Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **10-B** (2007), n.3, p. 549–558.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_549_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 549-558

Sums of Three Prime Squares.

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Sunto. – *Siano A, $\varepsilon > 0$ arbitrari. Supponiamo che x sia un numero positivo sufficientemente grande. Proviamo che il numero di interi n appartenenti ad $(x, x + x^{\theta}]$, e soddisfacenti alcune condizioni di congruenza naturali, che non si possono scrivere come somma di tre quadrati di primi è $\ll x^{\theta}(\log x)^{-A}$ con $7/16 + \varepsilon \le \theta \le 1$.*

Summary. – *Let A, $\varepsilon > 0$ be arbitrary. Suppose that x is a sufficiently large positive number. We prove that the number of integers $n \in (x, x + x^{\theta}]$, satisfying some natural congruence conditions, which cannot be written as the sum of three squares of primes is $\ll x^{\theta}(\log x)^{-A}$, provided that $7/16 + \varepsilon \le \theta \le 1$.*

1. – Introduction.

It is conjectured that every integer $n \in \mathbb{H} = \{m : m \equiv 3 \,(\mathrm{mod}\ 24),\ m \not\equiv 0 \,(\mathrm{mod}\ 5)\}$ can be written as the sum of three squares of primes. The number of possible exceptions up to x,

$$E(x) = \{n \le x \,:\, n \in \mathbb{H},\ n \ne p_1^2 + p_2^2 + p_3^2, \text{ for all prime } p_i,\ i = 1, 2, 3\},$$

was first estimated in 1938 by Hua [3], who showed that $E(x) \ll x(\log x)^{-A}$ for a certain constant $A > 0$. In 1961 Schwarz [11] demonstrated that any $A > 0$ is acceptable. In 1993 Leung and Liu [5] proved that $E(x) \ll x^{1-\delta}$ for some absolute constant $\delta > 0$. In 2000 Bauer, Liu and Zhan [1] gave a specific value for δ, namely $\delta = 9/160 - \varepsilon$, which was later improved to $\delta = 3/50 - \varepsilon$ by Liu and Zhan [6].

Liu and Zhan [7] were also the first to investigate the local properties of $E(x)$. In 1996 they proved that for any $A > 0$ one has

$$E(x + x^{\theta}) - E(x) \ll x^{\theta}(\log x)^{-A}, \tag{1}$$

provided that $3/4 + \varepsilon \le \theta \le 1$. An important tool in their proof is a new mean value estimate for non linear exponential sums over primes [7, Theorem 3]. Shortly

(*) The second author was supported by the Japan Society for the Promotion of Science, P03763.

thereafter, the first named author [8] replaced the constant 3/4 with 1/2 by using a tricky argument based on certain properties of the sequence of squares.

In this paper we refine Liu and Zhan's approach [7], and establish the following result.

THEOREM 1. – *Let* A, $\varepsilon > 0$ *be arbitrary. Then the inequality* (1) *holds, provided that* $7/16 + \varepsilon \le \theta \le 1$.

Theorem 1 can be derived by a standard argument (see [7, §1]) from the average estimate, which we formulate as Theorem 2 below. Let $\Lambda(n)$ and $\varphi(n)$ denote the von Mangoldt function and the Euler function, respectively, and write $e(\alpha) = e^{2\pi i\alpha}$ for real α. Define

$$R(n) = R(n, x, y) = \sum\sum\sum_{\substack{k^2+l^2+m^2=n \\ x-y<k^2+l^2\le x \\ y/4<m^2\le y}} \Lambda(k)\Lambda(l)\Lambda(m)\,,$$

$$s(q, a) = \sum_{\substack{h=1 \\ (h,q)=1}}^{q} e\left(\frac{ah^2}{q}\right), \qquad \sigma(n, P) = \sum_{q\le P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\frac{s(q, a)}{\varphi(q)}\right)^3 e\left(-\frac{an}{q}\right).$$

Our main result is the following.

THEOREM 2. – *Let* A, $\varepsilon > 0$ *be given,* $x^{7/12+\varepsilon} \le y \le x$ *and* $y^{3/4+\varepsilon} \le H \le y/4$. *Then there exists a constant* $B_0 = B_0(A) > 0$ *such that for* $B > B_0$ *we have*

$$\sum_{x<n\le x+H} \left| R(n) - \frac{\pi}{8}\sqrt{y}\,\sigma(n, (\log x)^B) \right|^2 \ll_{A,\varepsilon} Hy(\log x)^{-A}.$$

To prove Theorem 2 we apply the Hardy–Littlewood circle method, the main difficulties arising in the treatment of the major arcs error term, see Lemma 3. The minor arcs are handled by using a version of Liu and Zhan's estimate [7, Theorem 3].

2. – Lemmas.

Before we are able to establish Theorem 2, we require some auxiliary results. Our first lemma is a minor modification of [7, Theorem 3].

LEMMA 1. – *Let* X *be a large number,* $Y = X^\theta$ *and* $1/2 \le \theta < 1$. *Suppose that* $|\alpha - a/q| \le 1/q^2$ *with* $(a, q) = 1$. *Then*

$$Y \int_{|\lambda|\le 1/Y} \left| \sum_{X<n^2\le 4X} \Lambda(n)\, e((\alpha+\lambda)n^2) \right|^2 d\lambda$$

$$\ll X^{3/4}(\log X)^6 + (Yq^{-1/4} + YX^{-1/8} + Y^{3/4}q^{1/4})(\log X)^{13}\,.$$

LEMMA 2. – *Let D, $E>0$ and $0<\varepsilon<1/6$ be given. Let X be a large number, $Y=X^\theta$ and $7/12+\varepsilon\le\theta\le 1-\varepsilon$. Then, uniformly for $1\le a\le q\le(\log X)^D$ with $(a,q)=1$, we have*

$$\int\limits_{|\lambda|\le 1/Y}\left|\sum_{n^2\le X}\Lambda(n)\,e\left(\left(\frac{a}{q}+\lambda\right)n^2\right)-\frac{s(q,a)}{\varphi(q)}\sum_{n^2\le X}e(\lambda n^2)\right|^2 d\lambda \ll (\log X)^{-E}, \tag{2}$$

where the implied constant depends on D, E, ε, and θ.

PROOF. – By the orthogonality of the characters mod q, we see that the left-hand side of (2) is

$$\ll \sum_{\chi \bmod q}\ \int\limits_{|\lambda|\le 1/Y}\left|\sideset{}{^\#}\sum_{n^2\le X}\chi(n)\Lambda(n)e(\lambda n^2)\right|^2 d\lambda + Y^{-1}(\log X)^4,$$

where $\sum^\#$ indicates that, for $\chi=\chi_0$, $\chi(n)\Lambda(n)$ is to be replaced by $\Lambda(n)-1$. By Gallagher's lemma [2, Lemma 1] and the Siegel-Walfisz theorem (see, e.g., [10, Chapter IV, Satz 8.3]), the above sum becomes

$$\ll Y^{-2}\sum_{\chi \bmod q}\ \int\limits_Y^X\left|\sideset{}{^\#}\sum_{t<n^2\le t+Y/2}\chi(n)\Lambda(n)\right|^2 dt + qYX^{-1}+q(\log X)^{-E-D}.$$

Since for a sufficiently small $\varepsilon>0$,

$$(t+Y/2)^{1/2}-t^{1/2}\gg Yt^{-1/2}\gg (t^{1/2})^{1/6+\varepsilon},$$

Huxley's zero-density estimate for the Dirichlet L-functions [4] and the zero-free region [10, Chapter VIII, Satz 6.2] yield the desired result. □

The next statement can be derived easily from Lemma 2.

LEMMA 2′. – *Suppose that the hypothesis of Lemma 2 is satisfied. Then*

$$\int\limits_{|\lambda|\le 1/Y}\left|\sum_{n^2\le X}\Lambda(n)e\left(\left(\frac{a}{q}+\lambda\right)n^2\right)-\frac{s(q,a)}{\varphi(q)}\sum_{n\le X}\frac{e(\lambda n)}{2\sqrt{n}}\right|^2 d\lambda \ll (\log X)^{-E},$$

where the implied constant depends on D, E, ε, and θ.

LEMMA 3. – *Let D, $E>0$ and $7/12<\theta<1$ be given. Let X be a large number, $Y=X^\theta$. Then, uniformly for $1\le a\le q\le(\log X)^D$ with $(a,q)=1$, we have*

$$\sum\sum_{X-Y<m^2+n^2\le X}\Lambda(m)\Lambda(n)\left(\frac{a}{q}(m^2+n^2)\right)=\frac{\pi}{4}\left(\frac{s(q,a)}{\varphi(q)}\right)^2 Y+\mathcal{O}(Y(\log X)^{-E}), \tag{3}$$

where the implied constant depends on D, E, and θ.

PROOF. – We begin by noting that the double sum in the left-hand side of (3) can be written as

$$W := \int_{-1/2}^{1/2} S\left(\frac{a}{q}+\beta\right)^2 K(-\beta)\, d\beta\,,$$

where

$$S(\alpha) = \sum_{n^2 \le X} \Lambda(n) e(\alpha n^2)\,, \qquad K(\beta) = \sum_{X-Y<n\le X} e(\beta n) \ll \min(Y, \|\beta\|^{-1})\,.$$

In order to approximate $S(a/q+\beta)$, we define

$$T\left(\frac{a}{q}+\beta\right) = \frac{s(q,a)}{\varphi(q)}\, t(\beta)\,, \qquad t(\beta) = \sum_{n^2\le X} e(\beta n^2)\,.$$

For brevity, we write $S_\beta = S(a/q+\beta)$ and $T_\beta = T(a/q+\beta)$. Since $A^2 = B^2 + (A-B)(A+B)$, we have that

$$W = \left(\frac{s(q,a)}{\varphi(q)}\right)^2 \int_{-1/2}^{1/2} t(\beta)^2 K(-\beta)\, d\beta$$

$$+\, \mathcal{O}\left(\int_{-1/2}^{1/2} |S_\beta - T_\beta|(|S_\beta| + |T_\beta|)|K(\beta)|\, d\beta\right). \tag{4}$$

An elementary argument, arising in the Gauss circle problem, shows that the integral in the main term is equal to

$$\sum\sum_{X-Y<m^2+n^2\le X} 1 = \frac{\pi}{4}\, Y + \mathcal{O}(X^{1/2})\,. \tag{5}$$

Next we bound the error term in (4). Since

$$\int_{-1/2}^{1/2} \min(Y, |\beta|^{-1}) e(n\beta)\, d\beta \ll \min\left(\log Y, \frac{Y}{|n|}\right),$$

we have that

$$\begin{aligned} \int_{-1/2}^{1/2} |S_\beta|^2 |K(\beta)|\, d\beta &\ll \sum\sum_{m^2,\, n^2 \le X} \Lambda(m)\Lambda(n) \min\left(\log Y, \frac{Y}{|m^2-n^2|}\right) \\ &\ll \sum_{n^2\le X} \Lambda(n)^2 \log Y + \sum_{h\le X} \tau(h) \frac{Y}{h} (\log X)^2 \\ &\ll X^{1/2}(\log X)^2 + Y(\log X)^4 \\ &\ll Y(\log X)^4\,. \end{aligned} \tag{6}$$

Similarly,

$$\int_{-1/2}^{1/2} |T_\beta|^2 |K(\beta)|\, d\beta \ll Y(\log X)^2 . \tag{7}$$

Put $\tilde{Y} = Y(\log X)^{-8-2E}$, so that $\tilde{Y} \geq X^{\theta'}$ with some $\theta' > 7/12$, provided that X is large. Then Lemma 2 yields

$$\int_{|\beta|\leq 1/\tilde{Y}} |S_\beta - T_\beta|^2 |K(\beta)|\, d\beta \ll Y \int_{|\beta|\leq 1/\tilde{Y}} |S_\beta - T_\beta|^2\, d\beta \ll Y(\log X)^{-4-2E} .$$

In the remaining range $1/\tilde{Y} \leq |\beta| \leq 1/2$, we see that

$$|K(\beta)| \ll |\beta|^{-1} \leq \min(\tilde{Y}, |\beta|^{-1}) .$$

As in (6) and (7), we obtain that

$$\begin{aligned}\int_{1/\tilde{Y}\leq|\beta|\leq 1/2} |S_\beta - T_\beta|^2 |K(\beta)|\, d\beta &\ll \int_{|\beta|\leq 1/2} (|S_\beta|^2 + |T_\beta|^2) \min(\tilde{Y}, |\beta|^{-1})\, d\beta \\ &\ll X^{1/2}(\log X)^2 + \tilde{Y}(\log X)^4 \\ &\ll Y(\log X)^{-4-2E} .\end{aligned}$$

Hence,

$$\int_{-1/2}^{1/2} |S_\beta - T_\beta|^2 |K(\beta)|\, d\beta \ll Y(\log X)^{-4-2E} .$$

Combining this estimate with (6) and (7) by means of Schwartz's inequality, we derive that the error term in (4) is

$$\ll \Big(Y(\log X)^{-4-2E}\Big)^{1/2} \Big(Y(\log X)^4\Big)^{1/2} = Y(\log X)^{-E} . \tag{8}$$

Now the conclusion of Lemma 3 follows from (4), (5) and (8). □

3. – Proof of Theorem 2.

We have

$$R(n) = \int_0^1 S_1(a) S_2(a) e(-an)\, da ,$$

where

$$S_1(\alpha) = \sum\sum_{x-y<k^2+l^2\le x} \Lambda(k)\Lambda(l)e(\alpha(k^2+l^2))$$

and

$$S_2(\alpha) = \sum_{y/4<m^2\le y} \Lambda(m)e(\alpha m^2)\,.$$

Put

$$P = L^B, \qquad Q = HL^{-B},$$

where $L = \log x$ and $B = 4A + 84$. Define the set of major arcs $\mathfrak{M}$ as the union of all intervals $\{\alpha \in \mathbb{R} : |q\alpha - a| \le 1/Q\}$ with $1 \le a \le q \le P$ and $(a, q) = 1$. Denote the corresponding set of minor arcs by $\mathfrak{m} = [1/Q, 1+1/Q]\backslash\mathfrak{M}$. Hence,

$$R(n) = \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right) S_1(\alpha)S_2(\alpha)e(-\alpha n)d\alpha = R_{\mathfrak{M}}(n) + R_{\mathfrak{m}}(n)\,, \tag{9}$$

say.

Let us first consider $R_{\mathfrak{m}}(n)$. Recall that by Dirichlet's approximation theorem, for any $\alpha \in \mathfrak{m}$ there exists a rational number a/q such that $|\alpha - a/q| \le 1/q^2$, $(a, q) = 1$ and $P \le q \le Q$. Further, one has

$$\int_0^1 |S_1(\alpha)|^2\, d\alpha \ll L^4 \sum_{x-y<m\le x} \left(\sum\sum_{k^2+l^2=m} 1\right)^2 \ll yL^7\,. \tag{10}$$

Arguing as in [9, §2], we obtain

$$\begin{aligned}\sum_{x<n\le x+H} |R_{\mathfrak{m}}(n)|^2 &\ll HL \sup_{\alpha\in\mathfrak{m}} \int_{|\beta|\le 1/H} |S_2(\alpha+\beta)|^2\, d\beta \int_{\mathfrak{m}} |S_1(\alpha')|^2\, d\alpha' \\ &\ll (y^{3/4} + HL^{-B/4})L^{14}\, yL^7 \\ &\ll HyL^{-A}\,, \end{aligned} \tag{11}$$

by Lemma 1 and (10).

We proceed to $R_{\mathfrak{M}}(n)$. In order to approximate $S_2(\alpha)$, we define, for $\alpha = a/q + \beta \in \mathfrak{M}$,

$$T_2(\alpha) = \frac{s(q,a)}{\varphi(q)}\, t_2(\beta)\,, \qquad t_2(\beta) = \sum_{y/4<n\le y} \frac{e(\beta n)}{2\sqrt{n}}\,.$$

Note that $|s(q,a)| \ll q^{1/2+\varepsilon}$, and $|t_2(\beta)| \ll \min\Big(y, y^{-1}|\beta|^{-2}\Big)^{1/2}$ for $|\beta| \le 1/2$. Then

$$\int_{\mathfrak{M}} (|S_2(\alpha)|^2 + |T_2(\alpha)|^2)\,d\alpha$$

$$\ll \int_0^1 |S_2(\alpha)|^2\,d\alpha + \sum_{q\le P} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\frac{s(q,a)}{\varphi(q)}\right|^2 \int_{|\beta|\le 1/qQ} |t_2(\beta)|^2\,d\beta$$

$$\ll y^{1/2}L^2 + PL$$

$$\ll y^{1/2}L^2\,. \tag{12}$$

We next replace $S_2(\alpha)$ by $T_2(\alpha)$. The resulting cost is

$$\sum_{x<n\le x+H} \left| \int_{\mathfrak{M}} S_1(\alpha)(S_2(\alpha) - T_2(\alpha))e(-\alpha n)\,d\alpha \right|^2$$

$$\ll \left(H \max_{\substack{1\le a\le q\le P\\(a,q)=1}} \int_{|\beta|\le 1/qQ} \left| S_2\Big(\frac{a}{q}+\beta\Big) - T_2\Big(\frac{a}{q}+\beta\Big)\right|^2 d\beta \right.$$

$$\left. + P^2 \int_{\mathfrak{M}} |S_2(\alpha) - T_2(\alpha)|^2\,d\alpha \right) \int_{\mathfrak{M}} |S_1(\alpha')|^2\,d\alpha'$$

$$\ll \Big(HL^{-A-7} + P^2y^{1/2}L^2\Big)\,yL^7$$

$$\ll HyL^{-A}\,, \tag{13}$$

by Lemma 2′, (12) and (10).

Furthermore, we approximate $S_1(\alpha)$, and to this end we need to make the major arcs small. Put

$$\tilde{y} = yL^{-A-8}\,,$$

and let $\mathfrak{M}^\dagger$ be the union of all intervals $\{\alpha \in \mathbb{R} : |q\alpha - a| \le 1/\tilde{y}\}$ with $1 \le a \le q \le P$ and $(a,q) = 1$. Clearly $\mathfrak{M}^\dagger \subset \mathfrak{M}$. Observe that for $\alpha = a/q + \beta \in \mathfrak{M}\backslash\mathfrak{M}^\dagger$ one has $1/q\tilde{y} \le |\beta| \le 1/qQ$, and

$$|T_2(\alpha)|^2 \ll Lq^{-1}y^{-1}|\beta|^{-2}\,. \tag{14}$$

As before, we obtain

$$\sum_{x<n\le x+H}\left|\int_{\mathfrak{M}\setminus\mathfrak{M}^\dagger} S_1(\alpha)T_2(\alpha)e(-\alpha n)\,d\alpha\right|^2$$

$$\ll\left(H\max_{\substack{1\le a\le q\le P\\(a,q)=1}}\int_{1/q\tilde y\le|\beta|\le 1/qQ}\left|T_2\left(\frac{a}{q}+\beta\right)\right|^2 d\beta+P^2\int_{\mathfrak{M}\setminus\mathfrak{M}^\dagger}|T_2(\alpha)|^2\,d\alpha\right)$$

$$\times\int_{\mathfrak{M}\setminus\mathfrak{M}^\dagger}|S_1(\alpha')|^2\,d\alpha'$$

$$\ll\left(HL\tilde y y^{-1}+P^2PL\right)yL^7$$

$$\ll HyL^{-A}\,, \tag{15}$$

by (14), (12) and (10).

For $\alpha=a/q+\beta\in\mathfrak{M}^\dagger$, we can approximate $S_1(\alpha)$ by

$$T_1(\alpha)=\frac{\pi}{4}\left(\frac{s(q,a)}{\varphi(q)}\right)^2 t_1(\beta)\,,\qquad t_1(\beta)=\sum_{x-y<n\le x}e(\beta n)\,.$$

In fact, by partial summation and Lemma 3, we have that

$$\begin{aligned}|S_1(\alpha)-T_1(\alpha)|&\ll(1+|\beta|y)\,yL^{-2A-13}\\&\ll(1+y\tilde y^{-1})\,yL^{-2A-13}\\&\ll yL^{-A-5}\,,\end{aligned}\tag{16}$$

uniformly for $\alpha\in\mathfrak{M}^\dagger$. We also see that

$$\int_{\mathfrak{M}^\dagger}|T_1(\alpha)|^2\,d\alpha\ll\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\frac{s(q,a)}{\varphi(q)}\right|^4\int_{|\beta|\le 1/q\tilde y}|t_1(\beta)|^2\,d\beta\ll yL\,. \tag{17}$$

Hence, we infer that

$$\sum_{x<n\le x+H}\left|\int_{\mathfrak{M}^\dagger}(S_1(\alpha)-T_1(\alpha))T_2(\alpha)e(-\alpha n)\,d\alpha\right|^2$$

$$\ll H\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{|\beta'|\le 1/q\tilde y}\left|T_2\left(\frac{a}{q}+\beta'\right)\right|^2 d\beta'$$

$$\times \int_{|\beta|\le 1/q\tilde{y}} \left| S_1\left(\frac{a}{q}+\beta\right) - T_1\left(\frac{a}{q}+\beta\right)\right|^2 d\beta$$

$$+ P^2 \int_{\mathfrak{M}^\dagger} |T_2(a')|^2\, da' \int_{\mathfrak{M}^\dagger} |S_1(a) - T_1(a)|^2\, da$$

$$\ll H \sum_{q\le P} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\frac{s(q,a)}{\varphi(q)}\right|^2 L\,(q\tilde{y})^{-1}(yL^{-A-5})^2$$

$$+ P^2 PL \int_{\mathfrak{M}^\dagger} \left(|S_1(a)|^2 + |T_1(a)|^2\right) da$$

$$\ll HL^2L^{A+8}\, yL^{-2A-10} + P^3L^9y$$

$$\ll HyL^{-A}\,, \tag{18}$$

by (16), (12), (10) and (17).

It remains to consider

$$\int_{\mathfrak{M}^\dagger} T_1(a)T_2(a)e(-an)\,da$$

$$= \frac{\pi}{4} \sum_{q\le P} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left(\frac{s(q,a)}{\varphi(q)}\right)^3 e\left(-\frac{an}{q}\right) \int_{|\beta|\le 1/q\tilde{y}} t_1(\beta)t_2(\beta)e(-\beta n)\,d\beta\,.$$

We extend the interval of integration in the right-hand side to $|\beta| \le 1/2$. The resulting expression is equal to

$$\frac{\pi}{4}\ \sigma(n,P) \sum\sum_{\substack{l+m=n\\x-y<l\le x\\y/4<m\le y}} \frac{1}{2\sqrt{m}} = \frac{\pi}{4}\ \sigma(n,P)\left(\frac{\sqrt{y}}{2} + \mathcal{O}(1)\right).$$

By Schwartz's inequality and the dual form of the additive large sieve inequality we then conclude that

$$\sum_{x<n\le x+H} \left| \int_{\mathfrak{M}^\dagger} T_1(a)T_2(a)e(-an)\,da - \frac{\pi}{8}\sqrt{y}\,\sigma(n,P)\right|^2$$

$$\ll \int_{1/\tilde{y}\le|\beta|\le 1/2} |t_2(\beta)|^2 \sum_{x<n\le x+H} \left| \sum_{\substack{q\le P\\ q\tilde{y}|\beta|\ge 1}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left(\frac{s(q,a)}{\varphi(q)}\right)^3 e\left(-\frac{an}{q}\right)\right|^2 d\beta$$

$$\times \int\limits_{1/\tilde{y} \le |\beta'| \le 1/2} |t_1(\beta')|^2 \, d\beta' + HPL^2$$

$$\ll y^{-1} \left(\int_{1/\tilde{y}}^{1/2} \beta^{-2} \, d\beta \right)^2 (H + P^2) \sum_{q \le P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \frac{s(q,a)}{\varphi(q)} \right|^6 + HPL^2$$

$$\ll y^{-1} \tilde{y}^2 H + HPL^2$$

$$\ll HyL^{-A} \, .$$

Combining this with (9), (11), (13), (15) and (18), we complete the proof of Theorem 2. □

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Pervenuta in Redazione
il 12 settembre 2005