,1))$ the boundary value problem

$$\begin{cases} \dfrac{\partial^2u(x,y)}{\partial x^2}-\left(-\dfrac{\partial^2}{\partial y^2}+\lambda I\right)^2u(x,y)=f(x,y), & (x,y)\in(0,1)\times(0,1),\\ u(x,0)=u(x,1)=0=\dfrac{\partial^2u(x,0)}{\partial y^2}=\dfrac{\partial^2u(x,1)}{\partial y^2}, & 0<x<1,\\ \lambda(a_1u(0,y)+\beta_1u(1,y))+\delta_1\dfrac{\partial u}{\partial x}(0,y)+\gamma_1\dfrac{\partial u}{\partial x}(1,y)=f_1(y), & 0<y<1,\\ a_2u(0,y)+\beta_2u(1,y)=f_2(y), & 0<y<1,\end{cases}$$

where λ is a complex parameter and the coefficients $a_1,\ \beta_1,\ \delta_1,\ \gamma_1,\ a_2,\ \beta_2$ are given. We take the operator A in $L^2(0,1)$ according to $D(A) = H_0^1(0,1) \cap H^2(0,1)$, $(Au)(y) = -u''(y) = -\frac{d^2u}{dy^2}(y)$.

Then it is well known that A is the opposite of the generator of an analytic semigroup on $H = L^2(0,1)$ satisfying assumption 1 in Theorem 3. Therefore, if the numbers $a_1,\ \beta_1,\ \delta_1,\ \gamma_1,\ a_2,\ \beta_2$ are like in assumptions 2 and 3 in the quoted Theorem, then for all λ with $|\arg\lambda| \leq \frac{\pi}{2} - \delta$, where $0 < \delta < \frac{\pi}{2}$, $|\lambda| \mapsto \infty$, the preceding problem admits a unique solution $u \in W_p^2(0,1; D(A^2), H)$ provided that $f_1 \in (L^2(0,1),\ H_0^1(0,1) \cap H^2(0,1))_{1-\frac{1}{p},p}, f_2 \in (L^2(0,1),\ D(A^2))_{1-\frac{1}{2p},p}$.

Recall that, from [13, Théorème 3.2, p. 59], $a \in (H, D(A^2))_{1-\frac{1}{2p},p}$ if and only if $a \in D(A)$ and $Aa \in (H, D(A))_{1-\frac{1}{p},p}$. Therefore, by [18, Theorem 6, p. 45], our assumptions on f_1, f_2 reduce to

$$f_1 \in B_{2,p}^{2(1-1/p)}(0,1), \text{if } 1 < p < \frac{3}{4},$$

$$f_1 \in B_{2,p}^{2(1-1/p)}(0,1) \text{ and } f_1(j) = 0,\ j = 0,1, \text{if } \frac{3}{4} < p < \infty,$$

$$f_2 \in H_0^1(0,1) \cap H^2(0,1),\ \frac{d^2f_2}{dy^2} \in B_{2,p}^{2(1-1/p)}(0,1), \text{if } 1 < p < \frac{3}{4},$$

$$f_2 \in H_0^1(0,1) \cap H^2(0,1),\ \frac{d^2f_2}{dy^2} \in B_{2,p}^{2(1-1/p)}(0,1),\ \frac{d^2f}{dy^2}(j) = 0,\ j = 0,1, \text{if } \frac{3}{4} < p < \infty.$$

Here $B_{p,q}^s(0,1)$ denotes the Besov space.

4.2 – *Application* 2. – We consider the elliptic degenerate problem

$$\begin{cases} \dfrac{\partial^2 u(x,y)}{\partial x^2} - \left(y(1-y)\dfrac{\partial^2}{\partial y^2} - \lambda I\right)^2 u(x,y) = f(x,y), & (x,y) \in (0,1) \times (0,1), \\ \lim\limits_{y\to 0,1} y(1-y)\dfrac{\partial^2 u}{\partial y^2}(x,y) = 0 = \lim\limits_{y\to 0,1}\left(y(1-y)\dfrac{\partial^2}{\partial y^2}\right)u(x,y), & 0 < x < 1, \\ \lambda(a_1 u(0,y) + \beta_1 u(1,y)) + \delta_1 \dfrac{\partial u}{\partial x}(0,y) + \gamma_1 \dfrac{\partial u}{\partial x}(1,y) = f_1(y), & 0 < y < 1, \\ a_2 u(0,y) + \beta_2 u(1,y) = f_2(y), & 0 < y < 1. \end{cases}$$

Here we take $H = H^1(0,1)$, endowed with the inner product $(f,g) = \int_0^1 f(y)\bar{g}(y)dx + \int_0^1 f'(y)\bar{g}'(y)dy$.

Let us introduce an operator A by $D(A) = \{u \in H^1(0,1); y(1-y)u'' \in H_0^1(0,1)\}$, $(Au)(y) = -y(1-y)u''(y)$, where the derivative u'' is viewed in the sense of distributions. Then it is known [8, 9] that $-A$ generates an analytic semigroup in H, so that Theorem 3 applies immediately. In this case, interpretation of the interpolation spaces seems complicate, but one can assume $f_1 \in D(A)$ and $f_2 \in D(A^2)$.

4.3 – *Application* 3. – More generally than in Application 1, consider in $L^p(0,1;L^2(0,1))$ the problem

$$(7)\qquad \begin{aligned} &-\frac{\partial^2 u(x,y)}{\partial x^2} + a(y)\frac{\partial^{2m}}{\partial y^{2m}}\left(a(y)\,\frac{\partial^{2m}u(x,y)}{\partial y^{2m}}\right) - 2\lambda a(y)\,\frac{\partial^{2m}u(x,y)}{\partial y^{2m}} \\ &+\lambda^2 u(x,y) = f(x,y), \quad 0 < x, y < 1, \end{aligned}$$

$$(8)\qquad \frac{\partial^k u}{\partial y^k}(x,0) = 0 = \frac{\partial^k u}{\partial y^k}(x,1),\ 0 < x < 1,\ k = 0,\ldots,m-1,$$

$$(9)\qquad \frac{\partial^{k+2m} u}{\partial y^{k+2m}}(x,0) = 0 = \frac{\partial^{k+2m} u}{\partial y^{k+2m}}(x,1),\ 0 < x < 1,\ k = 0,\ldots,m-1,$$

$$\lambda(a_1 u(0,y) + \beta_1 u(1,y)) + \delta_1\frac{\partial u(0,y)}{\partial x} + \gamma_1\frac{\partial u(1,y)}{\partial x} = f_1(y),\ 0 < y < 1,$$

$$a_2 u(0,y) + \beta_2 u(1,y) = f_2(y),\ 0 < y < 1,$$

where

$$f \in L^p(0,1;L^2(0,1));\ f_1 \in (L^2(0,1),\ H^{2m}(0,1) \cap H_0^m(0,1))_{1-\frac{1}{p},p},$$

$$f_2 \in H^{2m}(0,1) \cap H_0^m(0,1),\ \frac{d^{2m}f_2}{dy^{2m}} \in (L^2(0,1), H^{2m}(0,1) \cap H_0^m(0,1))_{1-\frac{1}{p},p}.$$

Notice, see again [18, Theorem 6, p. 45], that Grisvard-Seeley type theorem applies, so that they read

$$\begin{aligned} &f_1 \in B_{2,p}^{2m(1-\frac{1}{p})}(0,1),\ f_1^{(j)}(0) = 0 = f_1^{(j)}(1) \\ &\qquad \text{for all } j \in \{0,1,\ldots,m-1\} \text{ such that } j < 2m\left(1-\frac{1}{p}\right) - \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} &f_2 \in H^{2m}(0,1) \cap H_0^m(0,1),\ \frac{d^{2m}f_2}{dy^{2m}} \in B_{2,p}^{2m(1-1/p)}(0,1),\ f_2^{(2m+j)}(0) = 0 = f_2^{(2m+j)}(1) \\ &\qquad \text{for all } j \in \{0,1,\ldots,m-1\} \text{ such that } j < 2m\left(1-\frac{1}{p}\right) - \frac{1}{2}. \end{aligned}$$

Now, we are able to establish the result as follows:

THEOREM 4. – *Let m be a positive integer, and suppose that*

1. *for $\delta > 0$ we have $|\arg a(y)| \leq \frac{\pi}{2} - \delta$ if m is odd and $|\arg a(y)| \geq \frac{\pi}{2} + \delta$ if m is even;*
2. $a(y) \in C^{2m}[0,1]; a(y) \neq 0; a(0) = a(1)$;
3. $\theta_1 = -\theta_2, \theta_2 \neq 0$, *with*

$$\theta_1 = \begin{vmatrix} a_1 & \beta_1 \\ a_2 & \beta_2 \end{vmatrix} \text{ and } \theta_2 = \begin{vmatrix} \delta_1 & \gamma_1 \\ a_2 & -\beta_2 \end{vmatrix};$$

4. *there exists $\delta \in \left(0, \frac{\pi}{2}\right)$ such that $\left|\arg(1 + \beta_1 \gamma_1^{-1})\right| \leq \delta$ for $\gamma_1 \neq 0$ and $\left|\arg(1 - a_1 \delta_1^{-1})\right| \leq \delta$ for $\delta_1 \neq 0$;*
5. *f, f_1, f_2 satisfy assumption* (10).

Then for all λ such that $|\arg \lambda| \leq \frac{\pi}{2} - \delta$ and $|\lambda|$ large enough, problem (7), (8), (9) *has a unique solution u such that*

$$\frac{\partial^2 u}{\partial x^2} = D_x^2 u \in L^p(0,1;L^2(0,1)), \ \frac{\partial^{4m} u}{\partial y^{4m}} = D_y^{4m} u \in L^p(0,1;L^2(0,1)).$$

In addition, the following coerciveness inequality with defect holds:

$$\begin{aligned} &\|D_x^2 u\|_{L^p(0,1;L^2(0,1))} + |\lambda| \|D_y^{2m} u\|_{L^p(0,1;L^2(0,1))} + \|D_y^{4m} u\|_{L^p(0,1;L^2(0,1))} \\ &\quad + |\lambda|^2 \|u\|_{L^p(0,1;L^2(0,1))} \\ &\leq C\Big\{ |\lambda| \, \|f\|_{L^p(0,1;L^2(0,1))} + \|f_1\|_{B_{2,p}^{2m(1-\frac{1}{p})}(0,1)} + |\lambda|^{2-\frac{1}{p}} \|f_1\|_{L^2(0,1)} \\ &\quad + |\lambda|^{3-\frac{1}{p}} \|f_2\|_{L^2(0,1)} + \|f_2\|_{H^{2m}(0,1)} + \|D_y^{2m} f_2\|_{B_{2,p}^{2m(1-1/p)}(0,1)} \Big\}. \end{aligned}$$

PROOF. – Define in $L^2(0,1)$ the following operator $A = -a(y) D_y^{2m} = -a(y) \frac{\partial^{2m}}{\partial y^{2m}}$ with

$$D(A) = H^{2m}(0,1) \cap H_0^m(0,1).$$

Hence $D(A^2)$ is given by

$$D(A^2) = \{u \in H^{4m}(0,1) \cap H_0^m(0,1) \text{ such that } D_y^{2m+k} u(0) = 0 = D_y^{2m+k} u(1) \text{ for } k = 0, \ldots, m-1\}.$$

Then problem (7), (8), (9) can be rewritten in the form (2). According to [16], [18, Theorem 1, p. 111], taking $H = L^2(0,1)$, the operator A satisfies all conditions of Theorem 3. □

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