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The Topology of the Spectrum for Gelfand Pairs on Lie Groups.

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Sunto. – *Data una coppia di Gelfand di gruppi di Lie, identifichiamo lo spettro con un opportuno sottoinsieme di $\mathbb{C}^n$ e dimostriamo l'equivalenza tra la topologia di Gelfand e la topologia euclidea.*

Summary. – *Given a Gelfand pair of Lie groups, we identify the spectrum with a suitable subset of $\mathbb{C}^n$ and we prove the equivalence between Gelfand topology and euclidean topology.*

1. – Introduction.

Let (G, K) be a Gelfand pair with G a connected Lie group and K a compact subgroup. The Gelfand spectrum Σ of $L^1(G)^\natural$, the commutative convolution algebra of bi-K-invariant integrable functions on G, is identified, as a set, with the set of bounded spherical functions. The Gelfand topology on Σ is, by definition, the weak-$*$ topology, which coincides with the topology of uniform convergence on compact sets.

Since G is a connected Lie group, the spherical functions on G are characterized as the joint eigenfunctions of the algebra $\mathbb{D}(G/K)$ of differential operators on G/K invariant by left G-translation. Being this algebra finitely generated, we idenfity Σ with a subset of $\mathbb{C}^s$ assigning to each function the s-tuple of its eigenvalues with respect to a finite set of generators. Hence one can define on Σ also the Euclidean topology induced from $\mathbb{C}^s$. In this article we prove that the two topologies coincide.

2. – Gelfand pairs and spherical functions.

We briefly recall the general theory of Gelfand pairs, that can be found in [4].

Let $(G, \cdot)$ be a locally compact group, with a fixed left Haar measure dx. Let $K \leq G$ be a compact subgroup with normalized Haar measure dk.

DEFINITION 1. – *A function* $f : G \to \mathbb{C}$ *is said to be* bi-invariant under K *if it is constant on double cosets of* K*, i.e., if:*

$$f(k_1 x k_2) = f(x) \quad \forall k_1, k_2 \in K, \, \forall x \in G$$

Let $C_c(G)^\natural$ (resp. $L^1(G)^\natural$) be the set of continuous compactly-supported (resp. L^1) functions $f : G \to \mathbb{C}$ that are bi-invariant under K. It is easy to verify that it is a subalgebra of $C_c(G)$ (resp. of $L^1(G)$) with respect to the convolution in G.

DEFINITION 2. – (G, K) *is said to be a* Gelfand pair *if* $C_c(G)^\natural$ *is a commutative algebra.*

One can easily prove that $C_c(G)^\natural$ is dense in $L^1(G)^\natural$, therefore $C_c(G)^\natural$ is a commutative algebra if and only if $L^1(G)^\natural$ is.

Given a function $\varphi \in C(G)$ (not necessarily compactly-supported), we consider the linear functional:

$$\chi_\varphi : C_c(G) \to \mathbb{C}$$
$$\chi_\varphi(f) = \int_G f(x)\varphi(x^{-1})dx$$

DEFINITION 3. – *A function* $\varphi \in C(G)$*,* $\varphi \neq 0$*, is said to be* spherical *if it is bi-invariant under* K *and* χ_φ *is a character of* $C_c(G)^\natural$*, i.e.:*

$$\chi_\varphi(f * g) = \chi_\varphi(f) \cdot \chi_\varphi(g) \quad \forall f, g \in C_c(G)^\natural$$

One proves that φ is spherical if and only if:

$$\int_K \varphi(xky)dk = \varphi(x)\varphi(y) \quad \forall x, y \in G \tag{1}$$

(see [4] prop. I.3 p. 319). In particular, this implies that $\varphi(1_G) = 1$.

THEOREM 1. – *The dual space of* $L^1(G)^\natural$ *is* $L^\infty(G)^\natural$*. In fact, every continuous functional on* $L^1(G)^\natural$ *has the form:*

$$\chi_\varphi : f \to \int_G f(x)\varphi(x^{-1})dx$$

with $\varphi \in L^\infty(G)^\natural$ *unique and such that* $\|\chi_\varphi\| = \|\varphi\|_\infty$.

PROOF. – If $\varphi \in L^\infty(G)^\natural$, χ_φ is a continuous functional on $L^1(G)$, hence on its closed subspace $L^1(G)^\natural$, and $\|\chi_\varphi\| \leq \|\varphi\|_\infty$.

For the converse, let χ be a continuous functional on $L^1(G)^\natural$. By the Hahn-Banach theorem, we can extend χ to all of $L^1(G)$ without alterating its norm. So, being $L^\infty(G)$ the dual of $L^1(G)$, we have:

$$\chi(f) = \int_G f(x)\psi(x^{-1})\,dx$$

for some $\psi \in L^\infty(G)$, with $\|\chi\| = \|\psi\|_\infty$. Let $\varphi = \psi^\natural$ be the radialization of ψ, i.e.:

$$\psi^\natural(x) = \iint_{K\times K} \psi(k_1 x k_2)\,dk_1 dk_2 \tag{2}$$

It is easy to see that $\varphi \in L^\infty(G)^\natural$, and $\|\varphi\|_\infty \leq \|\psi\|_\infty = \|\chi\|$. Moreover, ψ and φ induce the same functional on $L^1(G)^\natural$. In fact, if $f \in L^1(G)^\natural$, we have:

$$\begin{aligned}\int_G f(x)\varphi(x^{-1})dx &= \int_G f(x)\iint_{K\times K} \psi(k_1 x^{-1} k_2)dk_1 dk_2 dx\\ &= \iint_{K\times K}\int_G f(x)\psi(k_1 x^{-1} k_2)dx dk_1 dk_2\\ &= \iint_{K\times K}\int_G f(k_2 x k_1)\psi(x^{-1})dx dk_1 dk_2\\ &= \int_G f(x)\psi(x^{-1})dx\end{aligned} \tag{3}$$

Hence every continuous functional on $L^1(G)^\natural$ has the form χ_φ for $\varphi \in L^\infty(G)^\natural$, with $\|\varphi\|_\infty \leq \|\chi_\varphi\|$. Since we have also proved that $\|\chi_\varphi\| \leq \|\varphi\|_\infty$, we can conclude that $\|\chi_\varphi\| = \|\varphi\|_\infty$. We now prove that φ is unique: by linearity of χ_φ in φ, we have to prove that $\chi_\varphi = 0 \Rightarrow \varphi = 0$. But $\chi_\varphi = 0 \Leftrightarrow \|\chi_\varphi\| = 0 \Leftrightarrow \|\varphi\|_\infty = 0 \Leftrightarrow \varphi = 0$. □

THEOREM 2. (See [4] Th. I.5 p. 320 or [7] Lemma 3.2 p. 408). – *An element φ of $L^\infty(G)^\natural$ defines a character of $L^1(G)^\natural$ if and only if φ is a bounded spherical function.* □

COROLLARY 3. – *A bounded spherical function has ∞-norm equal to* 1.

PROOF. – If $\varphi \in L^\infty(G)^\natural$ is spherical, it determines a character χ_φ of $L^1(G)^\natural$, which is a commutative Banach algebra, with $\|\chi_\varphi\| = \|\varphi\|_\infty$. Hence, $\|\varphi\|_\infty = 1$. □

Let Σ be the spectrum of $L^1(G)^\natural$, i.e., for the previous theorem, the set of bounded spherical functions. We define the *Fourier spherical transform* (see [4] p. 333):

$$\hat{f} : \Sigma \to \mathbb{C}$$

$$\hat{f}(\varphi) = \chi_\varphi(f) = \int_G f(x)\varphi(x^{-1})\, dx$$

We can introduce on Σ the *Gelfand topology*, i.e., the weak-$*$ topology.

THEOREM 4. – *The Gelfand topology on Σ is equal to the topology of uniform convergence on compact sets (or locally uniform convergence).* □

(The proof is similar to the one given in [8] p. 10-11.)

3. – The case of Lie groups.

If G and K are Lie groups, we can characterize Gelfand pairs and spherical functions by a differential point of view. Given a differential operator D on a manifold M (see [7] p. 239) and a diffeomorphism ϕ of M, we say that D is ϕ-invariant if $D(f \circ \phi) = Df \circ \phi \; \forall f \in C^\infty_C(M)$.

On a Lie group G we have a special family of diffeomorphisms, the left translations by elements of G: $\phi_g(x) = gx$. Remembering that a Lie group always admits an *analytic* structure compatible with the operations, we can construct a unique analytic structure also on the space of left cosets G/K (with the quotient topology) such that the G-action on G/K:

$$L : G \times G/K \to G/K$$

$$L(x, gK) = xgK$$

is analytic (see [6] p. 113).

Let $C^\infty_K(G)$ be the set of functions in $C^\infty(G)$ such that $f(xk) = f(x)$ $\forall k \in K, g \in G$. We have an isomorphism of algebras between $C^\infty(G/K)$ and $C^\infty_K(G)$ given by the projection π.

We consider three algebras of differential operators (see [7] p. 274-287 and [6] p. 389-398):

$$\mathbb{D}(G) = \{\text{diff. op. on } G \text{ invariant by left } G\text{-translation}\}$$

$$\mathbb{D}_K(G) = \{\text{diff. op. in } \mathbb{D}(G) \text{ invariant also by right } K\text{-translation}\}$$

$$\mathbb{D}(G/K) = \{\text{diff. op. on } G/K \text{ invariant by left } G\text{-translation}\}$$

We also consider the algebra:

$$\mathbb{D}_K^K(G) = \mathbb{D}_K(G)/A, \quad A = \{D \in \mathbb{D}_K(G) : Df = 0 \ \forall f \in C_K^\infty(G)\}$$

We can think of $\mathbb{D}_K^K(G)$ as the algebra of differential operators in $\mathbb{D}_K(G)$ acting only on $C_K^\infty(G)$: infact, if D and E coincide on $C_K^\infty(G)$, we have $D - E \in A$.

One can prove that $\mathbb{D}_K^K(G) \cong \mathbb{D}(G/K)$, with the isomorphism given by the projection π (see [6] lemma 2.2 p. 390).

THEOREM 5. (See [7] p. 485 ex. 13). – *Let G be a connected Lie group and let K be a compact subgroup. Then, (G, K) is a Gelfand pair if and only if $\mathbb{D}_K^K(G)$ is a commutative algebra.* □

THEOREM 6. (See [7] prop. 2.2 p. 400). – *Let (G, K) be a Gelfand pair of Lie groups and $f \in C(G)$. Then, f is spherical if and ony if:*
- $f \in C^\infty(G)^\natural$;
- $f(1_G) = 1$;
- *f is an eigenfunction of all the operators in $\mathbb{D}_K^K(G)$:*

$$Df = \lambda_D f \quad \forall D \in \mathbb{D}_K^K(G)$$ □

REMARK. – The proof of the theorem shows that a spherical function is necessarily *analytic*.

It can be proved that, being K compact, $\mathbb{D}_K^K(G)$ is a *finitely-generated* algebra (see [6] cor. 2.8 p. 395 and th. 5.6 p. 421). Let $D_1, \ldots, D_s$ be generators. Of course, φ is an eigenfunction of all the operators in $\mathbb{D}_K^K(G)$ if and only if it is an eigenfunction of the generators. In this way, we can associate to each spherical function the s-uple of eigenvalues $(\lambda_1, \ldots, \lambda_s)$ with respect to the generators. We can also prove that this association is injective, because the analyticity implies that two spherical functions having the same eigenvalues $(\lambda_1, \ldots, \lambda_s)$ must coincide (see [7] cor. 2.3 p. 402).

4. – The topology of the spectrum.

In this way, we identify a spherical function, and in particular a bounded one, with a point in $\mathbb{C}^s$. So we identify the spectrum Σ of $L^1(G)^\natural$ with a subset $A \subseteq \mathbb{C}^s$. Now, on Σ we have the Gelfand topology, and on A we have the induced euclidean topology. It is natural to ask if these two topologies coincide.

LEMMA 7 (See [3] p. 218). – *Let X be a topological space such that every point has a countable fundamental system of neighborhoods. Then a subset of X is closed if and only if it is sequentially closed.*

In particular, two such topologies coincide if and only if they induce the same notion of convergence on sequences. □

LEMMA 8. – $L^1(G)^\natural$ *is separable.*

PROOF. – We have only to prove that $L^1(G)$ is separable, because a subset of a separable space is separable (see [1] prop. III.22 p. 47). To see this, we choose a denumerable base of G (which exists by definition of differential manifold) and we consider the subspace generated by the characteristic functions of these base-sets. Then we can argue as for $L^1(\mathbb{R}^n)$ (see [1] Th. IV.13 p. 62). □

REMARK 9. – Being K compact, one can construct on G/K a riemannian metric invariant by the left action of G: hence, Laplace-Beltrami operator Δ with respect to this metric is invariant by left G-translations (see [6] Prop. 2.1 p. 387), i.e., $\Delta \in \mathbb{D}(G/K)$. This implies that, if φ is a spherical function, $\pi : G \to G/K$ is the projection and $\varphi^\pi = \varphi \circ \pi^{-1}$, then φ^π is an eigenfunction of Δ, which is an elliptic operator.

THEOREM 10. – *The induced euclidean topology on A and the Gelfand topology on Σ coincide under the bijection $\varphi \in \Sigma \longleftrightarrow (\lambda_1, \dots, \lambda_s) \in A$.*

PROOF. – Of course A is a metric space, so every point of A has a denumerable fundamental system of neighborhoods. By corollary 3, $\Sigma \subseteq B\Big(\big(L^1(G)^\natural\big)'\Big)$ (where B is the unit ball). Being $L^1(G)^\natural$ separable for lemma 8, the weak-$*$ topology is metrizable on the unit ball (see [1] Th. III.25 p. 48), in particular the Gelfand topology on Σ is metrizable. So, applying lemma 7, we have to prove that the two topologies we are considering induce the same notion of convergence.

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of spherical functions, and let $\mathbb{D}_K^K(G) = \langle D_1, \dots, D_s\rangle$. Let, $\forall i \in \{1, .., s\}$:

$$D_i\varphi_n = \lambda_{i,n}\varphi_n$$

$$D_i\varphi = \lambda_i\varphi$$

We have to prove that if $\varphi_n \to \varphi$ locally uniformly, then $\lambda_{i,n} \to \lambda_i$ $\forall i \in \{1, .., s\}$. But $D_i\varphi_n(1_G) = \lambda_{i,n}\varphi_n(1_G) = \lambda_{i,n}$ and similarly $D_i\varphi(1_G) = \lambda_i$. So, being $D_1, \dots, D_s$ generators, we have to prove that:

$$\varphi_n \to \varphi \text{ loc. unif.} \Rightarrow D\varphi_n(1_G) \to D\varphi(1_G), \quad \forall D \in \mathbb{D}_K^K(G)$$

If f is a spherical function, it is continuous and non-zero by hypotesis, so it is easy to construct a function $\rho \in C_c^\infty(G)$ such that:

$$\int_G f(x)\rho(x)dx \neq 0$$

(We have to choose a point $x_0 \in G$ such that $f(x_0) \neq 0$, choose by continuity a neighborhood $U(x_0)$ such that $\Re f$ or $\Im f$ has constant sign on U, and construct $\rho \geq 0$ such that $\operatorname{supp}(\rho) \subseteq U$ and $\rho(x_0) = 1$). So we have, by the formula (1):

$$
\begin{aligned}
f(x)\int_G f(y)\rho(y)\,dy &= \int_G \rho(y)\left(\int_K f(xky)\,dk\right)dy \\
&= \int_K\int_G \rho(y)f(xky)\,dydk = \int_K\int_G \rho(k^{-1}x^{-1}y)f(y)\,dydk \\
&= \int_G\left(\int_K \rho(k^{-1}x^{-1}y)dk\right)f(y)\,dy
\end{aligned}
$$

Concretely, the last integral in dy is not extended to all of G: indeed, the domain of integration is the set of y such that $\exists k \in K : k^{-1}x^{-1}y \in \operatorname{supp}(\rho)$, i.e., $x \cdot K \cdot \operatorname{supp}(\rho)$, which is compact because the product in G is continuous.

So, if we restrict x to an open neighborhood V of 1_G with $\overline{V}$ *compact*, we can assume that, for all such x, the domain of integration is $\overline{V} \cdot K \cdot \operatorname{supp}(\rho)$. We put:

$$C = \overline{V} \cdot K \cdot \operatorname{supp}(\rho)$$

$$A = \frac{1}{\int_G f(y)\rho(y)\,dy}$$

$$\psi(x,y) = \int_K \rho(k^{-1}x^{-1}y)dk$$

C is compact, $A \neq 0$ and, being K compact, $\psi(x,y) \in C^\infty(G \times G)$. So, in particular, $\psi(\cdot, y) \in C^\infty(V)\ \forall y \in C$. We have, for $D \in \mathbb{D}(G)$:

$$f|_V(x) = A\int_C \psi(x,y)f(y)\,dy$$

$$Df|_V(x) = A\int_C \left[D^{(x)}\psi(x,y)\right]f(y)\,dy$$

$$Df|_V(1_G) = A\int_C \eta_D(y)f(y)\,dy$$

with $\eta_D(y) = \left(D^{(x)}\psi(x,y)|_{x=1_G}\right)$. But $\eta_D(y)$ is a continuous function, indeed $\psi \in C^\infty(G \times G)$, so $D^{(x)}\psi(x,y) \in C^\infty(G \times G)$, and, composing with the immersion $y \to (1_G, y)$ we still obtain a $C^\infty(G)$ function. So, the restriction of η_D to C is still continuous.

So, applying the previous formula to φ_n and φ, we obtain:

$$D\varphi_n(1_G) = A_n \int_{C_n} \eta_{n,D}(y)\varphi_n(y)\,dy$$

$$D\varphi(1_G) = A \int_{C} \eta_{D}(y)\varphi(y)\,dy$$

But, by construction, we can suppose $\eta_{n,D} = \eta_D$: in fact, we can begin the construction with $\rho_n = \rho$. For this, being $\varphi_n(1_G) = \varphi(1_G) = 1$, we choose a neighborhood $U(1_G)$ with compact closure such that $\Re\varphi|_U \geq \delta > 0$. Then, being by hypotesis $\varphi_n|_U \to \varphi|_U$ uniformly, we can suppose that $\Re\varphi_n|_U > 0\ \forall n \in \mathbb{N}$. So we take $\rho_n = \rho$ such that $\rho(1_G) = 1$ and $\rho = 0$ outside U. From this we deduce that $\eta_{n,D} = \eta_D$ and $C_n = C$.

If $\varphi_n \to \varphi$ uniformly on compact sets, in particular uniformly on C, being η_D continuous and hence bounded on C, we have that $\eta_D \cdot \varphi_n \to \eta_D \cdot \varphi$ uniformly on C. So $\int_C \eta_D(y)\varphi_n(y)\,dy \to \int_C \eta_D(y)\varphi(y)\,dy$. Moreover, $A_n \to A$, in fact:

$$\left|\int_G \varphi_n(x)\rho(x)dx - \int_G \varphi(x)\rho(x)dx\right| \leq \int_G |\varphi_n(x) - \varphi(x)|\,\rho(x)dx$$

$$= \int_{\mathrm{supp}\,(\rho)} |\varphi_n(x) - \varphi(x)|\rho(x)dx \leq K \int_{\mathrm{supp}\,(\rho)} |\varphi_n(x) - \varphi(x)|dx \to 0$$

So $D\varphi_n(1_G) \to D\varphi(1_G)$.

For the converse, we know that $\Sigma \subseteq B\Big(\big(L^1(G)^\natural\big)'\Big)$, which is compact for the weak-$*$ topology by the Alaoglu-Banach theorem (see [1] Th. III.15 p. 42). Being the Gelfand topology metrizable on $B\Big(\big(L^1(G)^\natural\big)'\Big)$, compactness is equivalent to compactness by sequences (see [2] prop. 4.4 p. 188). We indicate with $\stackrel{\bullet}{\to}$ the convergence with respect to the euclidean topology on A. So let us suppose that $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \Sigma$ is such that $\varphi_n \stackrel{\bullet}{\to} \varphi$. By compactness, we can extract a convergent subsequence (with respect to the Gelfand topology) $\varphi_{n_k} \to \tilde{\varphi}$, with $\tilde{\varphi} \in B((L^1(G)^\natural)')$. But necessarily $\tilde{\varphi} \in \Sigma \cup \{0\}$: indeed, $\chi_{\varphi_{n_k}}(f * g) \to \chi_{\tilde{\varphi}}(f * g)$ by definition on Gelfand topology, but $\chi_{\varphi_{n_k}}(f * g) = \chi_{\varphi_{n_k}}(f) \cdot \chi_{\varphi_{n_k}}(g) \to \chi_{\tilde{\varphi}}(f) \cdot \chi_{\tilde{\varphi}}(g)$.

By remark 9, the functions φ_n^π are solutions of the equation:

$$(\Delta - \lambda_{\Delta,n})\varphi_n^\pi = 0$$

with Δ elliptic. Morover, $\lambda_{\Delta,n} \to \lambda_\Delta$ with λ_Δ defined by $\Delta\varphi^\pi = \lambda_\Delta\varphi^\pi$. Choosing a local chart (U, ξ) in the origin of G/K, we have, by [5] Th. 8.32 p. 210, with $\Omega = U$, $\Omega' \subseteq \overline{\Omega'} \subseteq \Omega, f = g = 0$, $a = 0$ and denoting by $\|\cdot\|_s$ the Sobolev norm of order s:

$$\|(\varphi_n^\pi \circ \xi^{-1})|_{\xi(\Omega')}\|_1 \leq C\big(\|(\varphi_n^\pi \circ \xi^{-1})|_{\xi(\Omega)}\|_0\big) = C$$

and one can easily verify that C is independent by n because $\lambda_{\Delta,n} \to \lambda_\Delta$, hence the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ is bounded. This implies that the functions $\varphi_n^\pi|_{\Omega'}$, and in particular the functions $\varphi_{n_k}^\pi|_{\Omega'}$, are equicontinuous and, by Arzela-Ascoli theorem (see [9] Th. 11.28 p. 245), there is a subsequence $\varphi_{n_{k_h}}^\pi|_{\Omega'} \to \psi^\pi$ locally uniformly on G/K. It is easy to deduce from this that $\varphi_{n_{k_h}}|_{\pi^{-1}(\Omega')} \to \psi$ locally uniformly on G. In particular, $\psi(1_G) = 1$ because $\varphi_{n_{k_h}}(1_G) = 1\ \forall k \in \mathbb{N}$. We can choose $\Omega'' \subseteq \overline{\Omega''} \subseteq \pi^{-1}(\Omega')$ neighborhood of 1_G such that $\Re\psi|_{\Omega''} \geq \delta > 0$: in particular, $\varphi_{n_{k_h}}|_{\Omega''} \to \psi|_{\Omega''}$ uniformly. Then, if $\chi_{\Omega''}$ is the characteristic function of Ω'', we consider the function:

$$\xi(x) = (\chi_{\Omega''})^\natural(x^{-1})$$

with $(\chi_{\Omega''})^\natural$ defined according to formula (2) pag. 3. We have:

$$\begin{aligned}\hat{\xi}(\varphi_{n_{k_h}}) &= \int_G \xi(x)\varphi_{n_{k_h}}(x^{-1})dx = \int_G (\chi_{\Omega''})^\natural(x^{-1})\varphi_{n_{k_h}}(x^{-1})dx \\ &= \int_G (\chi_{\Omega''})^\natural(x)\varphi_{n_{k_h}}(x)dx = \int_G \chi_{\Omega''}(x)\varphi^\natural_{n_{k_h}}(x)dx \\ &= \int_G \chi_{\Omega''}(x)\varphi_{n_{k_h}}(x)dx = \int_{\Omega''} \varphi_{n_{k_h}}(x)dx\end{aligned}$$

$$\Re\left[\hat{\xi}(\varphi_{n_{k_h}})\right] \to \int_{\Omega''} \Re\psi(x)dx \geq \delta|\Omega''| > 0$$

Hence, by definition of Gelfand topology, it is not possible that $\varphi_{n_k} \to 0$, so that $\varphi_{n_k} \to \tilde{\varphi} \in \Sigma$.

But, for the first part of the theorem, it must be $\varphi_{n_k} \xrightarrow{\bullet} \tilde{\varphi}$, so $\tilde{\varphi} = \varphi$. Hence, we have proved that for every sequence $\varphi_n \xrightarrow{\bullet} \varphi$, we can find a subsequence $\varphi_{n_k} \to \varphi$ uniformly on compact sets. Let us suppose that $\varphi_n \not\to \varphi$: then, we can find a compact set $C \subseteq G$, $\varepsilon > 0$ and a subsequence φ_{n_k} such that $\sup_{x\in C} |\varphi_{n_k}(x) - \varphi(x)| > \varepsilon$ $\forall k \in \mathbb{N}$. But, of course, $\varphi_{n_k} \xrightarrow{\bullet} \varphi$, so, applying the previous argument, we can find a sub-subsequence $\varphi_{n_{k_j}} \to \varphi$ uniformly on compact sets, in particular uniformly on C: a contradiction. □

From the proof of the previous theorem one can also conclude that:

COROLLARY 11. – *A is closed in $\mathbb{C}^s$.*

PROOF. – Let $\{z_n\}_{n\in\mathbb{N}} = \{(\lambda_{1,n}, \ldots, \lambda_{s,n})\}_{n\in\mathbb{N}}$ be a sequence in A, with $z_n \to z = (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s$. Let $\varphi_n \in \Sigma$ be the spherical function associated to z_n. We have that $\lambda_{\Delta,n} = P(\lambda_{1,n}, \ldots, \lambda_{s,n})$ with P polynomial, hence $\lambda_{\Delta,n} \to \lambda_\Delta = P(\lambda_1, \ldots, \lambda_n)$: the sequence $\{\lambda_{\Delta,n}\}$ is then bounded, hence, arguing as in the proof of the theorem, we can extract a subsequence $\varphi_{n_k} \to \tilde{\varphi} \in \Sigma$. But necessarily $\varphi_{n_k} \stackrel{\bullet}{\to} \tilde{\varphi}$, hence $(\lambda_1, \ldots, \lambda_s)$ is the point of A assiociated to $\tilde{\varphi}$, so $z \in A$. $\square$

COROLLARY 12. – *If $\varphi_n \to \varphi$ in Σ then $D\varphi_n \to D\varphi$ uniformly on compact sets for every differential operator D.*

PROOF. – For every $x_0 \in G$, we have, for V neighborhood of x_0 with $\overline{V}$ compact, C compact and $A_n \neq 0$:

$$\varphi_n|_V(x) = A_n \int_C \psi(x, y)\varphi_n(y)\, dy$$

$$D\varphi_n|_V(x) = A_n \int_C \left[D^{(x)}\psi(x, y)\right]\varphi_n(y)\, dy$$

$$D\varphi_n|_V(x) = A_n \int_C \eta_D(x, y)\varphi_n(y)\, dy$$

with η continuous. Similarly:

$$D\varphi|_V(x) = A \int_C \eta_D(x, y)\varphi(y)\, dy$$

By continuity, η_D is bounded on $\overline{V} \times C$, so in particular on $V \times C$, so, being $\varphi_n|_C \to \varphi|_C$ uniformly, we have that $\int_C \eta_D(x, y)\varphi_n(y)\, dy \to \int_C \eta_D(x, y)\varphi_n(y)\, dy$ uniformly on x. Moreover, $A_n \to A$, so $D\varphi_n|_V \to D\varphi|_V$ uniformly. $\square$

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